COMP 531 - Fall 2022 - Assignment 1

Due: Sept 30, 11:59pm

- In solving these questions you may only consult the lecture notes, and the Sipser book, but you need to provide citations in that case.
- Each student must find and write their own solution. Copying solutions from any source, completely or partially, allowing others to copy your work, will not be tolerated, and will be reported to the disciplinary office. You are allowed to discuss the problems with each other without revealing your solution to each other.
- You must submit your solutions as **one readable pdf file** to mycourses.
- Your grade will be based on the mathematical correctness of your solution as well as the quality of your presentation.

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1. Consider the following problem. We are given an arrangement of an $n \times n$ chess board with chess pieces as input. We are asked "Can white win in at most 10 moves if she starts first?" More precisely, "If white starts, can she always win in at most 10 moves, independent of how black plays?"

   Which complexity class in the polynomial hierarchy captures this problem?

2. Prove that the following language is in L.

   $$\{ww^R : w \in \{0,1\}^*\},$$

   where w^R is the reverse of w.

3. Show that if PH does not collapse, that is $\mathsf{PH} \neq \Sigma_k$ for every k, then it does not have a complete problem under polynomial-time reductions.

4. Recall that a Boolean formula is an expression involving Boolean variables, and Boolean operations $\wedge, \vee, \neg$ (but no quantifiers $\exists, \forall$). Two Boolean formulas are called equivalent if they are on the same set of variables, and are true on the same set of truth assignments. Consider the following language

   $$\text{MinFormula} = \{\phi : \phi \text{ has no shorter equivalent formula}\}.$$

   - Prove that MinFormula $\in \mathsf{PSPACE}$.
   - Prove that if $\mathsf{P} = \mathsf{NP}$, then MinFormula $\in \mathsf{P}$.

5. (Exercise 7.2) Where is the error in the following incorrect proof that NonPATH $\in$ NL?

   Given an input $\langle G, s, t\rangle$, it is easy to see that there are no st-paths in G if and only if there exists a partition of the vertices into two parts A and B such that $s \in A$, $t \in B$, and there are no edges going from A to B. Let the vertices of G be $u_1, \ldots, u_n$, and let $x = (x_1, \ldots, x_n) \in \{0,1\}^n$ be such that $x_i = 0$ if $u_i \in A$ and $x_i = 1$ if $u_i \in B$. To verify that $\langle G, s, t\rangle \in$ NonPATH, the verifier is going to be provided with a certificate C consisting of k copies of x, where k is the number of edges in G:

   $$C = [\underbrace{x, x, \ldots, x}_{k \text{ times}}].$$

   Now as the verifier reads the i-th x, it verifiers that the edge e_i does not go from A to B. It will also use x to verify that $s \in A$ and $t \in B$. Once it finishes reading all of C it will have verified that none of the edges e_i goes from A to B. Thus it will successfully verify that $\langle G, s, t\rangle \in$ NonPATH.

6. (Exercise 6.8) Prove that if in the definition of the NL-certifier if relax the read-once condition on the certificate-tape to an ordinary read-only tape, then it will be powerful enough so that every problem in NP will have an NL-verifier.

7. Prove a Non-deterministic Time Hierarchy theorem: If T_2 is time-constructible and $T_1(n+1)\log T_1(n+1) = o(T_2(n))$, then $\mathsf{NTIME}(T_1(n)) \neq \mathsf{NTIME}(T_2(n))$.

   - First explain why the same diagonalization used for the deterministic time complexity does not work here without any changes.
   - Show how to fix the argument so that it works for the non-deterministic time complexity. (**Hint:** Use a *lazy diagonalization* argument: Only try to disagree at least once in an exponentially large interval. Consider a large intervals $(\ell_k, u_k]$, and for an input $1^n \in (\ell_k, u_k]$ if $n = u_k$, then deterministically run M_k on 1^{ℓ_k+1} for an appropriate number of steps, and negate its output. If $n < u_k$, then use a non-deterministic simulation.)

8.