COMP760, LECTURES 14-15: NOISE STABILITY IN GAUSSIAN SPACE, INVARIANCE PRINCIPAL, THRESHOLD FUNCTION

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In the previous lecture, we introduced the *Noise Stability* of boolean functions and stated theorem 2.22 about noise stability of the *Majority Function*. We start this lecture by proving the corresponding theorem.

1. Noise Stability in Gaussian Space

Theorem 1.1 (Noise Stability of Majority Function). *Let $Maj_n : (R^n, \gamma_n) \to \{0,1\}$*

$$Maj_n : x \to \begin{cases} 1 & \sum x_i \geq 0 \\ 0 & \sum x_i < 0 \end{cases}$$

, then we have

$$\mathbb{S}_\rho(Maj_n) = \frac{1}{4} + \frac{\arccos(\rho)}{2\pi}$$

Proof.

$$S_\rho(Maj_n) = \mathbb{E}1_{[\sum x_i \geq 0]}1_{[\sum y_i \geq 0]} = \mathbb{E}1_{[\rho \sum y_i + \sqrt{1-\rho^2}\sum g_i \geq 0]}1_{[\sum y_i \geq 0]}$$

where y_i's and g_i's are i.i.d. Gaussians. $\sum y_i$ has the same distribution as $\sqrt{n}h$, where h is a Gaussian in $\mathbb{R}$. Similarly, $\sum g_i$ has distribution the same as $\sqrt{n}h'$. Therefore, the expected value is equal to:

$$\mathbb{E}1_{[\rho h + \sqrt{1-\rho^2}h' \geq 0]}1_{[h \geq 0]} = \frac{1}{4} + \frac{\arccos \rho}{2\pi}$$

$\square$

Definition 1.2 (Gaussian Rearrangement). *Given $A \subset \mathbb{R}^n$ its Gaussian Rearrangement A^* is defined to be the interval (t, ∞) with $\gamma_1(t,\infty) = \gamma_n(A)$.*

Recall that γ_i is the Gaussian measure on $\mathbb{R}^*$.

Theorem 1.3 (Borrell 83). *Let $A, B \subseteq \mathbb{R}^n$. Then for any $0 \leq \rho \leq 1$ and $q \geq 1$ we have:*

$$\mathbb{E}(U_\rho A)^q B \leq \mathbb{E}(U_\rho A^*)^q B^*$$

In particular,

$$\mathbb{S}_\rho(A) = \mathbb{E}AU_\rho A \leq \mathbb{E}A^*U_\rho A^* = \mathbb{S}_\rho(A^*)$$

Hence, $\gamma_n(A) = \frac{1}{2}$ then $\mathbb{S}_\rho(A) \leq \mathbb{S}_\rho(Maj_n) = \frac{1}{4} + \frac{\arccos \rho}{2\pi}$.

These notes are scribed by Athena K. Moghaddam.

2. Invariance Principle

Theorem 2.1 (Invariance Principal I). *Let $Q(x_1,...,x_n) = \sum_{S\leq[n]} \alpha_S \prod_{i\in S} x_i$ satisfies:*

(1) $$deg(Q) \leq d$$

(2) $$\sum_{|S|>0} \alpha_S^2 = 1$$

(3) $$I_i := \sum_{S:i\in S} \alpha_S^2 \leq \tau \ \ \forall i : 1,...,n$$

Then:

$$Sup_t|prob[Q(\varepsilon_1,...,\varepsilon_n) \leq t] - prob[Q(g_1,...,g_n) \leq t]| \leq O(d\tau^{\frac{1}{8d}})$$

Where $\varepsilon_1,...,\varepsilon_n$ are i.i.d. ±1 uniform random variables and $g_1,...,g_n$ are i.i.d. Gaussians.

Definition 2.2 (Rademacher Random Variable). *A uniform ±1 random variable is called a rademacher random variable.*

Theorem 2.3 (Invariance Principal II).

$$|\mathbb{E}[\Psi(Q(\varepsilon_1,...,\varepsilon_n))] - \mathbb{E}[\Psi(Q(g_1,...,g_n))]| \leq O(d9^dB\tau)$$

for all $\Psi : \mathbb{R} \to \mathbb{R}$ in C^4 (four times differentiable) with $|\Psi^{(4)}(t)| < B$ for all t.

Remark that if we could take $\Psi : x \to \begin{cases} |x| & x \leq t \\ 0 & otherwise \end{cases}$ then theorem II would imply theorem I. One instead has to approximate functions with bounded fourth derivatives.

