Recognizable Series on Hypergraphs

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Outline

Objective and Method

- Grammatical Inference: estimate probability distributions on string/trees from samples

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Objective

Extend the notion of recognizable series to graphs and hypergraphs.

$\hookrightarrow$ by directly aiming for an algebraic characterization similar to linear representations of string/tree series.

Outline

Graphs

A *graph* $G = (V, E, \ell)$ on a ranked alphabet $\mathcal{F} = (\Sigma, \sharp)$

- Vertices V,
- Labeling function $\ell : V \to \Sigma$,
- Set of ports $P = \{(v, j) : v \in V, 1 \leq j \leq \sharp\ell(v)\}$,
- Edges $E \subset P \times P$ (partition of P) .

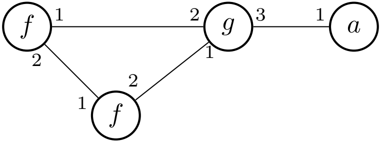


Figure : A graph on the ranked alphabet $\mathcal{F} = \{a(\cdot), f(\cdot,\cdot), g(\cdot,\cdot,\cdot)\}$.

$$V = \{1, 2, 3, 4\},\ \ell(1) = l(2) = f, \ell(3) = g, \ell(4) = a,$$
$$E = \big\{\{(1,1), (3,2)\}, \{(1,2), (2,1)\}, \{(2,2), (3,1)\}, \{(3,3), (4,1)\}\big\}$$

Tensors

Tensor $\mathfrak{T} \in \bigotimes^k \mathbb{R}^d = \mathbb{R}^d \otimes \cdots \otimes \mathbb{R}^d \;\simeq\;$ Multi-array $(\mathfrak{T}_{i_1 \dots i_k}) \in \mathbb{R}^{d \times \cdots \times d}$.

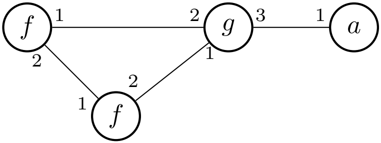
Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the canonical basis of $V = \mathbb{R}^d$, $\mathfrak{T}$ can be expressed as

$$\mathfrak{T} = \sum_{i_1, \dots, i_k \in [d]} \mathfrak{T}_{i_1 \dots i_k} \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k}$$

- $k = 1$: vector $\mathbf{v}_i$ $(1 \leq i \leq d)$
- $k = 2$: matrix $\mathbf{M}_{i_1 i_2}$ $(1 \leq i_1, i_2 \leq d)$
- $k = 3$: higher order tensor $\mathfrak{T}_{i_1 i_2 i_3}$ $(1 \leq i_1, i_2, i_3 \leq d)$

Graph Weighted Models (GWM)

- A graph G on the ranked alphabet $\mathcal{F} = \{g(\cdot,\cdot,\cdot), f(\cdot,\cdot), a(\cdot)\}$:



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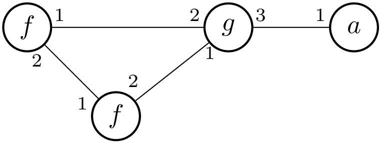

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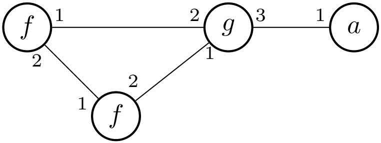

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- Computation of a GWM:
  1. Tensor product of all tensors associated to vertices in G:

$$\mathfrak{T}^f_{i_1 i_2} \mathfrak{T}^f_{i_3 i_4} \mathfrak{T}^g_{i_5 i_6 i_7} \mathfrak{T}^a_{i_8}$$

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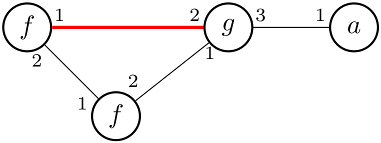

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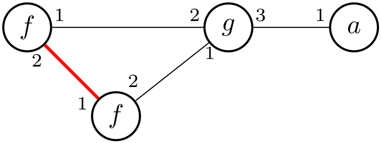

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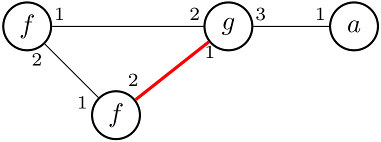

