COMP 360 - Winter 2023 - Assignment 2

Instructor: Hamed Hatami

Due: Feb 14th, 11:59pm.

General rules:

• You may not upload this assignment to any website.

• In solving these questions, you may consult books or other available notes. You can discuss high-level ideas with each other, but each student must find and write their own solution.

• You are not allowed to share your solutions with other students.

• Copying solutions from any source, entirely or partially, allowing others to copy your work, will not be tolerated and will be reported to the disciplinary office.

• You should upload (either typed or a clear and readable scan) of your solution to clouldmark. For the last question, you must upload your code to mycourses.

1. (3 points) You are working for the City of Montréal, which is undertaking a large project to redevelop a neighbourhood. This project can be broken down into many smaller individual jobs, some of which depend on each other but which can otherwise be completed in parallel. The city does not care about disrupting the area and simply wants to complete it as quickly as possible, but since there are so many jobs to complete they are having trouble coming up with an optimal schedule. There are $n$ jobs to complete, which take time $t_1, \ldots, t_n$, respectively. Each job $j \in \{1, \ldots, n\}$ also has a (potentially empty) set $S_j \subseteq \{1, \ldots, j-1\}$ of jobs which must be completed before work on $j$ can begin. Your task is to come up with a set of optimal starting times $s_1, \ldots, s_n \geq 0$ for jobs $1, \ldots, n$, respectively. Your goal is to optimize the time when all the individual jobs are completed.

Formulate this problem as a linear program.

2. (3 points) Time management is often a difficult task. Usually Tim studies $m$ hours per week. In this term, he has $n$ courses and course $i \in \{1, \ldots, n\}$ requires at minimum $k_i$ hours (e.g. going to classes, discussions). Assume there is a family of linear functions $f_i(x) = a_ix + b_i$ with $a_i \in [0, \infty)$ and $b_i \in \mathbb{R}$, which estimate the expected score of the course $i$ if Tim studies $x$ hours per week for that course. How can Tim allocate his time per week so that he can achieve the maximum expected average score of the whole term?

Write a linear program that solves this problem.

3. (a) (1 point) Draw the feasible region of the inequality $|x| + |y| \leq 1$.

(b) (1 point) Express $|x| + |y| \leq 1$ as a system of linear constraints.
(c) (3 points) In an arcade video game, the player controls a character that can move either up, down, left or right but not diagonally. In each round of the gameplay, the player is shown a map marked with \( n \) stars with coordinates \((a_1, b_1), \ldots, (a_n, b_n)\). The player chooses a location \((u, v)\) to place their character, and then a true target is revealed at one of the stars. The player moves the character from the selected location to the target, and more points are awarded if the character travels a shorter distance to the target.

Since the true target is revealed after the character is positioned, a safe strategy for the player is to minimize the longest travelling distance \( d \) from the selected position to any potential target. Write a linear program that models the problem of finding the minimum value of \( d \).

4. Consider the following linear program.

\[
\begin{align*}
\text{max } & \quad 3x - 4y + z + 3w \\
s.t. & \quad 5x - z + 3w \leq 6 \\
& \quad 2x - y + 2w \geq -7 \\
& \quad x \leq 0 \\
& \quad y \geq 0 \\
& \quad z \leq 0 \\
& \quad w \geq 0
\end{align*}
\]

(\( \ast \))

(a) (1 point) Write down the dual of the above linear program (without converting it to the standard form).

(b) (1 point) By plotting the boundaries defined by the dual of (\( \ast \)), show that the dual program is infeasible.

(c) (1 point) Provide a justification that (\( \ast \)) is feasible without solving the linear program.

(d) (1 point) Explain why the duality theorem is not violated in this example.

5. (5 points) The goal of this question is to write a python script which uses linear programming to solve the minimum cut problem. As input, you are given a .json file encoding the vertex and edge sets of a flow network. As an example, the json encoding of the following flow network is included:
Note that your script should work for any non-negative capacities, including non-integer ones (with at most 5 decimals).

Fill in the missing portions of the provided starter code so that your script prints a list of the vertices in the minimum cut. You may modify function signatures as you wish. Note that the flow returned by the linear program solver is not necessarily integer valued, so you should be careful with conditions such as $f(e) > 0$ or $f(e) < c_e$ (due to floating point rounding errors).

Your script should use the package PuLP for solving LPs. The starter code includes examples of PuLP syntax, and the full documentation can be found here: [https://coin-or.github.io/pulp/](https://coin-or.github.io/pulp/)


- Clarification: Your program should use the Linear Program Solver to find a Max-Flow. Then you need to use the corresponding residual graph to find the corresponding min-cut $(A^*, B^*)$. In constructing the residual graph you should be careful about the floating point rounding errors in conditions such as $f(e) > 0$ or $f(e) < c_e$

- For this question, you should upload your code to mycourses. (It was brought to my attention that crowdmark does not allow code uploads).