Proof. Let $Z_i = Q(g_1,...,g_i,\varepsilon_{i+1},...,\varepsilon_n)$. We claim that $|\mathbb{E}\Psi(Z_{i-1} - \mathbb{E}\Psi(Z_i)| \leq O(B9^dI_i^2)$. First we show that the theorem can be extracted from this claim. Indeed,

$$|\mathbb{E}\Psi(Z_0) - \mathbb{E}\Psi(Z_n)| \leq \sum_{i=1}^n |\mathbb{E}\Psi((Z_{i-1}) - \mathbb{E}\Psi(Z_i)|$$

$$\leq O(B9^d)\sum_{i=1}^n I_i^2 = O(B9^d)$$

$$(maxI_i)\sum I_i \leq O(B9^d\tau)\sum I_i = O(B9^d\tau)\sum_{|S|>0}|S|/alpha_S^2 \leq O(dB9^d\tau)\sum_{|S|>0}\alpha_S^2 = O(\tau B9^dd)$$

To prove the claim $Q(x_1,...,x_n) = \sum_{S:i\notin S}\alpha_S\prod_{j\in s}x_j \ + \ x_i\sum_{S:i\in S}\alpha_S\prod_{j\in s\setminus\{i\}}x_j =$ $r(x_1,...,x_{i-1},x_{i+1},...,x_n) + s(x_1,...,x_{i-1},x_{i+1},...,x_n)$, let $R = r(g_1,...,g_{i-1},\varepsilon_{i+1},...,\varepsilon_n)$ and $S = s(g_1,...,g_{i-1},\varepsilon_{i+1},...,\varepsilon_n)$. We have $Z_{i-1} = R + \varepsilon_iS$ and $Z_i = R + g_iS$. Now using Taylor's theorem:

$$|\mathbb{E}\Psi(Z_{i-1} - \mathbb{E}\Psi(Z_i)| \leq$$

$$|\mathbb{E}\Psi(R) + \varepsilon_iS\Psi'(R) + \frac{(\varepsilon_iS)^2}{2}\Psi''(R) + \frac{(\varepsilon_iS)^3\Psi^{(3)}(R)}{6} + E_1$$

$$-\mathbb{E}\Psi(R) - g_iS\Psi'(R) - \frac{(g_iS)^2}{2}\Psi''(R) - \frac{(g_iS)^3\Psi^{(3)}(R)}{6} - E_2|$$

Where $|E_1| \leq \frac{|\Psi^{(4)}(\xi)|(\varepsilon_iS)^4}{24} \leq \frac{B(\varepsilon_iS)^4}{24}$ for some ξ between R and $R+\varepsilon_iS$. Similarly, $|E_2| \leq \frac{B(g_iS)^4}{24}$. All terms get canceled except the error terms E_1 and E_2. So the expression is bounded by:

$$\mathbb{E}|\frac{B(\varepsilon_iS)^4}{24}| + \mathbb{E}|\frac{B(g_iS)^4}{24}| \leq \frac{B}{24}\mathbb{E}S^4 + \frac{3B}{24}\mathbb{E}S^4\frac{B}{6}\mathbb{E}S^4$$

by *Hypercontractivity*

$$\leq \frac{B9^d}{6}(\mathbb{E}S^2)^2 = \frac{B9^d}{6}\sum_{i\in S}\alpha_S^2 = \frac{B9^d}{6}I_i^2$$

□

Lecture 15

In previous lectures, we claimed that *Majority Function* is the stablest in Gaussian space. Now, we are going to prove this fact using the properties of *Threshold Function.*

Definition 2.4. $T_\rho Q = \sum \rho^{|S|}\alpha_S \prod_{i\in S} x_i$

Definition 2.5 (Threshold Function)**.** *For any* $\mu \in [-1,1]$*, the function* $Thr^{(\mu)} : (\mathbb{R},\gamma_1) \to \{-1,1\}$ *is defined as:*

$$Thr^{(\mu)} : x \to \begin{cases} 1 & x \geq t_0 \\ -1 & x < t_0 \end{cases}$$

with $\mathbb{E}Thr^{(\mu)} = \mu$*.*

Theorem 2.6 (Majority is stablest in Gaussian space)**.** *Let* $f : (\mathbb{R}^n, \gamma_n) \to [1,-1]$ *with* $\mathbb{E}f = \mu$*. Then:*

