Consider the max flow problem in a flow network $G$ with multiple sources $s_1, \ldots, s_k$ and multiple sinks $t_1, \ldots, t_r$. Here the conservation and the capacity condition are as before, but the value of a flow $f$ is defined to be

$$\text{val}(f) = f^{\text{out}}(s_1) + \ldots + f^{\text{out}}(s_k).$$

Show that this problem can be converted to the original max flow problem (with a single source and a single sink).

**Solution:** Construct a flow network $G'$ from $G$ by adding a sink $s$, a source $t$, and edges $ss_1, \ldots, ss_k$ and $t_1t, \ldots, t_rt$ all with $\infty$ capacities. In $G'$, $s$ is the only source and $t$ is the only sink, so $G'$ is a flow network.

We claim that max flow in $G'$ equals to max flow in $G$. Indeed let $f$ be a max-flow in $G$, with

$$\text{val}(f) = f^{\text{out}}(s_1) + \ldots + f^{\text{out}}(s_k).$$

We can extend $f$ to a valid flow for $G'$ by assigning $f(ss_i) = f^{\text{out}}(s_i)$ and $f(t_jt) = f^{\text{in}}(t_j)$ for all $i$ and $j$. Note that this is a valid flow for $G'$ and moreover its value is

$$f^{\text{out}}(s) = f(ss_1) + \ldots + f(ss_k) = f^{\text{out}}(s_1) + \ldots + f^{\text{out}}(s_k) = \text{val}(f).$$

This shows that Max flow in $G'$ is at least the max flow in $G$. On the other hand take a max flow $g$ in $G'$. Now if we restrict $g$ to the original graph $G$ we will obtain valid flow for $G$ whose value (by definition) is

$$g^{\text{out}}(s_1) + \ldots + g^{\text{out}}(s_k) = g^{\text{in}}(s_1) + \ldots + g^{\text{in}}(s_k) = g^{\text{out}}(s),$$

and that is exactly the max flow in $G'$. So we also showed that the max flow in $G$ is least the max flow in $G'$. 