Notes on Acyclic Orientations and the Shelling Lemma

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Abstract

In this paper we study two lemmas on acyclic orientations and totally cyclic orientations of a graph, which can be derived from the shelling lemma in vector subspaces. We give simple graph theoretical proofs as well as a proof by the interpretations of the shelling lemma in the special setting of graphs.

Furthermore we present similar interpretations of closely related theorems in vector subspaces, which do not seem to admit simple graph theoretical proofs.

1 Introduction

Let $G = (V, E)$ be a graph. In this paper, a graph is assumed to be free of loops and parallel edges. An orientation π of G is called *acyclic* if the directed graph (G, π) contains no directed cycle, and called *totally cyclic* if each edge is contained in some directed cycle. For any orientation π of G and for any edge $e \in E$ we denote by *flip*(π, e) the orientation obtained from π by reversing the orientation of e. Also, we use $D(\pi, \pi')$ to denote the set of all edges whose orientations are different in orientations π and π'.

In this note, we study the following two lemmas.

Lemma 1.1 (Acyclic Orientation Lemma) *Let G be a graph and let π and π' be any two distinct acyclic orientations of G. Then there exists an edge $e \in D(\pi, \pi')$ such that $\mathrm{flip}(\pi, e)$ is again acyclic.*

Lemma 1.2 (Totally Cyclic Orientation Lemma) *Let G be a 3 edge-connected graph, and let π and π' be any two distinct totally cyclic orientations of G. Then there exists an edge $e \in D(\pi, \pi')$ such that $\mathrm{flip}(\pi, e)$ is again totally cyclic.*

The main purposes of the present paper are to discuss how these lemmas can be considered as special cases of the shelling lemma in vector subspaces of R^E, and to give simple graph theoretical proofs of the two lemmas above.

For a given acyclic orientation π of a graph, an edge is called *flippable* if *flip*(π, e) is again acyclic. We use the same term for totally cyclic case as well: for a given totally cyclic orientation π of a graph, an edge is called *flippable* if *flip*(π, e) is again totally cyclic.

Here is another results on the average number of flippable edges for which no graph theoretical proof seems to be known.

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Lemma 1.3 (Flippable Edges in Acyclic Orientations) *Let G be a connected graph with n nodes. Let π be a random acyclic orientation of G. Then the expected number of flippable edges in π is at most $2(n-2)$, for $n \geq 3$.*

The above result is a specialization of a known result [11, 7] in arrangement of hyperplanes. When the same result applies to the totally cyclic case, we obtain the following statement which turns out to be rather trivial and can be strengthened by simple graph theoretical arguments. The strengthened lemma replaces "expected" with "maximum" below.

Lemma 1.4 (Flippable Edges in Totally Cyclic Orientations) *Let G be a 3 edge-connected graph with n nodes and m edges. Let π be a random totally cyclic orientation of G. Then the expected number of flippable edges in π is at most $2(m-n)$.*

The paper is organized as follows. In Section 2, we explain how all the lemmas above can be proved as special cases of some known results in vector subspaces. We give simple graph theoretical proofs of the first two lemmas, Lemma 1.1 and Lemma 1.2, in Section 3, which one can read independently without going through Section 2

2 Vector Subspaces and the Shelling Lemma

Lemma 1.1 and Lemma 1.2 look quite similar. Yet, we do not know any way to deduce one from the other, except for the planar case when two lemmas are dual to each other and thus are equivalent. A very natural and seemingly a unique way to relate these two lemmas is to consider them as two special cases of a general theorem, not quite in graphs, but in vector subspaces or more naturally in oriented matroids.

Let E be a finite set and let V be a vector subspace of R^E. For $x \in R^E$, denote by $\sigma(x)$ the sign vector of x, i.e.,

$$\sigma(x)_e = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases} \qquad e \in E.$$

The set $\mathcal{F}_\mathcal{V} = \sigma(\mathcal{V})$ of the sign vectors of vectors of V is known as the *oriented matroid* of V. Two elements e and f of E are *equivalent* in V if either $X_e = X_f$ for all $X \in \mathcal{F}_\mathcal{V}$ or $X_e = -X_f$ for all $X \in \mathcal{F}_\mathcal{V}$. An element $e \in E$ is a *loop* of V if $X_e = 0$ for all $X \in \mathcal{F}_\mathcal{V}$. A vector subspace is *simple* if it has neither equivalent elements nor loops.

Let V be a vector subspace of R^E. A sign vector $X \in \mathcal{F}_\mathcal{V}$ is called an *vertex* if it has a minimal nonempty support, and called a *tope* if it has a maximal support.

