# COMP 360 - Fall 2019 - Sample Final Exam

1. We have a supply of $m$ different kinds of material. Specifically, we have $b_j$ units of raw material of kind $j$. They are $n$ different kinds of products that we can make from these raw material. Each unit of product $i$ takes $a_{i1}$ unit of raw material 1, $a_{i2}$ units of raw material 2 etc. to create, and can be sold at the price $c_i$ dollars. It is possible to sell fractional amounts of any product. Our goal is to produce the most profitable set of products with our available material.

   (a) Formulate the above optimization problem as a linear program.
   **Solution**: We will create a variable $x_i$ representing the amounts of product $i$ produced. We then have the following linear program:

$$\begin{aligned} \max\ & c_1x_1 + c_2x_2 + ... + c_nx_n \\ \text{s.t.}\ & \sum_{i=1}^{n} a_{ik}x_i \leq b_k \ \forall\ 1 \leq k \leq m \\ & x_1, x_2, ..., x_n \geq 0 \end{aligned}$$

   (b) Write the dual of this program.
   **Solution**: We introduce variables $y_1, ..., y_m$ for each $b_i$. We then have the following linear program:

$$\begin{aligned} \min\ & b_1y_1 + b_2y_2 + ... + b_my_m \\ \text{s.t.}\ & \sum_{i=1}^{m} a_{ki}y_i \geq c_k \ \forall\ 1 \leq k \leq n \\ & y_1, y_2, ..., y_m \geq 0 \end{aligned}$$

2. Formulate the following problem as a linear program: Let $G = (V, E)$ be an undirected graph. We want to assign a positive number to every vertex of $G$ such that the total sum of these numbers is 1, and the largest load of a vertex in the graph is minimized. The load of a vertex is the sum of the number on that vertex and the numbers on its immediate neighbours.

   **Solution:**

$$\text{Min D}$$

$$\text{s.t. } D \geq l_v + \textstyle\sum_{(u,v)\in E} l_u \qquad \forall v \in V$$

$$\textstyle\sum_{v\in V} l_v = 1$$

$$l_v \geq 0 \qquad \forall v \in V$$

3. Either prove that the following problem is NP-complete or prove that it belongs to P.

   - Input: A CNF $\phi$ with 10 clauses (and $n$ variables).
   - Question: Is there a truth assignment that satisfies $\phi$?

   **solution:** The problem is in $P$. Observe that each clause is true if one of its literals is true. So we can try all combinations of 10 variables (out of $n$) together with their assignment, and see if there is one such combination that makes $\phi$ true. That is from each clause we pick one term, and set all of these to true (if they are consistent)/ There are at most $\binom{n}{10} \times 2^{10} = O(n^{10})$ such combinations. For each combination, it takes polynomial time to check. So the algorithm takes polynomial time.

4. Either prove that the following problem is NP-complete or prove that it belongs to P.

   - Input: A CNF $\phi$ on $2n$ variables.
   - Question: Is there a truth assignment that satisfies $\phi$ and assigns True to exactly $n$ variables?

   **solution:** The problem is NP-complete. Given an assignment, we can verify if it satisfies the CNF in polynomial time, just as in the SAT problem. So the problem is in NP.

   To show completeness, we reduce SAT to this problem. Let $\phi'$ be a CNF with $n$ variables $x_1, \ldots, x_n$. Let
   $$\phi = \phi' \bigwedge_{i=1}^{n} (x_i \vee y_i) \wedge (\bar{x}_i \wedge \bar{y}_i).$$
   If $\phi'$ has a satisfying assignment, then by assigning $y_i = \bar{x}_i$, we get a satisfying assignment of $\phi$ with exactly $n$ literals assigned true. (Because if exactly $s$ among $x_1, \ldots, x_n$ are assigned true, then exactly $n - s$ among $y_1, \ldots, y_n$ are assigned true.) On the other direction, if $\phi$ has a satisfying assignment, then clearly $\phi'$, as part of $\phi$, has a satisfying assignment.

5. Consider a graph $G = (V, E)$. The chromatic number of $G$ is the minimum number of colors required to color the vertices of $G$ properly. Let $\mathcal{I}$ be the set of all independent sets in $G$ (Note that every element in $\mathcal{I}$ is a set).