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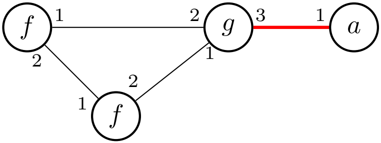

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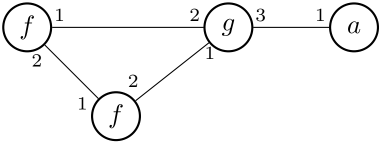

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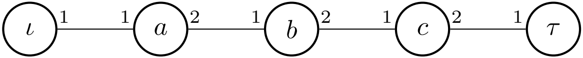
GWM: Examples

- $\mathcal{F} = \{\iota(\cdot), \tau(\cdot), a(\cdot,\cdot), b(\cdot,\cdot), c(\cdot,\cdot)\}$, GWM $\{\boldsymbol{\iota}, \mathbf{M}^a, \mathbf{M}^b, \mathbf{M}^c, \boldsymbol{\tau}\}$

  $G =$ 

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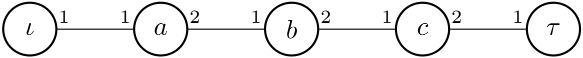
  $G =$ 

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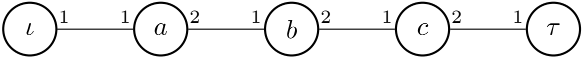
  $G =$ 

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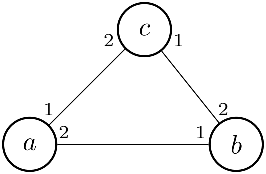
- $\mathcal{F} = \{a(\cdot,\cdot), b(\cdot,\cdot), c(\cdot,\cdot)\}$, $G =$ 

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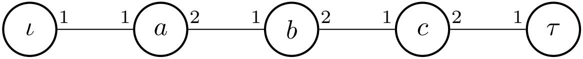
  $G =$ 

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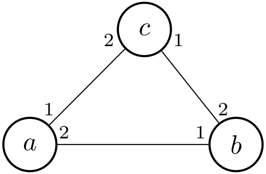
- $\mathcal{F} = \{a(\cdot,\cdot), b(\cdot,\cdot), c(\cdot,\cdot)\}$, $G =$ 

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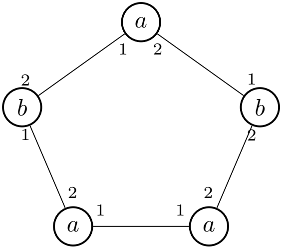
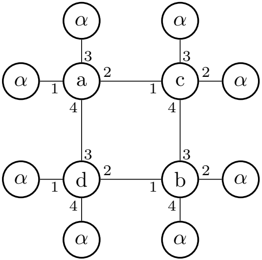
- Beyond strings: circular strings, 2D words/pictures...





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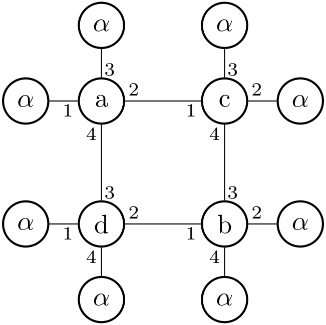
- Crosswords: Let r_h and r_v be two recognizable string series on Σ^*



$$\mapsto \quad r_h(ac) r_h(db) r_v(ad) r_v(cb)$$

Outline

Main results (1)

Proposition

GWMs are a direct generalization of linear representation of string/tree series.

Proposition

- *The sum of two recognizable series is recognizable*
- *The Hadamard product of two recognizable series is recognizable*

A main question:

Are series with finite support recognizable?

Recognizability of Finite Support Series

- Given a graph $\widehat{G}$, is there a GWM s.t. $r(G) = 1$ if $G = \widehat{G}$ and 0 otherwise?

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- Simple counter-example:
    - Circular strings on $\mathcal{F} = \{a(\cdot, \cdot)\}$, GWM $r : \langle d, \{\mathbf{M}_a \in \mathbb{R}^{d \times d}\}\rangle$.
    - $r(G_{a^n}) = Tr(\mathbf{M}_a^n)$ for all n.
    - If $\widehat{G} = G_a$, we want $Tr(\mathbf{M}_a) = 1$ and $Tr(\mathbf{M}_a^n) = 0$ for all $n \geq 2$.

