COMP 360

1. (10 Points) Consider a flow network $(G, s, t, \{c_e\})$.

   (a) Prove or Disprove: There always exists a minimum cut which contains the edge with smallest capacity.

   (b) Prove or Disprove: For every flow network with at most m edges, there always exists a sequence of at most m augmentations that leads to a maximum flow?

2. (10 Points) Write the dual of the following linear program:

$$\begin{array}{ll}
\max & x_1 + x_2 + 4x_3 \\
\text{s.t.} & x_1 + x_2 + x_3 \leq 8 \\
 & x_1 + 2x_2 \leq 4 \\
 & x_1, x_2, x_3 \geq 0
\end{array}$$

3. (10 Points) State the strong duality theorem for linear programs.

4. (10 Points) Show that the following problem belongs to P:

$$X = \{\langle G \rangle \mid G \text{ has an independent set of size } n - 10\}.$$

(Here n is the number of vertices of G, and an independent set is a set of vertices, no two of which are adjacent).

5. (10 Points) Write a linear program for solving the following problem: Given an $n \times n$ matrix A and an n-dimensional vector b, we want to find a vector

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

such that $Ax = b$ and that $(\min_{i=1}^{n} x_i)$ is maximized.

6. (10 Points) Show that the following problem is NP-complete:

$$X = \{\langle G \rangle \ : \ G \text{ has a cycle of length at least } n-1\}.$$

7. (10 Points) Show that the following problem is NP-complete:

- Input: An integer m and a graph $G = (V, E)$ where every vertex v has a cost $c_v \geq 0$.
- Question: Does G have an independent set whose total cost is exactly m?

8. (10 Points) Consider the MAX-CUT problem: Given a graph G, we want to partition its vertices into two sets A and B such that the number of edges between A and B is maximized. Show that the following is a $\frac{1}{2}$-factor approximation algorithm for MAX-CUT.

- Let $v_1, \ldots, v_n$ be the vertices of G.
- Set $A := \emptyset$ and $B := \emptyset$.
- For $i = 1, \ldots, n$:
- if v_i has more neighbors in A than in B, add it to B, otherwise add it to A.

9. (10 Points) Formulate the maximum independent set problem as an Integer Linear Program. (In the maximum independent set problem, given a graph $G = (V, E)$, we want to find the largest set S of vertices in G such that no two vertices in S are adjacent.)

10. (10 Points) Let G be the complete graph on n vertices. We want to color the *edges* of G with two colors Red and Blue such that the number of monochromatic triangles (i.e. triangles which are all colored all red or all blue) is minimized. If we color the edges at random, then what is the expected number of monochromatic triangles? (a triangle is a cycle with three vertices).