   (a) Prove that the solution to the following linear program provides a lower-bound for the chromatic number of $G$.

   $$\begin{array}{lll} \min & \sum_{I \in \mathcal{I}} x_I & \\ \text{s.t.} & \sum_{I: v \in I} x_I \geq 1 & \forall v \in V \\ & x_I \geq 0 & \forall I \in \mathcal{I} \end{array}$$

   **Solution:** Consider an optimal coloring of the vertices of $G$ with $k$-colors where $k$ is the chromatic number of $G$. Note that every color-class forms an independent set as vertices of the same color cannot be adjacent. Let $I_1, \ldots, I_k$ be the independent sets corresponding to the color-classes. Consider the following solution to the linear program: set $x_{I_1} = \ldots = x_{I_k} = 1$ and $x_I = 0$ for every other independent set $I$. This is a feasible solution as every vertex $v$ belongs to exactly one of $I_1, \ldots, I_k$, and hence $\sum_{I: v \in I} x_I = 1$. Moreover the objective value of the linear program for this feasible solution is $k$. Hence the optimal value is at most $k$.

(b) Write the dual of the above linear program.

**Solution:**

$$\begin{array}{lll} \max & \sum_{v\in V} y_v & \\ \text{s.t.} & \sum_{v\in I} y_v \leq 1 & \forall I \in \mathcal{I} \\ & y_v \geq 0 & \forall v \in V \end{array}$$

(c) Prove that every clique in $G$ provides a solution to the dual linear program.

**Solution:** Let $S$ be the set of the vertices of a clique in $G$. Set $y_v = 1$ if $v \in S$ and $y_v = 0$ if $v \notin S$. Since $S$ forms a clique, no independent set contains more than one vertex from $S$, and thus $\sum_{v\in I} y_v \leq 1$ for every $I \in \mathcal{I}$. On the other hand, the objective value of the linear program for this feasible solution is equal to $|S|$.

6. Show that if we strengthen linear programming by also allowing constraints of the form $\sum_{i,j=1}^{n} a_{ij}x_i x_j = b$ (for integers $b$ and $a_{ij}$), then the problem becomes NP-complete.

**Solution I:** A feasible solution can be used as a certificate and can be verified easily. To show that the problem is NP-complete, we reduce the VertexCover problem to this problem. Since $y^2 = 1$ and $x = \frac{y+1}{2}$ is equivalent to saying that $x \in \{0,1\}$, we can solve the VertexCover problem using the following program:

$$\begin{array}{lll} \min & \sum_u x_u & \\ \text{s.t.} & x_u + x_v \leq 1 & \forall uv \in E \\ & y_u^2 = 1 & \forall u \in V \\ & x_u - \frac{y_u}{2} = \frac{1}{2} & \forall u \in V \end{array}$$

**Solution II:** A feasible solution can be used as a certificate and can be verified easily. To show that the problem is NP-complete, we reduce the VertexCover problem to this problem. Note that the size of the minimum vertex cover is $\frac{K+n}{2}$ if $K$ is the solution to the following program, and $n$ is the number of vertices of $G$:

$$\begin{array}{lll} \min & \sum_u x_u & \\ \text{s.t.} & x_u + x_v \leq 1 & \forall uv \in E \\ & x_u^2 = 1 & \forall u \in V \end{array}$$

Indeed $x_u^2 = 1$ guarantees that $x_u \in \{-1,1\}$, and the constraints $x_u + x_v \leq 1$ guarantee that the set $S$ of the vertices that receive 1 form a vertex cover. Thus $K = \sum_u x_u = \sum_{u\in S} 1 + \sum_{u\notin S}(-1) = 2|S| - n$.

7. Show that if we strengthen linear programming by also allowing constraints of the form $|\sum_{i=1}^{n} a_i x_i| \geq b$ (for integers $b$ and $a_i$), then the problem becomes NP-complete.

**Solution:** Note that $|y| \geq 1$ and $y \leq 1$ and $-y \leq 1$ together imply that $y \in \{-1,1\}$. Hence in the solution to the previous question, we can replace $y^2 = 1$ with these three constraints.