Honours Analysis 2 (Math 255) Tutorial Notes

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These notes may continue to be updated when the TA-strike ends. I hope to be back and teaching soon!

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Contents

1 Metric Spaces and a Topological Introduction

Let us begin with a brief recall of the key concepts seen in class. A metric space is a pair (X, d) where X is a non-empty set and $d : X \times X \to [0, \infty)$ is a *metric* (or distance-function) satisfying the following:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

We note that (iii) is called the *triangle inequality*. As seen in class, any normed vector space V inherits a metric structure from its norm. Indeed, V is automatically a metric space when equipped with the metric $d(x, y) = \|x - y\|$. The most frequently encountered metric space is $\mathbb{R}^n$ with its usual norm and induced metric. However, as seen in class, there are more "exotic" instances of such spaces.

Example 1.1 (The Discrete Metric)**.** Given any non-empty set X, we can define the *discrete* metric on X by putting

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

It is left as a straightforward exercise to verify that d is indeed a valid metric on X.

Example 1.2. Consider the space $X = \mathbb{R}^{\mathbb{N}}$ of real sequences. That is, an element $x \in X$ is a sequence $x = (x_n)_{n \in \mathbb{N}}$ of real numbers. We show that

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

is a metric on X.

(i) It is clear that $d(x, x) = 0$ for any $x \in X$. Conversely, if $d(x, y) = 0$ then

$$2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} = 0$$

for each $n \in \mathbb{N}$. Thus $|x_n - y_n| = 0$ for all $n \in \mathbb{N}$ so $x_n = y_n$ at each index n; i.e. $x = y$.

(ii) The equality $d(x, y) = d(y, x)$ for all $x, y \in X$ is trivially true by symmetry of the absolute value function.

(iii) Finally, to establish the triangle inequality, we first make some observations on the function

$$f : [0, \infty) \to [0, 1), \qquad f(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}.$$

Clearly, f is an increasing function. Moreover, given $s, t \geq 0$ there holds

$$f(s) + f(t) = \frac{s}{1+s} + \frac{t}{1+t} \geq \frac{s}{1+s+t} + \frac{t}{1+s+t} = \frac{s+t}{1+s+t} = f(s+t). \tag{1.1}$$

Using these observations, we see that for arbitrary $x, y, z \in X$ there holds

$$\begin{aligned}
d(x,y) &= \sum_{n=1}^{\infty} 2^{-n} f(|x_n - y_n|) \\
&\leq \sum_{n=1}^{\infty} 2^{-n} f(|x_n - z_n| + |z_n - y_n|) && \text{since } f \text{ increases} \\
&\leq \sum_{n=1}^{\infty} 2^{-n} \left(f(|x_n - z_n|) + f(|z_n - y_n|) \right) && \text{by (1.1)} \\
&= \sum_{n=1}^{\infty} 2^{-n} f(|x_n - z_n|) + \sum_{n=1}^{\infty} 2^{-n} f(|z_n - y_n|) \\
&= d(x,z) + d(z,y).
\end{aligned}$$

1.1 Distance and Bases

As seen in class, the metric d induces a topological structure on X, i.e. a notion of open sets. That is, we declare a subset U of X to be open if, for each $x \in U$, there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) := \{y \in X : d(x,y) < \varepsilon\} \subseteq U.$$

These "ε-balls" constitute a basis for the topology induced by the metric d. That is, every open subset of X is the union of ε-balls.

Definition 1.1. Let X be a non-empty set. A given collection $\mathfrak{T}$ of subsets of X is called a *topology* on X provided the following conditions are met

1. $\varnothing, X \in \mathfrak{T}$;
2. $\mathfrak{T}$ is closed under arbitrary unions;
3. $\mathfrak{T}$ is closed under *finitely* many intersections.

The pair $(X, \mathfrak{T})$ is then called a *topological space*. We say that the topological space $(X, \mathfrak{T})$ is *metrizable* when there exists a metric d on X whose induced topology is $\mathfrak{T}$.

Definition 1.2. Let $(X, \mathfrak{T})$ be a topological space. A subset of X is said to be closed if its complement is open.

In a topological space, open balls are generalized to the concept of a *basis*.

Definition 1.3. Let X be a non-empty set. A family $\mathcal{B}$ of subsets from X is called a *basis* on X provided each of the following hold

1. For every $x \in X$ there exists some $B \in \mathcal{B}$ such that $x \in B$;
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ is a basis for X, we define the topology generated by $\mathcal{B}$ on X as the set

$$\mathfrak{T} := \{U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U\}. \tag{1.2}$$

That is, $\mathfrak{T}$ consists of all $U \subseteq X$ such that, for every $x \in U$, one can find $B \in \mathcal{B}$ with $x \in B \subseteq U$.

Put otherwise, given a basis $\mathcal{B}$ we see that the topology generated by $\mathcal{B}$ is the collection of arbitrary unions of basis elements. That is, a set U is open in the topology generated by $\mathcal{B}$ if and only if U is a (possibly uncountable) union of basis elements. In a metric space, this is the statement that open sets are arbitrary unions of balls. For instance, in $\mathbb{R}$ every open set is an arbitrary union of open intervals.

Example 1.3. The discrete metric on a non-empty set X induces a topology where every subset of X is open. Indeed, with the discrete metric, we have

$$B(x, 1/2) = \{x\}$$

for any $x \in X$. Thus, singletons are open. Since arbitrary unions of open sets (or open balls) are open, every subset of X is open. Equivalently, every subset of X is closed.

So, we see that the discrete metric induces the topology $(X, \mathcal{P}(X))$ where $\mathcal{P}(X)$ denotes the power set of X. This is known as the *discrete topology*.

