Bimatrix Games

The bimatrix game here is a non-cooperative two person game. Denote by M and N the finite sets of strategies of Player I and Player II ($M \cap N = \emptyset$).

The game is given by two matrices A and B in $\mathbb{R}^{M \times N}$ where a_{ij} (b_{ij}, respectively) represents the payoff to player I (II) when player I takes the strategy $i \in M$ and player the strategy $j \in N$. The payoffs can be considered as profit or loss for the player. We consider it to be profit in this note. The game is called zerosum if $A + B = \mathbf{0}$.

The famous prisoner's dilemma is a bimatrix game with payoff matrices

$$A = \begin{bmatrix} -6 & -1 \\ -10 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} -6 & -10 \\ -1 & -3 \end{bmatrix},$$

where both players have two strategies, confess or not confess. The payoff matrix represents the negative of the number of years in prison (the larger, the better). An American law gives a favorable pardon to the confessor if the other did not confess. This makes the prisoner's decision complex because of the noncooperative nature of the game. (Of course, if they can cooperate, they would decide not to confess to get a reasonable outcome together.)

Another classical example of bimatrix game is the battle of sexes, where a couple of a man and a woman have to decide where to go in the evening. The choices are either a soccer or a ballet. Even though the man strongly prefers to see a soccer, he still prefers to go to see a ballet if he has to go alone to see a soccer game. Symmetrically for the woman. Thus the payoff matrices are something like

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}.$$

. What would be a rational decision for each of them, if they have to decide independently?

A mixed strategy for player I is a probabilistic decision vector $x \in \mathbb{R}^M$ such that $x \geq \mathbf{0}$ and $\mathbf{1}^T x = 1$. We define the same for player II. Denote by X and Y the set of mixed strategies for player I and II. A pure strategy is a mixed strategy that has only one nonzero component (i.e. 0/1).

Nash introduced the notion of equilibria for the N-person nonzero sum games and showed the existence of an equilibrium by using Brouwer's fixed point theorem.

A pair of mixed strategies $(\overline{x}, \overline{y}) \in X \times Y$ is said to be Nash equilibrium point if

$$\overline{x}^T A \overline{y} \geq x^T A \overline{y} \text{ for all } x \in X, \tag{1}$$

$$\overline{x}^T B \overline{y} \geq \overline{x}^T B y \text{ for all } y \in Y. \tag{2}$$

In other words, a Nash equilibrium point is a pair of strategies that do not motivate any one of the players to change his/her strategy as long as the other stay with his/her strategy.

Here is the well-known theorem due to Nash [2] specialized to the two person case:

(a) Every bimatrix game admits a Nash equilibrium point.

Lemke-Howson [1] found an elegant pivot algorithm to compute a Nash equilibrium point. The polynomial computability of a Nash equilibrium point is still open, see the recent article [4] by von Stengel.

How many Nash equilibrium points can a "nondegenerate" bimatrix have? (Of course, it can have infinitely many e.g. the matrices are zero matrices. Nondegenerate means basically input matrices are sufficiently generic.) Can you generate all efficiently? The first question was studied in [3] and a construction using cyclic polytopes showed that there are games with asymptotically more than $2.414^n/\sqrt{n}$ equilibria.

Here again, the polyhedral computation can help in generating all Nash equilibria via vertex enumeration algorithms. In particular, there are two polytopes associated with a bimatrix game. To make this work, we must assume that both A and B are (strictly) positive matrices. It is easy to transform the game (by adding large positive constants to the payoffs) to satisfy this without changing the Nash equilibria.

$$P_1 = \{x \in \mathbb{R}^M \mid x \geq \mathbf{0}, B^T x \leq \mathbf{1}\},$$
$$P_2 = \{y \in \mathbb{R}^N \mid Ay \leq \mathbf{1}, y \geq \mathbf{0}\}.$$

It can be shown that the equilibrium points are in one-to-one correspondance with the certain pairs of extreme points of P_1 and P_2, see [4], if the both inequality systems are nondegenerate.

There are many interesting open questions to investigate, such as the efficient generation of Nash equilibria, in Bimatrix Game Theory.

Exercise.

Compute all Nash equilibrium pairs of extreme points for the game:

$$A = \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}.$$

(Hint: There are exactly three of them.)

What about for the game?

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 3 & 2 \\ 3 & 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Is there any equilibrium better than the others? Is it pure?
(Here is a web site that helps you verify your answer: http://banach.lse.ac.uk/form.html.)

References

[1] C.E. Lemke and Jr. J.T. Howson. Equilibrium points of bimatrix games. Journal of the Society for Industrial and Applied Mathematics, 12:413–423, 1964.

[2] J.F. Nash. Non-cooperative games. Annals of Mathematics, 54:286–295, 1951.

[3] B. von Stengel. New maximal numbers of equilibria in bimatrix games. Discrete Comput. Geom., 21(4):557–568, 1999.

[4] B. von Stengel. Computing equilibria for two-person games. In R. J. Aumann and S. Hart, editors, Handbook of game theory, volume 3, pages 1947–1987. North-Holland, Amsterdam, 2002. available from http://www.maths.lse.ac.uk/Personal/stengel/bvs-publ.html.