For any sign vector X on E and any $e \in E$, denote by *flip*(X, e) the sign vector obtained from X by reversing the sign on e. For a tope T, an element $e \in E$ is called *flippable* in T if *flip*(T, e) is again a tope. The following lemma is known as the shelling lemma.

Lemma 2.1 (Shelling Lemma) *Let V be a simple vector subspace of R^E, and let T and T' be two distinct topes. Then there exists an element $e \in E$ which is flippable in T and $T_e = -T'_e$.*

One can find a simple combinatorial proof of this lemma using the oriented matroid axioms in [6, 9]. A direct proof is possible but it appears to be more cumbersome to use the combinatorial nature of simple-ness of a vector subspace.

The name "shelling" comes from the following connection. Suppose a vector subspace V contains all positive vector $(+, +, \ldots, +)$. The set of non-negative vectors in $\mathcal{F}_\mathcal{V}$ ordered by set inclusion of their supports is isomorphic to the face lattice of a convex polytope. The shellability of this lattice was first shown in [4] by using the elegant notion of line shelling. The above lemma can be considered as a combinatorial abstraction of line shelling which is used to prove the shellability of abstract polytopes (tope lattices) of oriented matroids in [6, 9]. In particular, any ordered sequence (guaranteed to exist by Lemma 2.1) of flips to move from all positive tope $(+,+,\ldots,+)$ to all negative tope $(-,-,\ldots,-)$ provides an ordering of elements of E which induces a shelling ordering of the polytope.

The expected number of flippable elements were studied in [11, 7] in the context of arrangement of hyperplanes and oriented matroids. See also [3, 13]. One can easily translate the result to vector subspaces.

Lemma 2.2 (Average Flip Lemma) *Let V be a simple d-dimensional vector subspace of R^E with $d \geq 2$, and let T be a random tope. Then the expected number of flippable components in T is at most $2(d-1)$.*

To apply the above lemmas to orientations in graphs, we need a few definitions. For any graph $G = (V, E)$ and for any fixed orientation π^*, we denote by A the *incidence matrix* of the directed graph. It is the $V \times E$ matrix $A = [a_{ie} : i \in V \text{ and } e \in E]$ defined by

$$a_{ie} = \begin{cases} 1 & \text{if } i \text{ is the head of } e \\ -1 & \text{if } i \text{ is the tail of } e \\ 0 & \text{otherwise} \end{cases} \qquad i \in V \text{ and } e \in E.$$

Since we fix one orientation π^*, we will assume that every orientation is represented by a signed vector $\pi \in \{+,-\}^E$ in such a way that the set of negative components in π indicates the set of edges oriented differently from π^*.

Similarly, each simple cycle (as an ordered sequence of edges) in G is representable as a signed vector $C \in \{+,0,-\}^E$ in such a way that the positive (negative) components in C are the edges in the cycle traced along (against) the orientation π^*. Each cut $(S, V \setminus S)$ in G is representable as a signed vector $D \in \{+,0,-\}^E$ in such a way that the positive (negative) components in D are the edges in the cut directed in π^* from S to $V \setminus S$ (reversely). A cut is *minimal* if the set of edges in the cut is minimal over all cuts.

Consider the two vector subspaces of R^E:

$$\begin{aligned} Cyc(G) &= \{x \in R^E : Ax = 0\} \\ Cut(G) &= \{y \in R^E : y = A^T\lambda, \lambda \in R^V\}. \end{aligned}$$

The first one is called the *cycle space* and the second called the *cut space*. Clearly they are a dual pair of orthogonal subspaces.

The following proposition is well known and easily verified.

Proposition 2.3 *Suppose G has exactly k connected components. Then the dimensions of the spaces* $\mathrm{Cut}(G)$ *and* $\mathrm{Cyc}(G)$ *are $(n-k)$ and $(m-n+k)$, respectively.*

We state two important propositions relating the acyclic orientations, the totally cyclic orientations and the topes of the two subspaces. These relations were used to relate the number of acyclic orientations with counting polynomials of underlying matroids in [12, 8].

Proposition 2.4 *The set Π_a of acyclic orientations of a graph G is exactly the set of topes of* $\mathrm{Cut}(G)$.

Proposition 2.5 *The set Π_c of totally cyclic orientations of a graph G is exactly the set of topes of* $\mathrm{Cyc}(G)$ *whose supports are E.*

Now we are ready to prove the four lemmas 1.1 – 1.4.