Problem 1. *Let $\mathcal{B}$ be a basis on X. Show that the topology $\mathfrak{T}$ generated by $\mathcal{B}$ is precisely the intersection of all topologies* containing $\mathcal{B}$. *Put otherwise, $\mathfrak{T}$ is the smallest topology containing $\mathcal{B}$.*

Proof. Let $\mathcal{B}$ be a basis for a topology $\mathfrak{T}$ on X. We claim that $\mathfrak{T} = \bigcap_{\mathfrak{W}} \mathfrak{W}$ where the intersection is taken over all topologies $\mathfrak{W}$ containing $\mathcal{B}$. Since $\mathcal{B}$ is a basis for $\mathfrak{T}$, we know that every element of $\mathfrak{T}$ is simply the union of elements in $\mathcal{B}$. Consequently, if $\mathfrak{W}$ is a topology containing $\mathcal{B}$, we have $\mathfrak{T} \subseteq \mathfrak{W}$ since $\mathfrak{W}$ is closed with respect to unions. Especially, $\mathfrak{T} \subseteq \bigcap_{\mathfrak{W}} \mathfrak{W}$. Conversely, $\mathfrak{T}$ is a topology containing $\mathcal{B}$ and therefore appears as one of the $\mathfrak{W}$ indexing the intersection. This yields $\bigcap_{\mathfrak{W}} \mathfrak{W} \subseteq \mathfrak{T}$. □

Bases are sometimes useful for simplifying proofs. For instance, let us recall the definition of continuity:

Definition 1.4. Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces and consider a function $f : X \to Y$. We say that f is *continuous* if $f^{-1}(V)$ is open in X for every open set V in Y.

As seen in class, in presence of a basis continuity need not be verified for all open sets. We make this statement precise in the case where Y is a metric space:

Proposition 1.1. *Let X be a topological space and Y a metric space. A function $f : X \to Y$ is continuous if and only if for every ball open ball $B \subseteq Y$ the set $f^{-1}(B)$ is open in X.*

Proof. If f is continuous, there is nothing to show. Conversely, suppose that for any open ball $B \subseteq Y$, the set $f^{-1}(B)$ is open in X. Let $U \subseteq Y$ be an arbitrary open set. Since balls generate open sets, U is a union (possibly uncountable) of open balls. That is, we may write

$$U = \bigcup_{i \in \mathcal{I}} B_i$$

where $\mathcal{I}$ is an index set and each B_i is open. Then,

$$f^{-1}(U) = \bigcup_{i \in \mathcal{I}} f^{-1}(B_i).$$

Since each $f^{-1}(B_i)$ is open, it follows that $f^{-1}(U)$ is open as an arbitrary union of open sets. □

Remark 1.1. A similar statement can be made if Y is a topological space whose topology is generated by a basis. Modifying the statement/proof of the above proposition for this more general setting is left as an exercise.

A similar statement can be made for open maps. We begin with a definition:

Definition 1.5. Let X, Y be topological spaces and $f : X \to Y$ a function.

1. We say that f is an *open map* if $f(O)$ is open in Y for all open sets $O \subseteq X$.
2. We say that f is a *closed map* if $f(C)$ is closed in Y for all closed sets $C \subseteq X$.

Note that continuous functions need not be open or closed. For instance, consider the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \frac{1}{1 + x^2}.$$

Clearly, f is continuous. However, the image of the clopen[1] set $\mathbb{R}$ is $f(\mathbb{R}) = (0, 1]$, which is neither open nor closed. Hence, f is neither an open map nor a closed map.

Proposition 1.2. *Let X be a metric space space and Y a topological space. A function $f : X \to Y$ is open if and only if for every ball open ball $B \subseteq X$ the set $f(B)$ is open in Y.*

[1] A clopen set is a set which is both open and closed.

Proof. If f is open, there is nothing to show. Conversely, suppose that for any open ball $B \subseteq X$, the set $f(B)$ is open in Y. Let $O \subseteq X$ be an arbitrary open set. Since balls generate open sets, O is a union (possibly uncountable) of open balls. That is, we may write

$$O = \bigcup_{i \in \mathcal{I}} B_i$$

where $\mathcal{I}$ is an index set and each B_i is open. Then,

$$f(O) = \bigcup_{i \in \mathcal{I}} f(B_i).$$

Since each $f(B_i)$ is open, it follows that $f(O)$ is open as an arbitrary union of open sets. □

Remark 1.2. The above statement cannot be made for closed maps if we replace open sets by closed sets. This is because arbitrary unions of closed sets need not be closed. For instance, consider the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \frac{1}{1 + x^2}.$$

Note that f is continuous. Hence, by the preservation of intervals theorem, every closed bounded interval $[a, b]$ (i.e. the closed balls in $\mathbb{R}$) maps to a closed interval. However, the closed (and open) set $\mathbb{R}$ maps to $(0, 1]$ which is not closed.

A similar statement can be made if X is a topological space whose topology is generated by a basis. Modifying the statement/proof of the above proposition for this more general setting is left as an exercise.

1.2 Nearness and Separation

In a general topological space, the open sets are what constitute our notion of "nearness". In fact, given a topological space $(X, \mathfrak{T})$ and a point $x \in X$, a "neighbourhood" of x refers to any open set containing x. Adhering to this picture, we can use the open sets to "separate points" as well.

Definition 1.6. Let $(X, \mathfrak{T})$ be a topological space. Two points x and y are said to be *separated* if there exist disjoint open sets containing x and y, respectively. A topological space is called *Hausdorff* if all distinct points in X are separated.

We remark that all metric spaces are automatically Hausdorff. Indeed, let (X, d) be a metric space and fix two distinct points $x, y \in X$. Put $\varepsilon := d(x, y)/2$ and note that $\varepsilon > 0$ because $x \neq y$. Next, consider the two ε-neighbourhoods $B(x, \varepsilon)$ and $B(y, \varepsilon)$. Obviously, these are both open subsets of X containing x and y, respectively. Furthermore, these two balls are disjoint. Otherwise, there exists $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$ whence the triangle inequality yields

$$d(x, y) \leq d(x, z) + d(z, y) < 2\varepsilon = d(x, y),$$

which is a contradiction. Thus, (X, d) is Hausdorff.

