1. (4 points) Consider the following problem: We are given a number \( r \) and a maximization linear program \( LP \) (say in the standard form) as follows:

\[
\begin{align*}
\text{max} & \quad c^T \cdot \vec{x} \\
\text{s.t.} & \quad A \vec{x} \leq \vec{b} \\
& \quad \vec{x} \geq 0
\end{align*}
\]

That is the input is \([A, c, b, r]\), where \( A \) is an \( m \times n \) matrix, \( b \) is an \( m \)-dimensional vector, and \( c \) is an \( n \)-dimensional vector, and \( r \) is a number, and we want to know if \( \text{Opt}(LP) \geq r \)?

Without using the fact that Linear Programming can be solved efficiently using the ellipsoid method show that this problem belongs to both NP and CoNP.

2. We are given the coordinates of \( n \) points \( p_1, \ldots, p_n \) on the plane. We want to find non-overlapping disks centred on these points such that the sum of their radii is maximized.

(a) (2 points) Show that this problem can be solved as a linear program.

(b) (4 points) Consider a complete directed graph on the points \( p_1, \ldots, p_n \) where for every \( i \) and \( j \), there is an edge from \( p_i \) to \( p_j \), and one edge from \( p_j \) to \( p_i \), both with a cost equal to \( d(p_i, p_j) \), the distance between the two points. We want to cover the vertices of this graph with cycles (that is every vertex belongs to at least one cycle) so that the total sum of the cost of the edges that participate in these cycles is minimized (Note that we also allow cycles with two edges).

Use the linear programming duality to prove that the solution to this problem is at least twice the solution to the non-overlapping disk problem.
3. (4 points) We are given an $n \times m$ table. The rows correspond to $n$ students and the columns correspond to their favourite things (For example, food, book, sport, composer, movie, etc). For example if the entry in the food column of the $i$-th row is Samosa, it means that Samosa is the favourite food of the $i$-th student. We are also given a number $k$, and our goal is to select $k$ students so that every two of them share at least one favourite thing in life. Prove that this problem is NP-complete.

4. (4 points) We are given $n$ numbers (each in decimal representation), and a number $k$. We want to see if it is possible to select $k$ of these numbers so that if we add them up, the digit 2 will not appear in the decimal representation of the sum. Prove that this problem is NP-complete.

5. (4 points) Prove that the following problem is NP-complete.
   
   • Input: A CNF $\phi$.
   • Question: Does $\phi$ have a truth assignment that assigns True to exactly half the terms in each clause?

   (For example $x_1 = T, x_2 = F, x_3 = F, x_4 = F$ assigns True to half the terms in each clause of $\phi = (x_1 \lor \overline{x_2} \lor x_3 \lor x_4) \land (x_1 \lor x_2)$.)