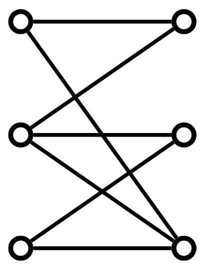
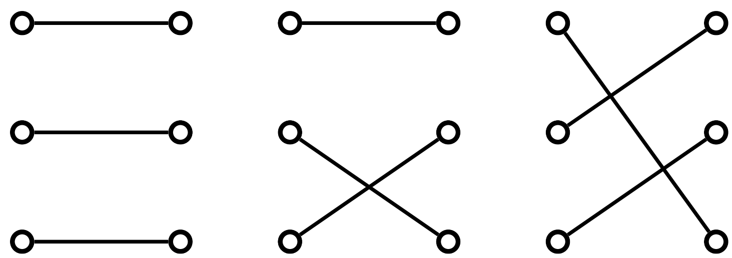
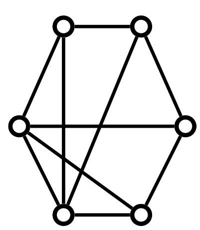
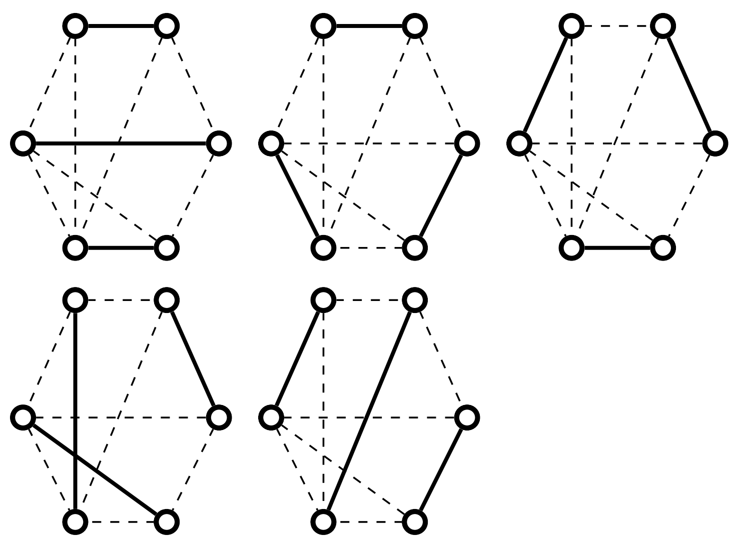
Matching Polytope

Let $G = (V, E)$ be a graph. A matching in G is a subset of edges $M \subseteq E$ such that every vertex meets at most one member of M. A matching M is perfect if every vertex meets exactly one member of M.

There are two natural enumerative problems associated with perfect matching, namely the counting problem and the listing problem. The counting problem is to count the number $\#_{\mathrm{MA}}(G)$ of perfect matchings, while the listing problem $\lambda_{\mathrm{MA}}(G)$ is to list explicitly all perfect matchings of G. It is known that the counting problem is #P-complete even for bipartite graphs. There are polynomial algorithms for the listing problem.

Input G	**Listing** $\lambda_{\mathrm{MA}}(G)$	**Counting** $\#_{\mathrm{MA}}(G)$
 bipartite graph		3
 general graph		5

Each subset S of E can be denoted by the incidence vector $\chi^S \in \{0,1\}^E$ defined by

$$\chi^S_e = \begin{cases} 1 & \text{if } e \in S, \\ 0 & \text{otherwise.} \end{cases}$$

We often identify a matching M with its incidence vector χ^M.

The (perfect) matching polytope $P_{\mathrm{MA}}(G)$ is defined as the convex hull of (the incidence vectors of) of perfect matchings of G, i.e.

$$P_{\text{MA}}(G) = conv\{\chi^M \mid M \text{ is a perfect matching of } G\}.$$

By definition, $P_{\text{MA}}(G)$ is a 0/1-polytope. This polytope has been studied extensively and in fact played an essential role in shaping the exciting evolution of combinatorial optimization theory, see Edmonds [2], Schrijver [6, Section 8.10] and Lovasz-Plummer [5].