Proposition 1.3. *Let $(X, \mathfrak{T})$ be a Hausdorff topological space (e.g. a metric space). All finite subsets of X are closed in X.*

Proof. Since finite unions of closed sets are again closed, it suffices to show that any singleton is closed. Let $x \in X$ be given and consider $\{x\}$. The complement of $\{x\}$ is the set $U = X \setminus \{x\}$. For each $y \in U$, since $(X, \mathfrak{T})$ is a Hausdorff space and $x \neq y$, we may select disjoint open sets V_y, W_y with $x \in V_y$ and $y \in W_y$. In particular, W_y is an open set containing y but not intersecting x. Put otherwise,

$$y \in W_y \subseteq X \setminus \{x\} = U.$$

Therefore, U is open as the union of the open sets $\{W_y : y \in U\}$.

□

We now provide an example of a topology that is not Hausdorff.

Example 1.4. Let X be a non-empty set, and endow X with the *finite complement topology*:

$$\mathfrak{T} := \{U \subseteq X : U^{\mathsf{c}} \text{ is finite or } U = \varnothing\}.$$

It is easy to check that $(X, \mathfrak{T})$ is indeed a topological space. Certainly, it follows from the definition that $\varnothing, X \in \mathfrak{T}$. Furthermore, given a family $\{U_\alpha\}_{\alpha \in I}$ of sets from $\mathfrak{T}$, we have for each $\alpha \in I$ that either $U_\alpha = \varnothing$ or U_α^{c} is finite. In the event that every $U_\alpha = \varnothing$, we see $\varnothing = \bigcup_{\alpha \in I} U_\alpha \in \mathfrak{T}$ by construction. If some $U_\beta \neq \varnothing$, then

$$\left(\bigcup_{\alpha \in I} U_\alpha \right)^{\mathsf{c}} = \bigcap_{\alpha \in U} U_\alpha^{\mathsf{c}} \subseteq U_\beta^{\mathsf{c}}$$

implies that the complement of $\bigcup_{\alpha \in I} U_\alpha$ is finite, and hence $\mathfrak{T}$ is closed under arbitrary unions. Finally, let $U_1, U_2 \in \mathfrak{T}$. If either is empty, it follows at once that $U_1 \cap U_2 \in \mathfrak{T}$. Otherwise, U_1^{c} and U_2^{c} are finite whence

$$(U_1 \cap U_2)^{\mathsf{c}} = U_1^{\mathsf{c}} \cup U_2^{\mathsf{c}}$$

is finite and $U_1 \cap U_2 \in \mathfrak{T}$. By induction, it readily follows that $\mathfrak{T}$ is closed under finite intersections.

We now show that X is <u>not</u> Hausdorff. Let x, y be distinct points in X and suppose, by way of contradiction, that there exist disjoint open sets U_x and U_y containing x and y, respectively. Since U_x and U_y are both open, they each contain all but finitely many points in x. That is,

$$U_x = X \setminus \{x_1, \ldots, x_k\} \quad \text{and} \quad U_y = X \setminus \{y_1, \ldots, y_l\}$$

for the appropriate $x_i, y_j \in X$. Since X is infinite, we can choose a point $\xi \in X$ distinct from each x_i and every y_j. Then, $\xi \in U_x \cap U_y$, which is absurd. In fact, we have shown that any two non-empty open sets must intersect!

This last example actually shows that the finite complement topology on an infinite set is not induced by any metric. Indeed, the metric topology is always Hausdorff!

Remark 1.3. The closure of the open ball need not be the closed ball of the same radius and center! Indeed, consider any set X with at least two points equipped with the discrete metric. Then, for any $x \in X$, we have by definition that

$$B(x, 1) = \{y \in X : d(x, y) < 1\} = \{y \in X : d(x, y) = 0\} = \{x\}$$

where this set is both open and closed (why?) in X. Consequently, $\mathrm{cl}\,(B(x, 1)) = B(x, 1) = \{x\}$. On the other hand, the closed ball of radius 1 about x is the entire space X!

1.3 More on Continuous Functions

Continuity is not always intuitive, particularly in non-metrizable spaces.

Problem 2. *Let X be an infinite set equipped with the finite complement topology and let Y be a Hausdorff space (e.g. a metric space). Then, every continuous function $X \longrightarrow Y$ is constant.*

Proof. Let $f : X \longrightarrow Y$ be a continuous mapping. By way of contradiction, assume that f is *not* constant, i.e. there exist distinct points $y_1, y_2 \in Y$ with $y_1 = f(x_1), y_2 = f(x_2)$ for suitable $x_1, x_2 \in X$. Using that Y is a Hausdorff space, we may select *disjoint* open sets $V_1, V_2 \subseteq Y$ with $y_1 \in V_1$ and $y_2 \in V_2$. By continuity, both $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$ are open in X. Furthermore, they are non-empty since they contain x_1 and x_2, respectively. Consequently, by a previously seen property of the finite complement topology, U_1 and U_2 must necessarily intersect. Thus, there is a point $z \in U_1 \cap U_2$ whence $f(z) \in V_1 \cap V_2$, contradicting our choice of V_1 and V_2. □

Thankfully, in metric spaces, much of the intuition developed from studying functions from the real line to itself will remain useful. For instance, let us recall the definition of density:

Definition 1.7. Let $(X, \mathfrak{T})$ be a topological space. A subset D of X is said to be *dense* in X provided $\mathrm{cl}\,(D) = X$. Or, equivalently, if every non-empty open subset of X intersects D.

In analysis I, it is seen that if two continuous functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ correspond on a dense set then they are in fact equal. This fact remains true for functions between metric spaces. Even more generally, we have the following result:

Example 1.5. Let X be a topological space and Y a Hausdorff space (e.g. a metric space). Let $f, g : X \longrightarrow Y$ be continuous functions.

1. Show that the set $\{x \in X : f(x) = g(x)\}$ is closed in X.

2. Show that if $f \equiv g$ on a dense subset D of X, then $f \equiv g$ on all of X.

Proof.