Proof. (of Lemma 1.1). By Proposition 2.4 and Lemma 2.1, it is enough to show that the cut space $Cut(G)$ is simple. This is a direct consequence of the assumption that every graph G is assumed to be free of loops and parallel edges. ∎

Proof. (of Lemma 1.3). Let G be a connected graph and $n \geq 2$. By Proposition 2.2, with setting $V = Cut(G)$, we immediately obtain that the average number of flips in a random acyclic orientation is at most $2(d-1)$ as long as $d \geq 2$. By Propositions 2.3 and $n \geq 3$, we have $d = (n-1) \geq 2$. This proves the result. ∎

Proof. (of Lemma 1.2). Let G be a 3 edge-connected graph. By Proposition 2.5 and Lemma 2.1, it is enough to show that the cycle space $Cyc(G)$ is simple. Since G is 3 edge-connected, there are three edge-disjoint paths between any two vertices. This means for each edge e there are two cycles containing e as a unique common edge. This implies that the cycle space is simple. ∎

As we mentioned, Lemma 1.4 can be easily proved and strengthened by graph theoretical arguments, see Section 3. However for the completeness of our discussion here, we prove it by vector subspace arguments.

Proof. (of Lemma 1.4). Let G be a 3 edge-connected graph. By Proposition 2.2, with setting $V = Cyc(G)$, we immediately obtain that the average number of flips in a random acyclic orientation is at most $2(d-1)$ as long as $d \geq 2$. By Propositions 2.3, we have $d = (m-n+1)$ which is always larger than or equal to 2 since G is connected. This proves the result. ∎

3 Simple Proofs

In this section, we present a simple graph theoretical proof of Lemma 1.1 and Lemma 1.2.

Proof. (of Lemma 1.1). First we remark that the lemma for the complete graph $G = K_n$ is true. Observe that an acyclic orientation of K_n is nothing more than a linear ordering of the vertices. We can assume one acyclic orientation, say π', corresponds to the canonical vertex order $(1, 2, \cdots, n)$. Then any other linear ordering π is a nontrivial permutation $(i_1, i_2, \cdots, i_n)$ of $\{1, 2, \cdots, n\}$, and there is always an neighbor pair reversed with respect to the canonical order, i.e. there exists j such that $i_j > i_{j+1}$. This corresponds to an edge which can be flipped to get another linear order which is closer to the canonical order.

Let G be a graph and let π and π' be two acyclic orientations. It is well known that any acyclic orientation is extendable to a linear ordering. Thus we can extend π and π' to acyclic orientations $\overline{\pi}$ and $\overline{\pi}'$ of the complete graph K_n extending G. Now, we apply the lemma to the acyclic orientations of K_n. Then one can move from $\overline{\pi}$ to $\overline{\pi}'$ by flipping one edge at a time with preserving all intermediate orientations to be acyclic. Along this path, the first edge in G to be flipped is one which can be flipped in π. ∎

Next we turn to Lemma 1.2. It appears to us that this lemma is somewhat harder to prove than the acyclic lemma. In fact, we do not know if there is any proof analogous to the proof above. Interestingly our proof below for Lemma 1.2, however, has an analogue for Lemma 1.1.

For a directed graph (G, π) and any $S \subset V$, we use $\delta^+(S)$ $(\delta^-(S))$ to denote the set of edges leaving S (entering S, respectively).

Proof. (of Lemma 1.2). Let G be a 3 edge-connected graph and let π and π' be any two distinct totally cyclic orientations of G. Clearly any totally cyclic orientation is a strongly connected orientation. Note that a directed graph is totally cyclic if and only if it does not contain any directed cut. Hence an edge (u, v) of a totally cyclic graph is not flippable if and only if there exists a *critical set* $S \subseteq V$ for it, i.e., a subset S with $\delta^+(S) = \{(u, v)\}$. Since G is 3 edge-connected, $|\delta^-(S)| \geq 2$ for any critical set S. First, we prove the following claim:

Claim: Let (u, v) be a non-flippable edge in (G, π). Let S be a critical set for (u, v) with $|\delta^-(S)|$ largest possible. Then any edge in $\delta^-(S)$ is flippable.

This claim implies Lemma 1.2, since, if $(u, v) \in D(\pi, \pi')$, then $\delta^-(S) \cap D(\pi, \pi') \neq \emptyset$ as π' is a totally cyclic orientation.