$$\mathbb{S}_\rho(f) \leq \mathbb{S}_\rho(Thr^{(\mu)})$$

Theorem 2.7 (Majority is stablest in discrete setting)**.** *Let* $f : \{0,1\}^n \to [-1,1]$ *and* $I_i(f) = \sum_{S\ni i}|\widehat{f(s)}|^2 \leq \tau$ *for every* i*. Then for every* $0 \leq \rho \leq 1$*,* $\mathbb{S}_\rho(f) \leq \mathbb{S}_\rho(Thr^{(\mu)}) + O_\rho(\frac{loglog\frac{1}{\tau}}{log\frac{1}{\tau}})$ *where* $\mu = \mathbb{E}f$

Proof. Express $f = \sum \widehat{f(S)}\chi_S$. Let $Q(x_1,...,x_n) = \sum_{s\subseteq[[n]} \widehat{f(S)}\prod_{i\in S} x_i$. Therefore, $f(x_1,...,x_n) = Q(\varepsilon_1,...,\varepsilon_n)$ where $\varepsilon_i = (-1)^{x_i}$. Let $(g_1,...,g_n)$ be an i.i.d. Gaussian. We have

$$\mathbb{S}_\rho(f) = \sum \rho^{|S|}|\widehat{f(S)}|^2 = \mathbb{S}_\rho(Q(g_1,...,g_n))$$

We would like to apply invariance principal to replace rademachers with Gaussians. However, since the degree of Q can be as large as n, we cannot apply invariance directly to Q. Instead, we apply a *smoothed* version of the theorem, which can be applied on $T_\beta Q$ for $\beta < 1$. Let $\rho = \rho'\beta^2$ where $\beta < 1$ is very close to 1. $(0 < 1-\beta << 1-\rho)$ to be determined later.

$$\mathbb{S}_\rho(f) = \sum \rho^{|S|}|\widehat{f(S)}|^2 = \sum(\rho'\beta^2)^{|S|}|\widehat{f(S)}|^2 = \mathbb{S}_{\rho'}(T_\beta Q(g_1,...,g_n).$$

Now using the smoothed invariance, $T_\beta Q(g_1,...,g_n)$ is close in distribution to $T_\beta Q(\varepsilon_1,...,\varepsilon_n)$ and hence it cannot be far from being in [-1,1] . To make this precise we define function ξ as follows:

$$\xi : t \to \begin{cases} 0 & |t| \leq 1 \\ (|t|-1)^2 & |t| > 1 \end{cases}$$

Note that ξ measures the L_2-distance of t from its truncated value in [-1, 1]. By invariance principle of random variables $R = T_\beta Q(\varepsilon_1, ..., \varepsilon_n)$ and $S = T_\beta Q(g_1, ..., g_n)$ satisfy $|\mathbb{E}\xi(R) - \mathbb{E}\xi(S)| \leq \tau^{\Omega(1-\beta)}$. Let S' be the truncation of S to the interval [-1, 1]:

$$S' = \begin{cases} S & |S| \leq 1 \\ 1 & S > 1 \\ -1 & S < -1 \end{cases}$$

By assumption, $Q(\varepsilon_1, ..., \varepsilon_n) \in [-1, 1]$ and since T_β is an averaging operator, $T_\beta Q(\varepsilon_1, ..., \varepsilon_n) \in [-1, 1]$ and hence $\xi(R) = 0$.
Thus,

$$\begin{aligned} \mathbb{E}|\xi(S)| &= \mathbb{E}(S - S')^2 \leq \tau^{\Omega(1-\beta)} \\ \Rightarrow |\mathbb{S}_{\rho'}(S) - \mathbb{S}_{\rho'}(S')| &= |\mathbb{E}SU_{\rho'}S - \mathbb{E}S'U_{\rho'}S'| \\ &\leq |\mathbb{E}SU_{\rho'}S - \mathbb{E}S'U_{\rho'}S| + |\mathbb{E}S'U_{\rho'}S - \mathbb{E}S'U_{\rho'}S'| \\ &\leq \|S - S'\|_2\|U_{\rho'}S\|_2 + \|S'\|_2\|U_{\rho'}(S - S')\|_2 \\ &\leq \|S - S'\|_2\|S\|_2 + \|S'\|_2\|S - S'\|_2 \leq \tau^(\Omega(1-\beta)). \end{aligned}$$