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Lemma

Let $\mathbf{M} \in \mathbb{R}^{d\times d}$. If $Tr(\mathbf{M}^n) = 0$ for all $n \geq 2$, then $Tr(\mathbf{M}) = 0$.

Tilings

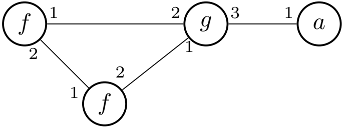

Figure : A graph G

Tilings

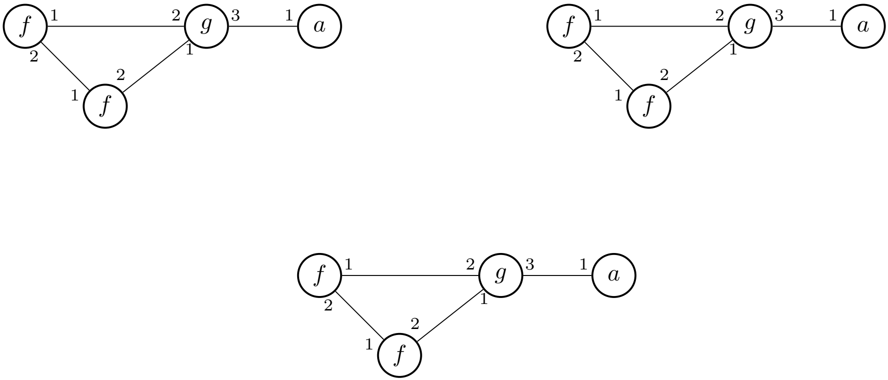

Figure : Graph G_2 with 3 connected components isomorphic to G.

Tilings

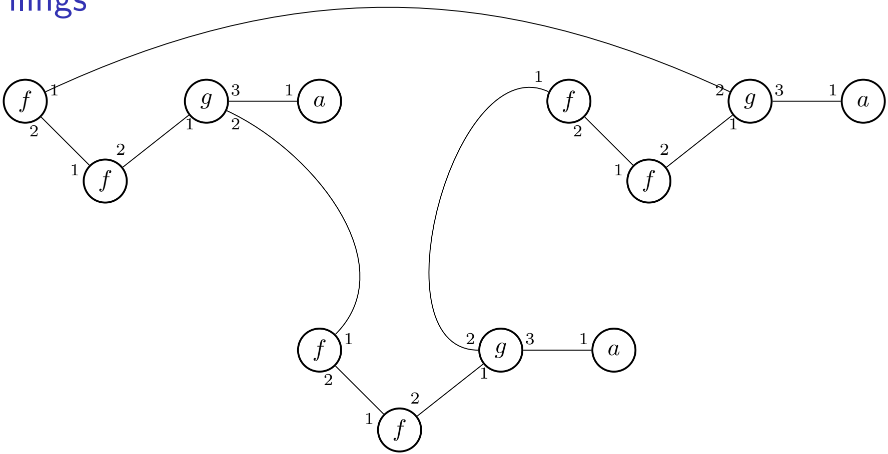

Figure : Graph G_3. Tiling made of three copies of the graph G.

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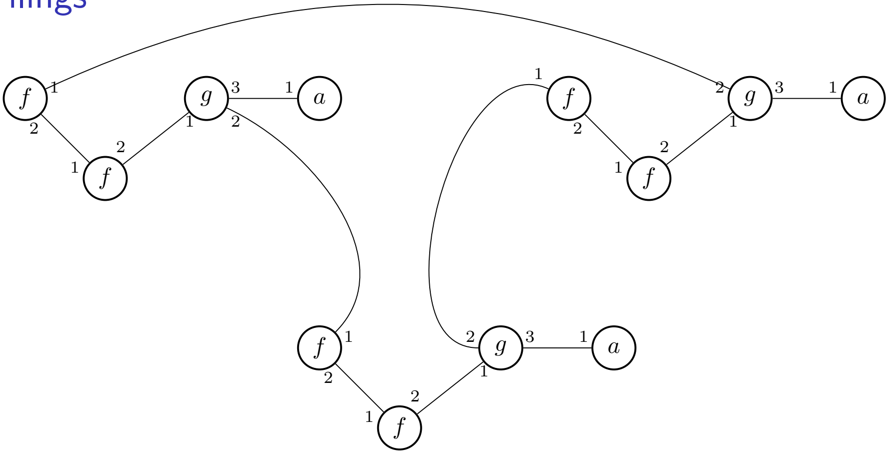

Figure : Graph G_3. Tiling made of three copies of the graph G.

