

# 1 Assignment #1

## 1. Induction Proof(arrangement of lines)

**Claim 1.** *Let  $L$  be a set of lines in general position<sup>1</sup> in the plane, with  $|L| > 2$ . Then, at least one of the regions formed by  $L$  is a triangle.*

*Proof.* Suppose  $|L| = 3$ . Since all lines are in general position, there exists three points of intersection among the 3 lines, and they form a closed triangle.

For the induction hypothesis, suppose the claim is true for  $|L| = n$ . Now, assume  $|L| = n + 1$ . Pick any line  $l$  from  $L$ , and consider the remaining  $n$  lines. Since we assumed the claim holds for  $|L| = n$ , there exists at least one triangle, say  $ABC$ . Now, put  $l$  back onto  $L$ . If  $l$  does not intersect the triangle,  $ABC$  still forms a triangle, and we are done. If  $l$  does intersect the triangle, it crosses precisely 2 sides of  $ABC$ . Then, the two intersection points at which  $l$  crosses  $ABC$ , and one of the vertices of  $ABC$  now forms a triangle.  $\square$

## 2. Induction Proof (number theory)

Find the function  $f(n)$  that generates, given  $n$ , the sum of the square of all positive integers 1 through  $n$ .

Expression:  $f(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ .

*Proof.* • Base case:  $n = 1$ .  $f(1) = 1/3 + 1/2 + 1/6 = 1$ , which is correct.

- Induction hypothesis: Suppose the formula for  $f(n)$  is correct for  $n \leq k$ .
- Consider  $n = k + 1$ . Then the expression gives:

$$\begin{aligned} f(k+1) &= 1/3(k+1)^3 + 1/2(k+1)^2 + 1/6(k+1) \\ &= 1/3k^3 + 3/2k^2 + 13/6k + 1 \end{aligned}$$

On the other hand, the definition of sum of squares gives:

$$\begin{aligned} f(k+1) &= f(k) + (k+1)^2 \\ &= 1/3k^3 + 1/2k^2 + 1/6k + (k^2 + 2k + 1) \\ &= 1/3k^3 + 3/2k^2 + 13/6k + 1 \end{aligned}$$

which completes the proof.  $\square$

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<sup>1</sup>Lines are in general position if: (1) no two lines are parallel, and (2) no three lines intersect at one point.

### 3. Induction Proof (circle map coloring)

**Claim 2.** *Let  $C$  be a set of circles in the plane. Then, the regions formed by  $C$  are 2-colorable.*

*Proof.* Suppose  $|C| = 1$ . Then, the claim is clearly true.

Suppose the claim is true for  $|C| = n$ , and consider the case where  $|C| = n + 1$ . Pick any circle  $c$  from  $C$ , and put it aside for now. Since there are  $n$  remaining circles, we can 2-color the regions using the induction hypothesis.

Now, put  $c$  back into  $C$ , and consider the coloring. All the regions that does not contain an arc from  $c$  still satisfy the coloring property (i.e. no neighboring region shares the same color). All the regions that does contain an arc from  $c$ , however, do not satisfy the coloring property. In particular, each of these regions has precisely 1 neighboring region that shares the same color.

The regions that contain an arc from  $c$  can be partitioned into two classes: (i) it lies inside  $c$ ; (ii) it lies outside  $c$ . For each region that lies inside  $c$ , we flip its color (i.e. change red to black, and vice versa). Now, we claim that the resulting coloring is valid. To see this, let  $r_1$  and  $r_2$  be two arbitrary neighboring regions. If they both lie outside of  $c$ , they must be colored in different colors since the border between them comes from  $C - \{c\}$ . If one lies inside  $c$  while the other lies outside, they are now colored in different colors, by the flipping operation. Finally, if they both lies inside  $c$ , they were originally colored in different colors before the flipping operation, and after the flipping operation, they now have opposite colors.  $\square$

## 2 Assignment #2

### 1. Algorithms for Turing Machines

### 3. Big “Oh” Notation

**Definition 1.** *If  $\exists c, n_0$  such that  $f(n) < cg(n)$  for all  $n \geq n_0$ , then  $f(n) \in O(g(n))$ .*

**Definition 2.** *If  $\exists c, n_0$  such that  $f(n) \geq cg(n)$  for all  $n \geq n_0$ , then  $f(n) \in \Omega(g(n))$ .*

(a)  $100n + \log n = O(n + \log^2 n)$

We know  $100n = O(n)$  and  $\log n = O(\log^2 n)$ . Lemma 3.1 then gives the sum is also  $O(n + \log^2 n)$ .

