COMP 531 - Winter 2021 - Assignment 4

Due: April 17, 11:59pm

- Solve all the problems.
- In solving these questions you may only consult the lecture notes, and the Sipser book, but you need to provide citations if you are using a fact that has not been covered in the lectures.
- Each student must find and write their own solution. Copying solutions from any source, completely or partially, allowing others to copy your work, will not be tolerated, and will be reported to the disciplinary office.
- You must submit your solutions as **one readable pdf file** to mycourses.
- Your grade will be based on the mathematical correctness of your solution as well as the quality of your presentation.

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1. (a) (1 point) Let $\epsilon > 0$ be a parameter, and ϕ be a DNF with s clauses. Prove that there exists a DNF ψ with width at most $O(\log(s/\epsilon))$ such that

$$\Pr_x[\phi(x) \neq \psi(x)] \leq \epsilon.$$

   (b) (3 points) Use the sunflower theorem to prove the following statement: For every monotone DNF ϕ with width w and size m, every $\epsilon > 0$, there exists a DNF formula ψ with width w and size at most $(w2^w \log(m/\epsilon))^{O(w)}$ such that $\psi(x) \geq \phi(x)$ for all x, and

$$\Pr_x[\phi(x) \neq \psi(x)] \leq \epsilon.$$

2. (5 points) In this exercise our goal is to prove that unlike majority, *approximate majority* can be solved by an AC^0 circuit. More precisely we want to construct a polynomial size circuit C of depth 3 such that if $|x| := \sum x_i \geq \frac{3n}{4}$, then $C(x) = 1$, and if $|x| < \frac{n}{4}$, then $C(x) = 0$. For $\frac{n}{4} \leq |x| < \frac{3n}{4}$, we do not care about the output of C.

   We construct a sequence of random circuit as follows:

   - C_0 is a single variable randomly chosen from $x_1, \ldots, x_n$.
   - $C_1 = \bigwedge_{i=1}^{10 \log n} D_i$, where D_i are independent copies of the random variable C_0.
   - $C_2 = \vee_{i=1}^{n^{15}} D_i$, where D_i are independent copies of the random variable C_1.
   - $C_3 = \bigwedge_{i=1}^{n^2} D_i$, where D_i are independent copies of the random variable C_2.

   For any fixed x, prove that if $|x| < \frac{n}{4}$, then

$$\Pr_{C_3}[C_3(x) = 1] < 2^{-n^2},$$

   and if $|x| \geq \frac{3n}{4}$, then

$$\Pr_{C_3}[C_3(x) = 0] < 2^{-n^2}.$$

   Conclude that there is a polynomial size depth 3 circuit C such that for all $|x| < \frac{n}{4}$, $C(x) = 0$, and for all $|x| \geq \frac{3n}{4}$, $C(x) = 1$.

3. (6 points) Let $k > 0$ be a fixed constant, and let the threshold function $t_k : \{0,1\}^n \to \{0,1\}$ be defined as $t_k(x) = 1$ if and only if $\sum x_i \geq k$. Prove that there is a (one-sided error) randomized parity decision tree T_R of constant depth that computes $t_k(x)$. That is

$$\Pr_R[T_R(x) = t_k(x)] = 1 \qquad \text{for all } x \text{ with } t_k(x) = 0.$$

$$\Pr_R[T_R(x) = t_k(x)] \geq \frac{1}{2} \qquad \text{for all } x \text{ with } t_k(x) = 1.$$

(Hint: First solve the problem for $k = 1$. For general k, randomly partition the variables into $k+1$ disjoint sets, and focus on each of these sets.)

4. (1 point) Let $f : \{-1,1\}^n \to \{0,1\}$. Prove that

$$\sum_S |\widehat{f}(S)| \leq 2^{n/2}\sqrt{\mathbb{E}_x[f(x)]},$$

and give an example to show that this inequality is sharp.

5. (2 point) Prove that if $\mathrm{dt}(f) \leq d$, then

$$\sum_S |\widehat{f}(S)| \leq 2^d.$$

6. For a function $f : \{-1,1\}^n \to \{0,1\}$, define

$$L_{1,k}(f) = \sum_{S:|S|=k} |\widehat{f}(S)|,$$

and let

$$L_1(f) = \sum_S |\widehat{f}(S)|.$$

(a) (1 point) Show that $L_{1,k}(f) \leq n^{k/2}$.

(b) (3 points) Let f be computed by a t-DNF. The switching lemma says that for a random restriction ρ with parameter p, we have

$$\Pr[\mathrm{dt}(f_\rho) \geq s] \leq (4pt)^s.$$

Use this to prove that for $p = \frac{1}{100t}$,

$$\mathbb{E}_\rho\left[L_1(f_\rho)\right] \leq 2.$$

(c) (3 points) Let f be computed by a t-DNF.. Use the previous part to show that for $p = \frac{1}{100t}$,

$$L_{1,k}(f) \leq 2p^{-k} = 2(100t)^k.$$

Note that this bound does not depend on n.