1. We achieve this by proving that the complement $\{x \in X : f(x) \neq g(x)\}$ is open in X. Choosing x from this set, we have $f(x) \neq g(x)$. Because Y is Hausdorff, we may select disjoint neighbourhoods U and V containing $f(x)$ and $g(x)$, respectively. Then, $f^{-1}(U)$ and $g^{-1}(V)$ are both open (by continuity) sets containing the point x. Consequently, $W := f^{-1}(U) \cap g^{-1}(V)$ is a neighbourhood of x. Furthermore, $W \subseteq \{x \in X : f(x) \neq g(x)\}$ by construction. It follows that $\{x \in X : f(x) \neq g(x)\}$ is the union of open sets and thence open, as was asserted.

2. Denote by E the set of points in X on which $f = g$. By hypothesis, we have $D \subseteq E \subseteq X$. Since E is closed, it follows that $\mathrm{cl}(D) \subseteq E \subseteq X$. Density of D then reduces our last equation to $X \subseteq E \subseteq X$ whence $E = X$.

□

1.4 A Quick Look at the Subspace Topology

We first recall the subspace topology

Definition 1.8. Let $(X, \mathfrak{T})$ be a topological space and $Y \subseteq X$ a subset. Then, the subspace topology on Y is given by

$$\mathfrak{T}_Y = \{V \cap Y : V \in \mathfrak{T}\}.$$

It is easy to show $(Y, \mathfrak{T}_Y)$ is indeed a topological space.

In a similar vein, we can see that closed sets in a subspace Y of a topological space $(X, \mathfrak{T})$ are all of the form $A \cap Y$ where A is closed in X. Indeed, we quickly verify this below.

Proposition 1.4. *Let $(X, \mathfrak{T})$ be a topological space and let Y be a subspace of X. Then, a subset A of Y is closed in Y if and only if it takes the form $A = F \cap Y$ for some closed subset F of X.*

Proof. Let $A \subseteq Y$ be a closed subset of Y. That is, $Y \setminus A$ is open in Y. By definition of the subspace topology, it follows that $Y \setminus A = U \cap Y$ for some open subset U of X (i.e. $U \in \mathfrak{T}$). Clearly, we may write $A = Y \cap (X \setminus U)$ where $X \setminus U$ is closed in X. Hence, every arbitrary closed subset of Y is necessarily of the asserted form. It remains only to show that every such set is closed in Y. To this end, let $F \subseteq X$ be closed in X. That is, $X \setminus F$ is open in X (i.e. $X \setminus F \in \mathfrak{T}$). Then, the set

$$A := Y \cap F$$

is closed in Y. Indeed, it is clear that $A \subseteq Y$. Furthermore, $Y \setminus A = Y \cap (X \setminus F)$ where $X \setminus F$ is open in the parent topology of X. Thus, by definition, $Y \setminus A$ is open in Y whence A is closed in Y. This completes the proof. □

Proposition 1.5. *Let $(X, \mathfrak{T})$ be a topological space and Y is a subspace of X.*

1. *Suppose Y is open in X. If U is open in Y then U is open in X.*

2. *Suppose Y is closed in X. If A is closed in Y then A is closed in X.*

Proof. We only prove the first point. Since U is open in Y, it must be of the form $U = V \cap Y$ for some V that is open in X. However, Y is open in X and topologies are closed under finite intersections. It follows that U is open in X. □

Consider the real line $\mathbb{R}$ with the usual topology (induced by the metric $d(x, y) = |x - y|$). Consider then $I = [0, 1]$ along with the subspace topology inherited from $\mathbb{R}$. Observe that $[0, 1], [0, 1), (0, 1]$ and $(0, 1)$ are all open in I. However, only the interval $(0, 1)$ is open in $\mathbb{R}$. On the other hand, by the above proposition, any set that is open in $J = (0, 1)$ with respect to the subspace topology is also open with respect to the parent $\mathbb{R}$-topology.

1.5 Characterizations of continuity

We now return to the concept of continuity; with the aim of developing several characterizations of continuity, As a primer, recall that a function $f : X \longrightarrow Y$ is said to be continuous provided

$$f^{-1}(U) \text{ is open in } X \text{ for all open sets } U \text{ in } Y.$$

Taking the complement, we see that

$$\left(f^{-1}(U)\right)^{\mathsf{c}} = f^{-1}(U^{\mathsf{c}}) \text{ is closed in } X \text{ for all open sets } U \text{ in } Y.$$

Thus, an equivalent characterization of continuity is as follows: $f : X \longrightarrow Y$ is continuous provided

$$f^{-1}(C) \text{ is closed in } X \text{ for all closed sets } C \text{ in } Y.$$

Theorem 1.6. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces and fix a function $f : X \longrightarrow Y$. The following statements are equivalent.*

1. *f is continuous.*
2. *For every $A \subseteq X$, there holds $f(\mathrm{cl}\,(A)) \subseteq \mathrm{cl}\,(f(A))$.*
3. *If B is closed in Y, then $f^{-1}(B)$ is closed in X.*
4. *For every $x \in X$ and every neighbourhood V of $f(x)$ in Y, there exists a neighbourhood U of x such that $f(U) \subseteq V$.*

Proof. We begin with the implication $(1) \implies (2)$. Let $A \subseteq X$ be given and suppose that $y \in f(\mathrm{cl}\,(A))$. This means that there exists $x \in \mathrm{cl}\,(A)$ such that $y = f(x)$. Let now V be a neighbourhood of $y = f(x)$ and notice that $f^{-1}(V)$ is a neighbourhood of x, by continuity of f. Since x belongs to the closure of A, this means that $f^{-1}(V)$ intersects A. Given that $f(f^{-1}(V)) \subseteq$

V, we conclude that V intersects $f(A)$. As V was an arbitrary neighbourhood of y, we have $y \in \mathrm{cl}\,(f(A))$.

Let us now argue for (2) $\implies$ (3). Let $B \subseteq Y$ be a closed subset and let A denote $f^{-1}(B)$. We claim that $A = \mathrm{cl}\,(A)$, and for this the only non-trivial inclusion is $\mathrm{cl}\,(A) \subseteq A$. Let $x \in \mathrm{cl}\,(A)$ so that, by hypothesis,

$$f(x) \in f(\mathrm{cl}\,(A)) \subseteq \mathrm{cl}\,(f(A)) \subseteq \mathrm{cl}\,(B) = B.$$

This certainly gives $x \in f^{-1}(B) = A$. Since x was arbitrary, we conclude that $\mathrm{cl}\,(A) \subseteq A$ as was required.