(b)  $\log n = \Theta(\log n^2)$

We have  $\log n^2 = 2 \log n = \Theta(\log n)$ . (Note that  $\Theta$  works both ways)

(c)  $n^2 / \log n = \Omega(n \log^2 n)$

Multiplying both sides by  $\log n$  gives  $n^2$  and  $n \log^3 n$ . Assuming the base of the logarithm is 3,  $c = 1, n_0 = 27$  shows  $n^2 = \Omega(n \log^2 n)$ .

(d)  $\log^{\log n} n = \Omega(n / \log n)$

Multiplying both sides by  $\log n$  gives  $\log^{\log n + 1} n$  and  $n$ . We now show  $\log^{\log n + 1} n = \Omega(n)$ . By plugging in  $c = 1$  and  $n_0 = 8$ , the claim holds.

(e)  $n^{0.5} = \Omega(\log^5 n)$

Just plug in  $c = 1$  and  $n_0 = 100$  into the definition completes the proof.

(f)  $n2^n = O(3^n)$

Just plug in  $c = 1$  and  $n_0 = 10$  into the definition completes the proof.

## 4. Minimum Spanning Trees

**Claim 3.** *Let  $S$  be a set of  $n > 2$  points in the plane in general position, such that  $S = B \cup R$  and  $B \cap R = \emptyset$  ( $B$  for blue, and  $R$  for red). Then, for every pair  $b \in B$  and  $r \in R$  that has the minimum distance between a blue point and a red point, there is a minimum spanning tree of  $S$  containing  $(b, r)$ .*

*Proof.* Let  $d$  denote the minimum distance between a blue point and a red point. Take any pair of points  $b \in B$  and  $r \in R$  such that  $\text{distance}(b, r) = d$ . We shall show that there is an MST( $S$ ) that contains  $(b, r)$  as an edge.

First, take an arbitrary MST  $T$ . If  $T$  contains  $(b, r)$ , we are done. So we assume otherwise. Now, consider the unique  $b - r$  path  $P$  in  $T$  (because  $T$  is a tree, this path is unique). Since  $b \in B$  and  $r \in R$ , this path must contain at least one edge, say  $(b', r')$ , whose endpoints are colored differently. We remove  $(b', r')$  from  $T$ , and then insert  $(b, r)$  to obtain the resulting graph  $T'$ . It suffices to show that  $T'$  is also an MST of  $S$ .

To show  $T'$  is a spanning tree, observe that the deletion of  $(b', r')$  disconnects the original spanning tree  $T$  into 2 connected components (2 trees, in particular). Then, by inserting  $(b, r)$ , the 2 components are now connected. Furthermore, joining 2 trees by an edge cannot introduce a cycle, and hence the resulting graph  $T'$  is a tree.

Finally, to show  $T'$  is a minimum spanning tree, see how the weight of the tree changed:

$$w(T') = w(T) - \text{distance}(b', r') + \text{distance}(b, r)$$

Since  $\text{distance}(b, r) = d$ , it must be that  $\text{distance}(b, r) \leq \text{distance}(b', r')$ , and therefore  $w(T') \leq w(T)$ . Since  $T$  was an MST, it follows that  $T'$  must also be minimum.  $\square$

### 3 Random Problems

#### One Mouth Theorem

**Claim 4.** *A mouth of a non-convex polygon  $P$  is 3 consecutive vertices  $a, b$ , and  $c$ , such that the closed triangle  $abc$  does not contain any vertex of  $P$  other than  $a, b$ , and  $c$ . For every non-convex polygon, it contains at least one mouth.*

*Proof.* Consider the convex hull  $CH(P)$ . Since  $P$  is non-convex, there is at least one edge  $(a, b)$  of  $CH(P)$  that doesn't belong to  $P$ . Now, consider the sequence of vertices along the boundary of  $P$  that aren't contained in  $CH(P)$ , starting from  $a$  to  $b$ . This in turn forms a simple polygon  $Q$ . Recall the two-ear theorem from previous lecture:

*“Except for triangles, every simple polygon contains at least two non-overlapping ears.”*

Thus,  $Q$  contains at least 1 ear, which corresponds to the mouth of  $P$ . □