COMP 531 - Fall 2021 - Assignment 4

Due: Nov 26, 11:59pm

- Each student must find and write their own solution. Copying solutions from any source, completely or partially, allowing others to copy your work, will not be tolerated, and will be reported to the disciplinary office. You are allowed to discuss the problems with each other without revealing your solution to each other.
- You must submit your solutions as **one readable pdf file** to mycourses.
- Your grade will be based on the mathematical correctness of your solution as well as the quality of your presentation.

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1. (a) Let $f : \{-1,1\}^n \to \{0,1\}$. Prove that

$$\sum_S |\widehat{f}(S)| \leq 2^{n/2}\sqrt{\mathbb{E}_x[f(x)]}.$$

Show that this inequality is sharp for the point-mass $f(x) = 1 \Leftrightarrow x = \vec{0}$.

(b) Estimate

$$\mathbb{E}_f\left[\sum_S |\widehat{f}(S)|\right],$$

where the expected value is over all functions $f : \{-1,1\}^n \to \{0,1\}$.

(c) Prove that if $\mathrm{dt}(f) \leq d$, then

$$\sum_S |\widehat{f}(S)| \leq 2^d.$$

2. For a function $f : \{-1,1\}^n \to \{0,1\}$, define

$$L_{1,k}(f) = \sum_{S:|S|=k} |\widehat{f}(S)|,$$

and let

$$L_1(f) = \sum_S |\widehat{f}(S)|.$$

(a) Show that $L_{1,k}(f) \leq n^{k/2}$.

(b) Let f be computed by a t-DNF. The switching lemma says that for a random restriction ρ with parameter p, we have

$$\Pr[\mathrm{dt}(f_\rho) \geq s] \leq (4pt)^s.$$

Use this to prove that for $p = \frac{1}{100t}$,

$$\mathbb{E}_\rho\left[L_1(f_\rho)\right] \leq 2.$$

(c) Let f be computed by a t-DNF. Use the previous part to show that

$$L_{1,k}(f) \leq 2p^{-k} = 2(100t)^k.$$

Note that this bound does not depend on n.

3. In this exercise our goal is to prove that unlike majority, *approximate majority* can be solved by an AC^0 circuit. More precisely we want to construct a polynomial size circuit C of depth 3 such that if $|x| := \sum x_i \geq \frac{3n}{4}$, then $C(x) = 1$, and if $|x| < \frac{n}{4}$, then $C(x) = 0$. For $\frac{n}{4} \leq |x| < \frac{3n}{4}$, we do not care about the output of C.

   We construct a sequence of random circuit as follows:

   - C_0 is a single variable randomly chosen from $x_1, \ldots, x_n$.
   - $C_1 = \bigwedge_{i=1}^{10 \log n} D_i$, where D_i are independent copies of the random variable C_0.
   - $C_2 = \vee_{i=1}^{n^{15}} D_i$, where D_i are independent copies of the random variable C_1.
   - $C_3 = \bigwedge_{i=1}^{n^2} D_i$, where D_i are independent copies of the random variable C_2.

   For any fixed x, prove that if $|x| < \frac{n}{4}$, then

$$\Pr_{C_3}[C_3(x) = 1] < 2^{-n^2},$$

   and if $|x| \geq \frac{3n}{4}$, then

$$\Pr_{C_3}[C_3(x) = 0] < 2^{-n^2}.$$

   Conclude that there is a polynomial size depth 3 circuit C such that for all $|x| < \frac{n}{4}$, $C(x) = 0$, and for all $|x| \geq \frac{3n}{4}$, $C(x) = 1$.

4. Let $f : \mathcal{X} \times \mathcal{Y} \to \{-1, 1\}$. In the lectures we have seen that $\log(\frac{1}{\mathrm{Disc}(f)}) \leq \mathrm{R}_{1/3}(f)$. In this exercise, we are interested in the opposite direction. Prove that $\mathrm{R}_{1/3}(f) \leq O(\frac{1}{\mathrm{Disc}(f)^2})$.

5. Let A be an $n \times n$ matrix with ± 1 entries with sign-rank $d = O(1)$. Prove that A contains an $\Omega(n) \times \Omega(n)$ monochromatic rectangle. Hint: First write $A = \mathrm{sgn}(\langle u_i, v_i \rangle)$ for $u_i, v_j \in \mathbb{R}^d$, and then convert the vectors u_i into isotropic position.