We show (3) $\implies$ (1). Let $U \subseteq Y$ be open. Then, $f^{-1}(U^{\mathrm{c}})$ is closed as the pre-image of a closed set. Put otherwise,

$$f^{-1}(U^{\mathrm{c}}) = f^{-1}(U)^{\mathrm{c}}$$

is closed. That is, $f^{-1}(U)$ is open in X.

It now only remains to check that (1) $\iff$ (4). First, assume that f is continuous and fix $x \in X$ and a neighbourhood V of $f(x)$. By continuity, $f^{-1}(V)$ is open and contains x. Thus, taking $U := f^{-1}(V)$ gives one implication. Conversely, fix an open set V in Y. For any $x \in f^{-1}(V)$, we can choose an open set U_x containing x such that $f(U_x) \subseteq V$. But, this gives $U_x \subseteq f^{-1}(V)$. Since $U_x \ni x$, it is then easy to see that

$$\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V).$$

Since unions of open sets are open, $f^{-1}(V)$ is open and this concludes the proof. □

1.6 Compactness

We now turn towards the next pivotal topic of study: the notion of compactness. This begins with the simple definition of an open covering.

Definition 1.9. Let $(X, \mathfrak{T})$ be a topological space and let K be a subset of X. A covering of K is a family of subsets of X whose union contains K. An *open cover* of K is a family $\mathcal{U}$ of open sets in X such that $K \subseteq \bigcup_{U \in \mathcal{U}} U$.

We now come to the definition of compactness.

Definition 1.10. Let $(X, \mathfrak{T})$ be a topological space and K a subset of X. The set K is called *compact* in X if for every open cover $\mathcal{U}$ of K by open subsets of X, there exists a finite sub-collection of $\mathcal{U}$ whose union contains K. When the context is understood, we will simply say that K is compact.

A topological space $(X, \mathfrak{T})$ is called compact if X is compact as a subset of X. In this case, we would simply say that X is a compact topological space (or subspace). Equivalently, we have the following.

Definition 1.11. Let $(X, \mathfrak{T})$ be a topological space. We say that X is a compact space if for every open cover $\mathcal{U}$ of X, by elements of $\mathfrak{T}$, there exists a finite sub-collection of $\mathcal{U}$ whose union is equal to X.

Theorem 1.7. *Let $(X, \mathfrak{T})$ be a compact topological space. If $F \subseteq X$ is closed, then F is compact in X. Hence, F is a compact subspace of X.*

Proof. Let $\mathcal{U}$ be an open covering of F by subsets of X. Since F^{c} is open in X, the family $\mathcal{U} \cup \{F^{\mathsf{c}}\}$ is itself an open covering of X, by subsets of X. Using that $(X, \mathfrak{T})$ is compact, we may extract a finite sub-collection

$$U_1, \ldots, U_n, F^{\mathsf{c}}, \quad \text{with } U_j \in \mathcal{U},$$

such that $F^{\mathsf{c}} \cup \bigcup_1^n U_n$ contains X, and hence F. Since F and F^{c} are disjoint, we get that $F \subseteq \bigcup_1^n U_j$. This means that we have found a finite sub-collection of $\mathcal{U}$ that covers F. Since $\mathcal{U}$ was arbitrary, we conclude that F is compact in X. □

Theorem 1.8. *Let $(X, \mathfrak{T})$ be a Hausdorff topological space. If $K \subseteq X$ is compact, then K is closed in X.*

Proof. We will prove that $K^{\mathsf{c}} = X \setminus K$ is open in X. To this end, let us first fix a point $x_0 \in K^{\mathsf{c}}$. For each $y \in K$, we may choose disjoint open sets $U_y \ni y$ and $V_y \ni x_0$ in X. Notice then that the collection

$$\mathcal{U} := \left\{U_y : y \in K\right\}$$

is an open cover of K in X. By compactness, may choose finitely many points $y_1, \ldots, y_n$ in K so that $K \subseteq \bigcup_1^n U_{y_j}$. Now, the set

$$V_{x_0} := V_{y_1} \cap \cdots \cap V_{y_n}$$

is open in X and contains x_0. By construction, V_{x_0} does not intersect $\bigcup_1^n U_j$. Since $\bigcup_1^n U_j \supseteq K$, we have found an open set V containing x_0 with the property thats $V_{x_0} \subseteq K^{\mathsf{c}}$. But then, $K^{\mathsf{c}} = \bigcup_{x_0 \in K^{\mathsf{c}}} V_{x_0}$ whence K^{c} is open. □

Proposition 1.9. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces. Assume further that X is compact and Y is Hausdorff. Any continuous function $X \to Y$ is closed.*

Proof. Let f be a continuous function $X \to Y$ and fix a closed subset A of X. Since X is compact, A is itself compact. By continuity, the image $f(A)$ is compact in Y. Using that Y Hausdorff, we see that $f(A)$ is closed in Y. □

Example 1.6. Consider once again any infinite set X equipped with the co-finite (finite complement topology) we have seen in the past, i.e. give X the topology

$$\mathfrak{T} := \{U \in \mathcal{P}(X) : X \setminus U \text{ is finite or } U = \varnothing\}.$$

We have seen that the resulting space $(X, \mathfrak{T})$ is "strange" in the sense that it defies much of our common intuition. In particular, we have seen that *any two* non-empty open sets $U, V \subseteq X$

must necessarily intersect. We concluded from this that $(X, \mathfrak{T})$ was *not* Hausdorff nor metrizable. However, even when the underlying set X is infinite, *every* subset of X is compact. Certainly, let $A \subseteq X$ be given and let $\{U_\alpha\}_{\alpha\in I}$ be an arbitrary open cover of A. Note that if A is empty then it is automatically contained in any one of these covering sets. Consequently, we may assume without loss of generality that $A \neq \emptyset$. Next, choose any non-empty $U_\beta \in \{U_\alpha\}_{\alpha\in I}$ and note that U_β is open. Now, if $A \subseteq U_\beta$ then we are done as we hav exhibited a finite subcover. If $A \nsubseteq U_\beta$ then $A \cap U_\beta^c \neq \emptyset$. In fact, $A \cap U_\beta^c$ consists of finitely many points, say, $x_1, \ldots, x_k \in X$ since U_β is open with respect to the co-finite topology. Because every $x_i \in A$, there exists some member U_{α_i} from our cover $\{U_\alpha\}_{\alpha\in I}$ containing x_i. However, this means that $A \subseteq U_\beta \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$ whence A is compact.