Proof of the claim: Let $(x, y) \in \delta^-(S)$ and suppose that (x, y) is not flippable. Thus there exists a critical set T for (x, y), and $|\delta^-(T)| \geq 2$. The strong connectivity implies then the following equivalence:

$$V = S \cup T \Longleftrightarrow S \cap T \neq \emptyset.$$

i) Suppose $S \cap T \neq \emptyset$. Let $a \in S \cap T$. Since (G, π) is strongly connected, there exists a directed path from a to x. Since $a \in S \cap T$, $x \in T \setminus S$ and (x, y) is the only edge leaving T, this path is entirely contained in $G(T)$ and uses (u, v). Thus $\{u, v\} \subseteq T$. But then there is no path leaving $S \cup T$, since the only edge leaving S enters T and the only edge leaving T enters S. Thus $V = S \cup T$.

ii) Suppose $S \cap T = \emptyset$. Since T is critical, all edges in $\delta^-(S)$ different from (x, y) must have their tails in $V \setminus (S \cup T)$. Since $|\delta^-(S)| \geq 2$, there exists at least one such edge and thus $V \setminus (S \cup T) \neq \emptyset$. Moreover the existence of a path from x to $V \setminus (S \cup T)$ implies that $v \notin T$.

Thus the claim is proved.

Now we set $S' := S \cap T$ if $S \cap T \neq \emptyset$, and $S' := S \cup T$ otherwise. Then it is clear from this equivalence that $\delta^+(S') = \{(x, y)\} = \delta^-(S) - \delta^-(S')$ and $\delta^-(T) = \delta^-(S') - \delta^-(S)$ whose cardinality is at least two. Hence $|\delta^-(S')| \geq |\delta^-(S) + 1|$, contradicting the choice of S. ∎

It should be remarked that an claim analogous to the one in the proof above is also true for the acyclic orientations, namely: Let (u, v) be a non-flippable edge in an acyclic orientation of a simple graph G. Then every edge in a longest path from u to v is flippable.

Finally, we show how Lemma 1.4 can be proved graph theoretically in a stronger form in which "expected" is replaced with "maximum".

Proof. (of Lemma 1.4). Let G be a 3 edge-connected graph and let π be any totally cyclic orientation of G. Let $k_1 = 2(m - n)$ and let k_2 be the total number of flippable edges in π. Also, let s be the number of nodes with degree 3. Since G is 3 edge-connected, $2m \geq 3s + 4(n - s)$ and thus $m \geq 2n - s/2$ and $k_1 \geq m - s/2$. On the other hand, since for each node of degree 3, at least one of the three incident edges is not flippable, we have $k_2 \leq m - s/2$. It follows that $k_2 \leq k_1$, a strengthening of the lemma. ∎

4 Concluding Remarks

We have presented some interesting relations between two different objects, graph and geometry. Their interactions are mutual, and it is hard (perhaps, nonsense) to say one has a richer structure than the other. In fact, certain properties are easier to see in the geometric setting than the graph theoretical setting, while at the same time there are other properties that can be observed much more naturally in the graph setting.

There are yet another results on vector subspaces that can be interpreted in the present graph orientation setting. These are questions on the connectivity of "flip" graphs. More precisely, the *acyclic flip graph* (*totally cyclic flip graph*) of a graph G is the graph whose vertices are the acyclic orientations (the totally cyclic orientations, respectively) and whose edges are the pairs of vertices which are connected by one flip. The *tope graph* of a vector subspace is defined similarly. The connectivity of the tope graph is studied in the setting of oriented matroids in [5], and shown to be d connected for any d-dimensional vector subspace V of R^E. In fact, this connectivity is exact and cannot be higher since there is always a tope with exactly d neighbors due to [10]. These results immediately imply the following corollaries.

Corollary 4.1 (Connectivity of Acyclic Flip Graph) *Let G be a connected graph with n nodes. Then the acyclic flip graph is $(n-1)$ connected, and has a vertex of degree $(n-1)$.*

Corollary 4.2 (Connectivity of Totally Cyclic Flip Graph) *Let G be a 3 edge-connected graph with n nodes and m edges. Then the totally cyclic flip graph is $(m-n+1)$ connected, and has a vertex of degree $(m-n+1)$.*

These connectivity results can be considered as special cases of the d-connectivity of the graph of a convex d-dimensional polytope due to [2], since the tope graph of a vector subspace is polytopal (in fact, zonotopal). As it was the case for Lemmas 1.3, we do not know of simple graph theoretical proofs for the corollaries above, and thus we leave them as open problems.

Note that the two basic lemmas, Lemmas 1.3 and 1.4, suggest efficient ways to generate all acyclic (totally cyclic) orientations of a given graph by the reverse search algorithm. In fact, a tope enumeration algorithm is given in [1], which can be specialized to the two graph cases.

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