We present here two of the most important results on the perfect matching polytope concerning simple H-representations of the matching polytope. The first one is for the case of bipartite graphs and the second one for the general nonbipartite case.

Bipartite case

When a graph G is bipartite, there is a very simple H-description, which is essentially a well-known result on doubly stochastic matrices by Birkhoff[1946] and von Neumann [1953], see [6]. Let us denote by $\delta(v)$ the set of edges incident to a vertex v in G.

(a) The perfect matching polytope of a bipartite graph $G = (V_1, V_2, E)$ is given by

$$\begin{aligned} P_{\text{MA}}(G) = \{x \in \mathbb{R}^E \mid x_e \geq 0 \text{ for all } e \in E, \text{ and} \\ \sum_{e \in \delta(v)} x_e = 1 \text{ for all } v \in V\}. \end{aligned}$$

The size of the H-representation is small, polynomially bounded by the size of the graph. This theorem (a) and the #P-completeness of the problem of couting bipartite matchings actually imply that the problem of counting the vertices of an H-polyhedron is #P-hard, that was first proved by Linial [4] using a different reduction.

It is quite easy to see that the inequalities in the H-representation are valid inequalities for the perfect matching polytope. Clearly the nonegativity of x_e is satisfied by the incidence vector of any perfect matching. The inequality :$\sum_{e \in \delta(v)} x_e = 1$ says that the sum of the weights x_e on the edges e incident to any particular vertex v is exactly one and this is clearly satisfied by the incidence vector of any perfect matching. Denoting the RHS of the equation (a) by $Q(G)$, we thus have the containment: $P_{\text{MA}}(G) \subseteq Q(G)$. The harder part of (a) is the other containmnent, namely,

(b) every point in $Q(G)$ is a convex combination of perfect matchings.

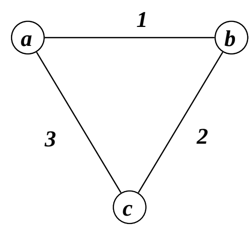
One can prove this without using any deep result. In fact, it is not hard to show that for any fractional point x in $Q(G)$, one can find two points x' and x'' in $Q(G)$ different from x such that $x = (x' + x'')/2$. Here x' and x'' are not necessarily integral. (Hint: the set of edges e having a fractional weight x_e must contain a cycle in G.)

General case

Let $G = (V, E)$ is a general graph, not necessary bipartite. Then the natural question is: Does the H-polytope (called <u>fractional matching polytope</u>, representing the matching polytope in the bipartite case):

$$P_{\mathrm{FMA}}(G) := \{x \in \mathbb{R}^E \mid x_e \geq 0 \text{ for all } e \in E, \text{ and } \sum_{e \in \delta(v)} x_e = 1 \text{ for all } v \in V\}$$

represent the matching polytope $P_{\mathrm{MA}}(G)$ in general? One can easily see that this is not the case. For example, if G is a triangle graph as given below, the fractional matching polytope consists of a single point $(1/2, 1/2, 1/2)$, while the matching polytope is the empty set.



It is easy to see that for any subset $S \subseteq V$ of odd cardinality $|S| = 2k+1$, the induced subgraph $G(S) = (S, E(S))$ can contain at most k edges from any matchings. Note that $E(S)$ is the subset of edges of G whose endpoints are contained in S. For example, any induced subgraph forming a triangle can contain at most one edge of any perfect matching. Edmonds' matching theorem states these inequalities are sufficient to determine the matching polytope:

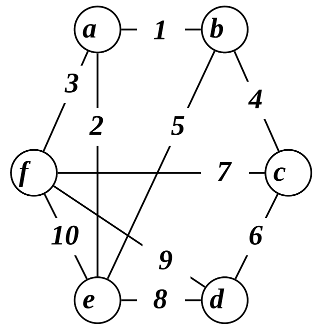
(c)

$$P_{\mathrm{MA}}(G) = \{x \in \mathbb{R}^E \mid x_e \geq 0 \text{ for all } e \in E, \sum_{e \in \delta(v)} x_e = 1 \text{ for all } v \in V, \text{ and } \sum_{e \in E(S)} x_e \leq k \text{ for all } S \subseteq V \text{ of odd cardinality} |S| = 2k+1\}.$$

The inequalities for odd cardinality sets S are known as the blossom inequalities. Clearly there are exponentially many such inequalities and it is not clear how many of them are redundant.