1.6.1 Finite Intersection Property

Definition 1.12. A topological space $(X, \mathfrak{T})$ is said to have the *finite intersection property* if every family $\mathfrak{C}$ of closed subsets of X having the property that

$$C_1 \cap C_2 \cap \cdots \cap C_n \neq \emptyset,$$

for all finite sub-collections $\{C_1, \ldots, C_n\} \subseteq \mathfrak{C}$, also satisfies $\bigcap_{C\in\mathfrak{C}} C \neq \emptyset$.

We have just shown that compact spaces have this property. We now prove that all topological spaces having this property are, in fact, compact.

Theorem 1.10. *Let $(X, \mathfrak{T})$ be a topological space having the finite intersection property. Then, $(X, \mathfrak{T})$ is compact.*

Proof. We shall prove the contrapositive. If $(X, \mathfrak{T})$ is not compact, then one can find an open covering $\mathcal{U}$ of X having no finite sub-covering of X. Consider now the set

$$\mathfrak{C} := \{X \setminus U : U \in \mathcal{U}\}$$

which is a family of closed subsets in X. Since no finite sub-collection of $\mathcal{U}$ can cover X, finite intersections of elements in $\mathfrak{C}$ are non-empty. However, $\bigcup_{U\in\mathcal{U}} U = X$ which means that $\bigcap_{C\in\mathfrak{C}} C$ is empty. Hence, $(X, \mathfrak{T})$ does not have the finite intersection property. □

1.7 Connectedness

Lemma 1.11. *Let $(X, \mathfrak{T})$ be a topological space and Y a connected subspace of X. If (A, B) forms a separation of X, then Y is contained in one of A or B.*

Proof. Let A and B be two non-empty disjoint open sets whose union is X. In the subspace topology of Y, the sets $A \cap Y$ and $B \cap Y$ are open. Of course,

$$(A \cap Y) \cup (B \cap Y) = (A \cup B) \cap Y = Y$$

and $(A \cap Y) \cap (B \cap Y) = \emptyset$ since A and B are disjoint. This means that $A \cap Y$ and $B \cap Y$ are disjoint open subsets of Y whose union is Y. Since Y is connected, one of these is empty. Without harm, assume that $B \cap Y$ is empty. This implies that $Y \subseteq A$, as was required. □

Proposition 1.12. *Let $(X, \mathfrak{T})$ be a topological space. Then X is connected if and only if the only clopen sets in X are X and $\emptyset$.*

Proof. Let $A \subseteq X$ be a clopen set that is neither empty nor the whole space X. Since A is closed, its complement A^c is open in X. Since A and A^c are disjoint, we have found a separation of the space X. This means that $(X, \mathfrak{T})$ is disconnected. Conversely, assume that $(X, \mathfrak{T})$ is disconnected. We may then choose non-empty disjoint open sets A and B whose union is X. This means that $A^c = B$ so that A is clopen. Since $A \neq X$ and $A \neq \emptyset$, the proof is complete. □

Theorem 1.13. *Let $(X, \mathfrak{T})$ be a topological space and let $\{Y_\alpha\}_{\alpha \in I}$ be an indexed family of connected subspaces of X. If $\bigcap_{\alpha \in I} Y_\alpha$ is non-empty, then $\bigcup_{\alpha \in I} Y_\alpha$ is connected.*

Proof. The claim amounts to proving that $\bigcup_{\alpha \in I} Y_\alpha$ does not admit a separation. By way of contradiction, assume we can find two non-empty disjoint open sets A and B whose union is $\bigcup_{\alpha \in I} Y_\alpha$. Let p be a point belonging to $\bigcap_{\alpha \in I} Y_\alpha$. Without loss of generality, assume that $p \in A$. Since A and B are disjoint, $p \notin B$. Since every Y_α is connected, it must lie entirely within one of A or B, by the previous lemma. As $p \in A$, we get that $Y_\alpha \subseteq A$ for each $\alpha \in I$. That is, $\bigcup_{\alpha \in I} Y_\alpha \subseteq A$. This leaves us with $B = \emptyset$, which is a contradiction. □

Problem 3. *Let X be a space and Y a connected subspace of X. Will* $\operatorname{Int}(Y)$ *and ∂Y necessarily be connected? Need the converse also hold true?*

Proof. This is a tricky question that is useful to keep in mind. Both implications do not hold, but it will require some contrived examples to demonstrate this fact.

1. Let $B_1(1, 0)$ denote the closed ball of radius 1 centered at $(1, 0)$ in $\mathbb{R}^2$. Let $B_1(-1, 0)$ denote the closed ball of radius 1 centered at $(-1, 0)$. These two connected sets share the point $(0, 0)$, and hence their union is connected. On the other hand, the interior of this set is the union of two disjoint *open* balls, which is disconnected. In a similar vein, the connected set $(0, 1)$ has $\{0, 1\}$ as its boundary, which is clearly disconnected.

2. The converse direction also need not hold true. Recall that $\mathbb{Q}$ is a *totally disconnected* subspace of $\mathbb{R}$. Since $\mathbb{Q}^c$ is dense in $\mathbb{R}$, it is clear that $\operatorname{Int}(\mathbb{Q}) = \emptyset$. Thus, $\operatorname{Int}(\mathbb{Q})$ is connected. By definition, one has $\partial\mathbb{Q} = \operatorname{cl}(\mathbb{Q}) \cap \operatorname{cl}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$, which is also connected.