For any graph $\widehat{G}$, if $r(\widehat{G}) \neq 0$ then there exists a tiling G of $\widehat{G}$ s.t. $r(G) \neq 0$.

Main results (2)

Theorem

Given a graph $\widehat{G}$, there exists a recognizable series r such that $r(G) \neq 0$ if and only if G is a tiling of $\widehat{G}$.

Corollary

For any family of graph which does not allow tilings, graph series with finite support are recognizable.

Family of *rooted graphs* over $\mathcal{F}$: there exists $a_0 \in \Sigma$ s.t. for any $G \in \mathcal{F}$, there exists exactly one vertex $v \in V_G$ such that $\ell(v) = a_0$.

Outline

Ongoing Work: Learning GWMs

- Let $r : \langle d, \{\mathfrak{T}^x \in \bigotimes^{\#x} \mathbb{R}^d\}_{x \in \mathcal{F}} \rangle$ be a GWM.

  Given $(G_1, r(G_1)), (G_2, r(G_2)), \cdots$, can we recover the tensors $\{\mathfrak{T}^x \in \bigotimes^{\#x} \mathbb{R}^d\}_{x \in \mathcal{F}}$?

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- Spectral learning for recognizable series on strings.
  - Low-rank factorization of Hankel matrix $\mathbf{H} \in \mathbb{R}^{\Sigma^* \times \Sigma^*}$, $\mathbf{H}_{u,v} = r(uv)$.

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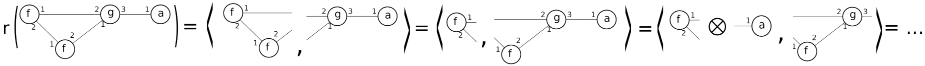
- Learning GWMs
  - Graph cuts:



  - Hankel Matrices/Tensors in $\mathbb{R}^{\mathcal{G}_{\mathcal{F},2} \times \mathcal{G}_{\mathcal{F},2}}$, $\mathbb{R}^{\mathcal{F}_1 \times \mathcal{G}_{\mathcal{F},1}}$, $\mathbb{R}^{\mathcal{F}_2 \times \mathcal{F}_1 \times \mathcal{G}_{\mathcal{F},3}}$, ...

  $\rightarrow$ Preliminary results show that low-rank factorizations of the Hankel tensors can be used to recover the GWM parameters (circular strings and 2D-words).

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- A bridge between graphical models and recognizable series?
- Algorithms to compute/approximate/learn (e.g. message passing).

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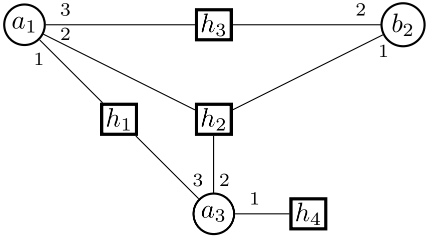
- Direct generalization of recognizable series on strings and trees.
- Characterization of the recognizability of finite support series.

- Generalization of the spectral method for hypergraph recognizable series?
- A bridge between graphical models and recognizable series?
- Algorithms to compute/approximate/learn (e.g. message passing).

Thank you for your attention.

Hypergraphs

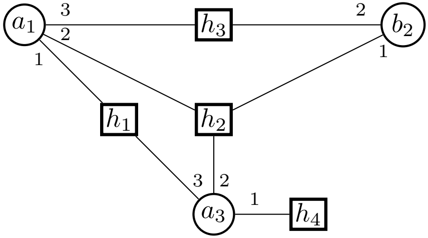


$\mathcal{F} = \{(a,3),(b,2)\}$ $\quad V = \{1,2,3\}$ $\quad \ell(1) = \ell(3) = a,\ \ell(2) = b$

A *hypergraph* $G = (V, E, \ell)$ on a ranked alphabet $\mathcal{F} = (\Sigma, \sharp)$

- V set of vertices,
- $\ell : V \to \Sigma$ labeling function,
- $P = \{(v,j) : v \in V, 1 \leq j \leq \sharp\ell(v)\}$ set of ports of G,
- $E = (h_k)_{1 \leq k \leq n_E}$ a partition of P set of hyper-edges of G.