By Borrell's theorem, $\mathbb{S}_{\rho'}(S') \leq \mathbb{S}_{\rho'}(Thr^{\mu'})$ where $\mu' = \mathbb{E}S'$. Now, we just have to show that $\mu = \mu'$:

$$|\mu - \mu'| = |\mathbb{E}(S - S')| \leq \|S - S'\|_2 \leq \tau^(\Omega(1-\beta))$$

$$\Rightarrow |\mathbb{S}_{\rho'}(Thr^{\mu} - \mathbb{S}_{\rho'}(Thr^{\mu'}| \leq O(\frac{1-\beta}{1-\rho})$$

$$\Rightarrow \mathbb{S}_\rho(f) = \mathbb{S}_\rho(Thr^{(\mu)}) + O(\tau^{(\Omega(1-\beta)} + \frac{1-\beta}{1-\rho}$$

and by optimizing the last expression over β the result yields to the theorem claim. $\square$

3. Applications of "Majority is Stablest" Theorem

Definition 3.1 (Condorcet Method for Ranking 3 Candidates)**.** *In an election with n voters and 3 candidates, A, B and C, each voter submits 3 bits representing his preferences. The first bit indicates whether he prefers A to B; The second one shows his preference between B and C and the third one shows the same fact over C and A. These preferences are aggregated into 3 strings* $x, y, z \in (-1, 1)^n$. *A boolean function* $f : \{-1, 1\}^n \mapsto -1, 1$ *is applied to x, y and z and the aggregated preference is represented by* $(f(x), f(y), f(z))$.

Definition 3.2 (Condorcet Paradox)**.** *If f is the Majority function we have an irrational outcome, in which all 3 aggregated bits are 1 or all are -1 representing preferences* $A < B < C < A$ *or* $A > B > C > A$.

Definition 3.3. *A triple* $(a, b, c) \in \{-1, 1\}^3$ *is called rational, if it corresponds to a non-cyclic ordering.*

Theorem 3.4 (Ken Arrow's Impossibility Theorem)**.** *The only functions f that never give irrational outcomes are dictator functions* $f(x) = x_i$ *or* $f(x) = 1 - x_i$ *for some i.*

Note that every voter has 6 possible rational rankings. Suppose that every voter votes independently at random from the 6 possible choices. Let $x, y, z \in \{-1, 1\}^n$ be the corresponding random string. Note that:

$$1_{[a_1=a_2=a3]} = \frac{1}{4} + \frac{1}{4}a_1a_2 + \frac{1}{4}a_1a_3 + \frac{1}{4}a_2a_3$$

$$\Rightarrow \Pr[(f(x), f(y), f(z))] = 1 - \mathbb{E}1_{[f(x)=f(y)=f(z)]} = \frac{3}{4} - \frac{1}{4}\mathbb{E}f(x)f(y) - \frac{1}{4}\mathbb{E}f(x)f(z) - \frac{1}{4}\mathbb{E}f(y)f(z)$$
$$= \frac{3}{4} - \frac{3}{4}\mathbb{E}f(x)f(y) = \frac{3}{4} - \frac{3}{4}\sum \widehat{f(S)}\widehat{f(T)}\mathbb{E}\chi_S(x)\chi_T(y)$$

Now we know that,

$$\mathbb{E}\chi_S(x)\chi_T(y) = (\prod_{i \in S\cap T} \mathbb{E}x_i y_i)(\prod_{i \in S\setminus T} \mathbb{E}x_i)(\prod_{i \in T\setminus S} \mathbb{E}y_i)$$

Since $\mathbb{E}y_i = \mathbb{E}x_i = 0$ and $\mathbb{E}x_i y_i = \frac{2}{6} - \frac{4}{6} = -\frac{1}{3}$, so $\mathbb{E}\chi_S(x)\chi_T(y) = \begin{cases} 0 & S \neq T \\ (\frac{-1}{3})^{|S|} & S = T \end{cases}$. Hence,

$$\Pr[(f(x), f(y), f(z)) \text{ is rational}] = \frac{3}{4} + \frac{3}{4}\sum (\frac{-1}{3})^{|S|}|\widehat{f(S)}|^2 \leq \frac{3}{4} + \frac{3}{4}\mathbb{S}_{\frac{1}{3}}(f)$$

Corollary 3.5. *If f satisfies $I_i(f) = o_n(1)$ and $\mathbb{E}f = 0$, then rationality of $f \leq \frac{3}{4} + \frac{3}{4}\arcsin\frac{1}{3} + o_n(1) \leq 0.9123 + o_n(1)$.*

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