This completes the problem. □

Problem 4. *Let $A \subseteq X$ and assume that C is a connected subspace of X that intersects both A and A^c. Prove that C also intersects ∂A.*

Proof. We begin by showing that

$$X = \text{Int}(A) \sqcup \text{Int}(A^c) \sqcup \partial A.$$

Note that the use of '$\sqcup$' is justified because ∂A never intersects $\text{Int}(A)$ (this is Problem **??**). Let now $x \in X$ but assume that $x \notin \text{Int}(A) \sqcup \text{Int}(A^c)$; we will show that $x \in \partial A$. Since $x \notin \text{Int}(A)$, any neighbourhood of x has non-empty intersection with A^c. Similarly, $x \notin \text{Int}(A^c)$ means that any neighbourhood of x intersects A. However, both of these statements mean that

$$x \in \text{cl}(A) \cap \text{cl}(X \setminus A) = \partial A.$$

Having now verified the aforementioned identity, the proof is easily within reach. If C does not intersect ∂A, then it is contained within the union $\text{Int}(A) \sqcup \text{Int}(A^c)$, which therefore forms a separation of C. ◻

Example 1.7. Let $(X, \mathfrak{T})$ be any infinite set X endowed with its co-finite topology. Then, X is connected. By way of contradiction, let us suppose that X is disconnected. Then, there exist disjoint non-empty open sets $U, V \subseteq X$ such that $X = U \sqcup V$.[2] However, we have seen (in earlier examples) that any two non-empty open sets must necessarily intersect (since X is infinite with the co-finite topology). Thus, $U \cap V \neq \varnothing$ and we have our contradiction.

If instead X is a finite set with the co-finite topology, then it follows by inspecting the definition of the co-finite topology that every subset of X is open. Put otherwise, when X is finite, the co-finite topology is precisely the discrete topology $\mathcal{P}(X)$ which *always* produces a disconnected space. Thus, imposing the co-finite topology on a set X results in a connected space if and only if X is infinite.

By popular demand, we also elaborate upon Hatchers' proof that every closed and bounded interval in $\mathbb{R}$ is compact. Observe that, as a consequence of Theorem 1.7, it follows that every closed and bounded subset of $\mathbb{R}$ is compact.

Theorem 1.14. *A closed and bounded interval $I = [a, b]$ is compact in $\mathbb{R}$ (with the usual topology).*

Proof. Since the case $a = b$ is trivial, we may assume $a < b$. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of I and define

$$J := \{x \in [a, b] : [a, x] \text{ is contained in finitely many elements of } \mathcal{U}\}.$$

Since $\mathcal{U}$ covers I, let us observe that $a \in I$ implies the existence of some member $U_\alpha \in \mathcal{U}$ such that $a \in U_\alpha$. Furthermore, since U_α is open, there exists some $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq U_\alpha$. In particular, every $x \in [a, a + \delta)$ is an element of J and so J is non-empty.

Define $s := \sup J$ and note that, by definition, $a < s \leq b$. Next, let us show that $s \in J$. Indeed, since $s \in I$, there is some member $U_\beta \in \mathcal{U}$ having $s \in U_\beta$. Using that U_β is open, we can find some

[2] We use the notation '$\sqcup$' rather than the usual '$\cup$' to emphasize the disjoint nature of the union.

$\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \subseteq U_\beta$. Without loss of generality, we may assume that $\varepsilon > 0$ is so small that

$$s - \varepsilon > a. \tag{1.3}$$

In fact, since $s - \varepsilon$ is *not* an upperbound of J, there also exists $x \in [a, b]$ such that $s - \varepsilon < x \leq s$ and the interval $[a, x]$ is contained in the union of finitely many members of $\mathcal{U}$; let us call this finite sub-collection $\mathcal{V}$. Then,

$$[a, s] \subseteq [a, s-\varepsilon] \cup (s-\varepsilon, s+\varepsilon) \subseteq [a, x] \cup (s-\varepsilon, s+\varepsilon)$$

is contained in the finite sub-collection of $\mathcal{U}$ given by $\mathcal{V} \cup \{U_\beta\}$. Thus, we have shown that $s \in J$ and the proof will be complete if we can show that $b = s$. Arguing by contradiction, let us assume that $a < s < b$. Then, by following the argument used above[3], we will have that $s + \frac{\varepsilon}{2} \in J$, contradicting the fact that s is *by definition* an upperbound of J. This proves that $b \in J$ and so I is contained in the union of finitely many open sets from the covering $\mathcal{U}$. Since $\mathcal{U}$ was an arbitrary open cover, this proves that I is compact. □

Lemma 1.15. *Give $\mathbb{R}$ the usual topology and let $Y \subseteq X$ be a connected subspace. Then, Y is an interval.*[4]

Proof. We shall establish the contrapositive, i.e. we show that if Y is *not* an interval then it is disconnected. If $Y \neq \varnothing$ is not an interval, then there exist points $x, y \in Y$ (with $x < y$) such that $[x, y] \nsubseteq Y$. That is, there exists some point $c \in (x, y)$ such that $c \notin Y$. Then, $Y = ((-\infty, c) \cap Y) \sqcup ((c, \infty) \cap Y)$. □

Theorem 1.16. *Let $n \geq 2$ be an integer. There does not exist a continuous injection $\mathbb{R}^n \to \mathbb{R}$. In particular, $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}$ if and only if $n = 1$.*

Proof. We argue by contradiction; let $f : \mathbb{R}^n \to \mathbb{R}$ be an injective continuous function. Note that since $\mathbb{R}^n$ is convex, it is path connected, and therefore connected. By continuity, the image $f(\mathbb{R}^n)$ is a connected subspace of $\mathbb{R}$. Invoking the previous lemma, it follows that $f(\mathbb{R}^n)$ is an interval. In fact, since f is injective, the interval $f(\mathbb{R}^n)$ has end-points a, b with $a < b$ (otherwise, $a = b$ and $f(\mathbb{R}^n)$ would contain only one point which contradicts injectivity). Thus, we may choose a point $y_0 = f(x_0)$ in the interior of $f(\mathbb{R}^n)$, i.e. $a < y_0 < b$, it is clear that the subspace $f(\mathbb{R}^n) \setminus \{y_0\}$ is disconnected.