Hypergraphs



$$P = \Big\{(1,1),(1,2),(1,3), \quad (2,1),(2,2), \quad (3,1),(3,2),(3,3)\Big\}$$

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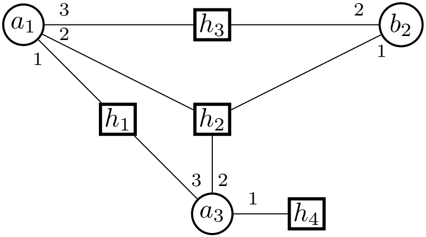
Hypergraphs



$$E = \Big\{ h_1 : \{(1,1),(3,3)\}, h_2 : \{(1,2),(2,1),(3,2)\}, h_3 : \{(1,3),(2,2)\}, h_4 : \{(3,1)\} \Big\}$$

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Hypergraph Weighted Model

$$\langle \mathcal{F}, d, \{\mathfrak{T}^x\}_{x\in\Sigma}, \odot, \boldsymbol{\beta}\rangle$$

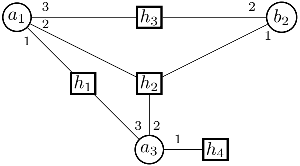
where

- $\mathcal{F}$ is a ranked alphabet,
- $d \in \mathbb{N}_+$ is the dimension of the representation, $V = \mathbb{R}^d$,
- $\mathfrak{T}^x \in V^{\otimes \sharp x}$, tensor associated with symbol x
- $\odot : V \times V \to V$ is a symmetric associative product
- $\boldsymbol{\beta}$ is a linear form on V.

Example of reduction operators:

- $\odot_{id}$ is defined by $\mathbf{e}_i \odot_{id} \mathbf{e}_j = \delta_{ij}\mathbf{e}_i$,
- β_1 is defined by $\beta_1(\mathbf{e}_i) = 1$ for $1 \leq i \leq d$.

HWM: Computation



- HWM $\langle \mathcal{F}, d, \{\mathcal{A}, \mathcal{B}\}, \odot, \boldsymbol{\beta} \rangle$
  1. Tensor product of all tensors associated to vertices

$$\sum_{i_1 \cdots i_8} \mathcal{A}_{i_1 i_2 i_3} \mathcal{A}_{i_4 i_5 i_6} \mathcal{B}_{i_7, i_8} \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_8}$$

  2. Reduction with $\odot$ directed by the hyperedges

$$\sum_{i_1 \cdots i_8} \mathcal{A}_{i_1 i_2 i_3} \mathcal{A}_{i_4 i_5 i_6} \mathcal{B}_{i_7, i_8} (\mathbf{e}_{i_1} \odot \mathbf{e}_{i_6}) \otimes (\mathbf{e}_{i_2} \odot \mathbf{e}_{i_5} \odot \mathbf{e}_{i_7}) \otimes (\mathbf{e}_{i_3} \odot \mathbf{e}_{i_8}) \otimes \mathbf{e}_{i_4}$$

  3. Contraction with $\boldsymbol{\beta}$

$$\sum_{i_1 \cdots i_8} \mathcal{A}_{i_1 i_2 i_3} \mathcal{A}_{i_4 i_5 i_6} \mathcal{B}_{i_7, i_8} \boldsymbol{\beta}^\top (\mathbf{e}_{i_1} \odot \mathbf{e}_{i_6}) \boldsymbol{\beta}^\top (\mathbf{e}_{i_2} \odot \mathbf{e}_{i_5} \odot \mathbf{e}_{i_7}) \boldsymbol{\beta}^\top (\mathbf{e}_{i_3} \odot \mathbf{e}_{i_8}) \boldsymbol{\beta}^\top \mathbf{e}_{i_4}$$