COMP 360 - Fall 2011 - Final Exam

1. (10 Points) Let (A, B) be a minimum cut in a flow network, and suppose that there are exactly two edges e_1 and e_2 going from A to B.

   (a) Prove: Decreasing the capacity of e_1 by 1 and at the same time increasing the capacity of e_2 by 1 cannot increase the value of the maximum flow.

   (b) Disprove: Decreasing the capacity of e_1 by 1 and at the same time increasing the capacity of e_2 by 1 cannot decrease the value of the maximum flow.

2. (5 Points) Write the dual of the following linear program:

$$\begin{array}{ll} \min & x_1 \\ \text{s.t.} & x_1 + x_2 \geq 8 \\ & x_1 - x_2 = 4 \\ & x_1 \geq 0 \text{ and } x_2 \text{ is universal} \end{array}$$

3. (5 Points) If a maximization linear program is unbounded, then what can be said about its dual? (Is it unbounded, feasible and bounded, or infeasible?) Explain.

4. (5 Points) Show that the following problem belongs to P:

$$X = \{\langle G\rangle \mid G \text{ has an independent set of size } 10\}.$$

5. (5 Points) Describe the complexity class PSPACE.

6. (10 Points) Write a linear program for solving the following problem: Given an $n \times n$ matrix A and an n-dimensional vector b, we want to find a vector

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

such that $Ax = b$ and that $\max_{i=1}^{n} |x_i|$ is minimized (Hint: Try to use the constrains of the form $y \geq a$ and $y \geq -a$).

7. (5 Points) Show that the following language is NP-complete:

$$X = \{\langle G, k \rangle \ : \ G \text{ has a cycle of length at least } k\}.$$

8. (10 Points) A kite is a graph on an even number of vertices, say $2k$, in which k of the vertices form a clique and the remaining k vertices are connected in a tail that consists of a path joined to one of the vertices of the clique. Prove that KITE is NP-complete, where

$$\text{KITE} = \{\langle G, k\rangle \ : \ G \text{ has a subgraph which is a kite on } 2k \text{ vertices}\}.$$

9. (10 Points) Show that the following language is NP-complete:

$$Z = \{\langle\{w_1, \ldots, w_n\}, k\rangle \mid w_1, \ldots, w_n \text{ and } k \text{ are positive integers and } \exists S \subseteq \{w_1, \ldots, w_n\} \text{ such that } \textstyle\sum_{w \in S} w = 2^k\}.$$

10. Consider the MAX-SAT problem: Given a CNF formula ϕ on variables $x_1, \ldots, x_n$, find a truth assignment to the variables that maximizes the total number of satisfied clauses.

    (a) (10 Points) Show that the following is a $\frac{1}{2}$-factor approximation algorithm for MAX-SAT: Let σ_{true} be the truth assignment that assigns True to every variable, and σ_{false} be the truth assignment that assigns False to every variable. Compute the number of clauses satisfied by σ_{true} and σ_{false}, and output the better assignment.

(b) (5 Points) Give a tight example: An input instance where this algorithm performs as bad as the $\frac{1}{2}$ factor.

11. (10 Points) Recall the triangle elimination problem: We are given a graph $G = (V, E)$, and want to find the smallest possible set of vertices $U \subseteq V$ such that the subgraph of G induced by $V - U$ does not contain any triangles (i.e. cycles of length 3). Consider the following algorithm:

   - Set U to be the empty set.
   - While there are still triangles in the graph:
   -     find a triangle and add all its three vertices to U and remove them from G.

   Prove that this is a 3-factor approximation algorithm.

12. (10 Points) Given a set U and subsets $S_1, \ldots, S_{2^m} \subseteq U$, each of size $m+2$, we want to color the elements of U with two colors Red and Blue such that every set S_i contains both colors. Consider the following randomized algorithm:

- For $i = 1, \ldots, 1000$:
- Pick a random coloring of U with two colors Red and Blue.
- If every set S_i contains both colors, then return the coloring.
- EndFor.

Show that the probability that the algorithm does not succeed in finding the desired coloring is at most 2^{-1000}.