COMP 360, SHORTEST PATH AS A LINEAR PROGRAM

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Consider the following problem: We are given a directed graph $G = (V, E)$, with two distinct nodes s and t marked as source and sink respectively. Furthermore to every edge e, a cost $\alpha_e \geq 0$ is assigned. The graph is considered as a flow network (but with no capacity conditions). Our goal is to find a flow of value 1 that has minimum cost $\sum_{e \in E} \alpha_e f(e)$.

Let P be the shortest (cost) path from s to t. Let D be the cost (equivalently length if α_e are interpreted as lengths of edges) of this path. Note that augmenting this path, and thus assigning $f(e) = 1$ to edges on this path, and $f(e) = 0$ to edges $e \notin P$ gives us a valid flow with value 1 and a cost Δ. We would like to show that this is actually optimal.

Let us formulate this problem as a linear program: Our variables are $f(e)$ for $e \in E$.

$$
\begin{array}{lrll}
\min & \sum_{uv \in E} \alpha_{uv} f(uv) & & \\
\text{s.t.} & \sum_{vu_0 \in E} f(vu_0) - \sum_{u_0 w \in E} f(u_0 w) & = 0 & \forall u_0 \in V - \{s, t\} \\
& \sum_{vs \in E} f(vs) - \sum_{sw \in E} f(sw) & = -1 & \\
& \sum_{vt \in E} f(vt) - \sum_{tw \in E} f(tw) & = 1 & \\
& f(uv) & \geq 0 & \forall uv \in E
\end{array}
$$

As we have shown above there is a feasible solution to this linear program with cost Δ. So we know that

$$\text{Opt}(LP) \leq \Delta.$$

Next we need to prove a lower-bound of Δ for $\text{Opt}(LP)$. One way to do this is to find a feasible solution to the dual with cost Δ. Hence let us write the dual of this LP.

The variables are x_u for $u \in V$. They correspond to the constraints in the primal linear program, first for $u \in V - \{s, t\}$, then the constraint for s, and then the one for t.

$$
\begin{array}{lrll}
\min & x_t - x_s & & \\
\text{s.t.} & x_v - x_u & \geq \alpha_{uv} & \forall uv \in E \\
& x_u & \geq 0 & \forall u \in E
\end{array}
$$

For every vertex u, let d_u be the cost of the shortest path from s to u. In particular $d_s = 0$ and $d_t = \Delta$. Note that for every edge uv we have $d_v \leq d_u + \alpha_{uv}$ since to go from s to v, one can first go to u at cost d_u and then take the edge uv with the additional cost of α_{uv} (Recall that this is what's used in Dijkstra and Bellman-Ford algorithms). This shows that setting $x_u = d_u$ is a feasible solution to the dual LP, and moreover, this solution has cost $x_t - x_s = \Delta - 0 = \Delta$. We conclude that

$$\Delta \leq \text{Opt}(DLP).$$

Now by weak duality, we have

$$\Delta \leq \text{Opt}(DLP) \leq \text{Opt}(LP) \leq \Delta,$$

which implies

$$\text{Opt}(DLP) = \text{Opt}(LP) = \Delta.$$

References

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