COMP 360 - Fall 2019 - Sample Final Exam

1. Either prove that the following problem is NP-complete, or show that it belongs to P:
   - Input: A graph G.
   - Question: Is G a Hamiltonian *bipartite* graph?

2. Either prove that the following problem is NP-complete, or show that it belongs to P:
   - Input: A CNF ϕ.
   - Question: Is there a truth assignment that satisfies none of the clauses in ϕ.

3. Consider the following optimization version of the Subset-Sum problem: Given positive integers $\{w_1, \ldots, w_n\}$ and a positive integer m. We want to find a set $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} w_i \leq m$ and is maximized. Show that the following is a $\frac{1}{2}$-factor approximation algorithm:
   - Set $S := \emptyset$.
   - Sort the numbers such that $w_1 \geq w_2 \geq \ldots \geq w_n$.
   - For $i = 1, \ldots, n$:
   - if it is possible add i to S without violating $\sum_{i \in S} w_i \leq m$, then add i to S.

4. Problem 10 of Chapter 11 textbook: Suppose you are given an $n \times n$ grid graph G. Associated with each node v is an integer weight $w(v) \geq 0$. You may assume that all the weights are distinct. Your goal is to choose an independent set S of nodes of the grid, so that the sum of the weights of the nodes in S is as large as possible. (The sum of the weights of the nodes in S will be called its total weight.) Consider the following greedy algorithm for this problem.
   - Start with $S := \emptyset$.
   - While some node remains in G:
     - Pick a node v of maximum weight.
     - Add v to S.
     - Delete v and its neighbors from G
   - Endwhile.

   Show that this algorithm returns an independent set of total weight at least $\frac{1}{4}$ times the maximum total weight of any independent set in the grid graph G.

5. Consider a directed bipartite graph $G = (V, E)$. We want to eliminate all the directed cycles of length 4 by removing a smallest possible set of <u>vertices</u>.

(a) Let $\mathcal{C}_4$ denote the set of all cycles of length 4 in the graph. Show that the following integer program models the problem:

$$\begin{array}{lll} \min & \sum_{v\in V} x_v & \\ \text{s.t.} & \sum_{u\in C} x_u \geq 1 & \forall C \in \mathcal{C}_4 \\ & x_u \in \{0,1\} & u \in V \end{array}$$

(b) Why does the optimal solution to the following relaxation provides a lower bound for the optimal answer to the above integer linear program? In other words why it is not necessary to have the constraints $x_u \leq 1$ in the relaxation?

$$\begin{array}{lll} \min & \sum_{v\in V} x_v & \\ \text{s.t.} & \sum_{u\in C} x_u \geq 1 & \forall C \in \mathcal{C}_4 \\ & x_u \geq 0 & \forall u \in V \end{array}$$

(c) Give a simple 4-factor approximation algorithm for the problem based on rounding the solution to the above linear program.

(d) (Let L and R denote the set of the vertices in the two parts of the bipartite graph. (Every edge has one endpoint in L and one endpoint in R). Let x^* denote an optimal solution to the linear program in Part (b). We round x^* in the following way:

For every $u \in V$,

- if $u \in R$ and $x^*_u \geq 1/2$, set $\widehat{x}_u = 1$.
- if $u \in L$ and $x^*_u > 0$, set $\widehat{x}_u = 1$.
- Otherwise set $\widehat{x}_u = 0$.

Show that $\widehat{x}$ is a feasible solution to the integer linear program.

(e) Consider the dual of the relaxation:

$$\begin{array}{lll} \max & \sum_{C\in\mathcal{C}_4} \; y_C & \\ \text{s.t.} & \sum_{C\in\mathcal{C}_4, u\in C} \; y_C \leq 1 & \forall u \in V \\ & y_C \geq 0 & \forall C \in \mathcal{C}_4 \end{array}$$

and let y^* be an optimal solution to the dual. Use the complementary slackness to prove the following statement: For every $C \in \mathcal{C}_4$ either we have $|\{u : \widehat{x}_u = 1\}| \leq 3$ or $y^*_C = 0$.

(f) Use the complementary slackness and the previous parts to show that our rounding algorithm is a 3-factor approximation algorithm.