Results of computation with polyhedral computation codes

Consider the fractional matching polytope $P_{\text{FMA}}(G)$ of the nonbipartite graph below.



The polytope is given by two types of inequalities. The first type are nonnegativities: $x_e \geq 0$ for all $e = 1, 2, \ldots, 10$, and the remaining inequalities are of type: $\sum_{e \in \delta(v)} x_e = 1$ $(v \in V)$. For example, for the vertex $a \in V$, this equality says:

$$x_1 + x_2 + x_3 = 1,$$

and, of course we have 6 equalities of this type. One can write the inequalities in (new) Polyhedra format as the following, where the first 6 rows of the matrix represent the 6 equalities.

```
H-representation
linearity 6  1 2 3 4 5 6
begin
   16  11  integer
  1 -1 -1 -1  0  0  0  0  0  0  0
  1 -1  0  0 -1 -1  0  0  0  0  0
  1  0  0  0 -1  0 -1 -1  0  0  0
  1  0  0  0  0  0 -1  0 -1 -1  0
  1  0 -1  0  0 -1  0  0 -1  0 -1
  1  0  0 -1  0  0  0 -1  0 -1 -1
  0  1  0  0  0  0  0  0  0  0  0
  0  0  1  0  0  0  0  0  0  0  0
  0  0  0  1  0  0  0  0  0  0  0
  0  0  0  0  1  0  0  0  0  0  0
  0  0  0  0  0  1  0  0  0  0  0
  0  0  0  0  0  0  1  0  0  0  0
  0  0  0  0  0  0  0  1  0  0  0
  0  0  0  0  0  0  0  0  1  0  0
  0  0  0  0  0  0  0  0  0  1  0
  0  0  0  0  0  0  0  0  0  0  1
end
```

Here "linearity" specifies as an option in Polyhedra format a set of equalities in H-representation. Above, it specifies 6 equations that are given in the 1st to the 6th consecutive rows in the total of 16 rows. The remaining (unspecified) rows are inequalities.

The linearity option is understood only by recent versions of cddlib and lrslib. With older codes, one must specify an equality with a pair of opposite inequalities. For example, scdd (simple cdd in exact GMP arithmetic, Version 0.92) [3] takes the above file as input and outputs a V-representation of the polytope.

```
V-representation
begin
 6 11 rational
 1 0 0 1 0 1 1 0 0 0 0
 1 0 0 1 1 0 0 0 1 0 0
 1 1/2 1/2 0 0 1/2 1/2 1/2 0 1/2 0
 1 0 1 0 1 0 0 0 0 1 0
 1 1 0 0 0 0 0 1 1 0 0
 1 1 0 0 0 0 1 0 0 0 1
end
```

Avis' lrs [1] does essentially the same transformation by a completely different algorithm.

The fractional matching polytope has one fractional vertex, and five integral vertices corresponding to the all five perfect matchings. Which blossom inequalities do we need to define the matching polytope? In general how many of the blossom inequalities are essential (i.e. facet-defining inequalities)?

Every fractional matching polytope is half-integral, i.e., every vertex has only 0, 1 or 1/2 components.

References

[1] D. Avis. lrs Homepage. http://cgm.cs.mcgill.ca/~avis/C/lrs.html.

[2] J. Edmonds. Path, trees, and flowers. Canadian J. Math., 17:449–467, 1965.

[3] K. Fukuda. cdd, cddplus and cddlib homepage. McGill University, Montreal, Canada, 2002. http://www.cs.mcgill.ca/~fukuda/software/cdd_home/cdd.html.

[4] N. Linial. Hard enumeration problems in geomety and combinatorics. SIAM J. Alg. Disc. Meth., 7:331–335, 1986.

[5] L. Lovasz and M. Plummer. Matching Theory. North Holland, 1986.

[6] A. Schrijver. Theory of Linear and Integer Programming. John Wiley & Sons, New York, 1986.