1. Either prove that the following problem is NP-complete, or show that it belongs to $P$:
   - Input: A graph $G$.
   - Question: Is $G$ a Hamiltonian bipartite graph?

2. Either prove that the following problem is NP-complete, or show that it belongs to $P$:
   - Input: A CNF $\phi$.
   - Question: Is there a truth assignment that satisfies none of the clauses in $\phi$.

3. Consider the following optimization version of the Subset-Sum problem: Given positive integers $\{w_1, \ldots, w_n\}$ and a positive integer $m$. We want to find a set $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} w_i \leq m$ and is maximized. Show that the following is a $\frac{1}{2}$-factor approximation algorithm:
   - Set $S := \emptyset$.
   - Sort the numbers such that $w_1 \geq w_2 \geq \ldots \geq w_n$.
   - For $i = 1, \ldots, n$:
     - if it is possible add $i$ to $S$ without violating $\sum_{i \in S} w_i \leq m$, then add $i$ to $S$.

4. Problem 10 of Chapter 11 textbook: Suppose you are given an $n \times n$ grid graph $G$. Associated with each node $v$ is an integer weight $w(v) \geq 0$. You may assume that all the weights are distinct. Your goal is to choose an independent set $S$ of nodes of the grid, so that the sum of the weights of the nodes in $S$ is as large as possible. (The sum of the weights of the nodes in $S$ will be called its total weight.) Consider the following greedy algorithm for this problem.
   - Start with $S := \emptyset$.
   - While some node remains in $G$:
     - Pick a node $v$ of maximum weight.
     - Add $v$ to $S$.
     - Delete $v$ and its neighbors from $G$
   - Endwhile.

Show that this algorithm returns an independent set of total weight at least $\frac{1}{4}$ times the maximum total weight of any independent set in the grid graph $G$.

5. Consider a directed bipartite graph $G = (V,E)$. We want to eliminate all the directed cycles of length 4 by removing a smallest possible set of vertices.
(a) Let $C_4$ denote the set of all cycles of length 4 in the graph. Show that the following integer program models the problem:

$$\begin{align*}
\min & \quad \sum_{v \in V} x_v \\
\text{s.t.} & \quad \sum_{u \in C} x_u \geq 1 \quad \forall C \in C_4 \\
& \quad x_u \in \{0, 1\} \quad u \in V
\end{align*}$$

(b) Why does the optimal solution to the following relaxation provides a lower bound for the optimal answer to the above integer linear program? In other words why it is not necessary to have the constraints $x_u \leq 1$ in the relaxation?

$$\begin{align*}
\min & \quad \sum_{v \in V} x_v \\
\text{s.t.} & \quad \sum_{u \in C} x_u \geq 1 \quad \forall C \in C_4 \\
& \quad x_u \geq 0 \quad \forall u \in V
\end{align*}$$

(c) Give a simple 4-factor approximation algorithm for the problem based on rounding the solution to the above linear program.

(d) (Let $L$ and $R$ denote the set of the vertices in the two parts of the bipartite graph. (Every edge has one endpoint in $L$ and one endpoint in $R$). Let $x^*$ denote an optimal solution to the linear program in Part (b). We round $x^*$ in the following way:

For every $u \in V$,
- if $u \in R$ and $x^*_u \geq 1/2$, set $\hat{x}_u = 1$.
- if $u \in L$ and $x^*_u > 0$, set $\hat{x}_u = 1$.
- Otherwise set $\hat{x}_u = 0$.

Show that $\hat{x}$ is a feasible solution to the integer linear program.

(e) Consider the dual of the relaxation:

$$\begin{align*}
\max & \quad \sum_{C \in C_4} y_C \\
\text{s.t.} & \quad \sum_{C \in C_4, u \in C} y_C \leq 1 \quad \forall u \in V \\
& \quad y_C \geq 0 \quad \forall C \in C_4
\end{align*}$$

and let $y^*$ be an optimal solution to the dual. Use the complementary slackness to prove the following statement: For every $C \in C_4$ either we have $|\{u : \hat{x}_u = 1\}| \leq 3$ or $y^*_C = 0$.

(f) Use the complementary slackness and the previous parts to show that our rounding algorithm is a 3-factor approximation algorithm.