COMP 531 - Fall 2022 - Assignment 2

Due: Oct 25, 11:59pm

- In solving these questions you may only consult the lecture notes, and the Sipser book, but you need to provide citations in that case.
- Each student must find and write their own solution. Copying solutions from any source, completely or partially, allowing others to copy your work, will not be tolerated, and will be reported to the disciplinary office. You are allowed to discuss the problems with each other without revealing your solution to each other.
- You must submit your solutions as **one readable pdf file** to mycourses.
- Your grade will be based on the mathematical correctness of your solution as well as the quality of your presentation.

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1. Prove that
$$\mathsf{NP} \subseteq \mathsf{BPP} \iff \mathsf{NP} = \mathsf{RP}.$$
2. Prove that if there is a polynomial time oracle Turing Machine M^{SAT} that takes two CNF's ϕ_1 and ϕ_2 and outputs their satisfiability correctly using *only one* oracle query to SAT, then $\mathsf{P} = \mathsf{NP}$.
3. Let $\text{SIZE}_{\text{dt}}(f)$ denote the smallest number of leaves in a decision tree computing f. Prove that for every $f : \{0,1\}^n \to \{0,1\}$ if ρ is a random restriction with parameter p (that is $\Pr[*] = p$, $\Pr[1] = \Pr[0] = \frac{1-p}{2}$), we have
$$\mathbb{E}_\rho[\text{SIZE}_{\text{dt}}(f_\rho)] \leq \text{SIZE}_{\text{dt}}(f)^{\log_2(1+p)}.$$
4. A randomized decision tree T of depth k is a probability distribution over deterministic decision trees of depth k. Suppose that for f :

$\{0,1\}^n \to \{0,1\}$ there is a randomized decision tree T of depth k such that for every $x \in \{0,1\}^n$, we have

$$\Pr[T(x) \neq f(x)] \leq \frac{1}{3}.$$

Let $S \subseteq \{0,1\}^n$ be of size $n^{O(1)}$. Prove that there is a deterministic decision tree of depth $O(k \log(n))$ that compures f correctly on all the inputs in S.

5. A language $L \subseteq \{0,1\}^*$ is sparse if there is a polynomial $p(n)$ such that $|L \cap \{0,1\}^n| \leq p(n)$ for every n. Show that every sparse language is in P/poly.