However, the restriction

$$f|_{\mathbb{R}^n \setminus \{x_0\}} : \mathbb{R}^n \setminus \{x_0\} \to f(\mathbb{R}^n) \setminus \{y_0\} = f(\mathbb{R}^n) \setminus \{f(x_0)\}$$

is again a continuous function. Since $\mathbb{R}^n \setminus \{x_0\}$ is connected (why?), its continuous image under f must also be connected. That is, $f(\mathbb{R}^n) \setminus \{y_0\}$ is connected – which contradicts our argument above. □

[3] It is the exact same argument, but in (1.3) choose $\varepsilon > 0$ so small that $a < s - \varepsilon < s + \varepsilon < b$. Note that this is possible in the case because $s < b$ by assumption.

[4] We use the following definition: a subset $I \subseteq \mathbb{R}$ is an interval if and only if, for each $x, y \in I$ with $x < y$, one has $[x, y] \subseteq I$.

1.8 Homeomorphy

Let us now take a brief side-step to introduce an important concept that will bring with it some very convenient mathematical language. Although we will not really spend time on this during the tutorials themselves, these notes will hopefully be of use to those who read this section.

Definition 1.13. Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces. A bijection $f : X \longrightarrow Y$ is called a *homeomorphism* provided both $f : X \longrightarrow Y$ and $f^{-1} : Y \longrightarrow X$ are continuous.

Remark 1.4. We remark that a homeomorphism is automatically both an open map and a closed map. Indeed, let $f : X \longrightarrow Y$ be a homeomorphism of topological spaces $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$. Then, given any open set $O \subseteq X$, we have that $f(O) = (f^{-1})^{-1}(O)$ is open in Y by continuity of f^{-1}. Similarly, we see that f is closed.

We remark that the inverse function $f^{-1} : Y \longrightarrow X$ exists because f (in the definition above) is assumed to be bijective, a priori. Having introduced a new category of function, we now use this to determine a notion of equivalence (or isomorphism!) for topological spaces.

Definition 1.14. Two topological spaces $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ are said to be *homeomorphic* if there exists a homeomorphism $X \longrightarrow Y$. In this case, we often abbreviate this by simply writing $X \cong Y$.[5]

Before proceeding, several key remarks are in order. We try to present these quickly by summarizing them below and most of their proofs are left as straight-forward exercises to the reader.

(i) Any space $(X, \mathfrak{T})$ is homeomorphic to itself because the identity map $\mathrm{id}_X : X \longrightarrow X$ given by $\mathrm{id}_X(x) = x$ is always a homeomorphism.

(ii) If $f : X \longrightarrow Y$ is a homeomorphism, then so is its inverse $f^{-1} : Y \longrightarrow X$. Thus, $X \cong Y \iff Y \cong X$.

(iii) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are homeomorphisms, then so is the composite mapping $g \circ f : X \longrightarrow Z$ (with inverse $f^{-1} \circ g^{-1} : Z \longrightarrow X$). Thus, if $X \cong Y$ and $Y \cong Z$ then $X \cong Z$.

(iv) These last three properties (resp. reflexivity, symmetry, transitivity) imply that being homeomorphic defines an *equivalence relation* on topological spaces. That is, we declare two topological spaces $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ to be "the same" when they are homeomorphic. [6]

(v) In what follows let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be two homeomorphic topological spaces.

- X is (path) connected if and only if Y is (path) connected;
- X is compact if and only if Y is compact;

[5] For those using LaTeX, the symbol '$\cong$' is the congruence symbol given by the command '\cong'.

[6] We warn the reader that these spaces are deemed equivalent only in a context that is purely topological; and X may carry additional structure that Y does not – or vice versa.

- X is metrizable if and only if Y is metrizable;
- X is completely metrizable if and only if Y is completely metrizable;
- X is Hausdorff if and only if Y is Hausdorff.

However, not everything is preserved via homeomorphism. For instance, if (X, d_X) and (Y, d_Y) are metric spaces with a homeomorphism $f : X \to Y$, we cannot guarantee that $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$ for $x_1, x_2 \in X$ (i.e. f may not be distance preserving[7]). A simple example of such a case is the homeomorphism

$$f : [0,1] \to [0,2], \quad x \mapsto 2x,$$

where $[0,1]$, $[0,2]$ inherit the metric of $\mathbb{R}$.

Theorem 1.17. *Let $(X, \mathfrak{T})$ be a compact topological space and $(Y, \mathfrak{W})$ a Hausdorff topological space. Let $f : X \to Y$ be a bijective continuous function. Then, f is a homeomorphism.*

Proof. The only non-trivial property to prove is the continuity of f^{-1}. By Theorem 1.6, this is equivalent to showing that f maps closed sets to closed sets (that is, f is a closed map). This is true by Proposition 1.9. Since the proof is short, we re-state it here:

Fix a closed subset A of X. Since X is compact, A is itself compact. By continuity, the image $f(A)$ is compact in Y. Using that Y Hausdorff, we see that $f(A)$ is closed in Y. □

Example 1.8. If we forgo the compactness assumption on the domain space, all bets are off in the previous theorem's conclusion. For instance, let $X = \mathbb{R}$ with the discrete topology and $Y = \mathbb{R}$ with the usual (Euclidean) topology. Consider the identity map $f(x) = \mathrm{id}_X(x) = x$ from $X \to Y$. Since every subset of X is open with respect to its discrete topology, it is automatic that f is continuous. Furthermore, f is obviously a bijection. However, $f^{-1} : Y \to X$ is not continuous. To see this, notice that $\left(f^{-1}\right)^{-1}(\{0\}) = f(\{0\}) = \{0\}$ is not open in Y, whereas $\{0\}$ is open in X. It follows that f^{-1} is discontinuous and is thus f is *not* a homeomorphism.

Theorem 1.18. *Let $X_1, \ldots, X_n$ be connected topological spaces and let X denote the product $X_1 \times \cdots \times X_n$ with the product topology. Then, X is connected.*

Proof. By induction, it suffices to show that $X_1 \times X_2$ is connected whenever $X_{1,2}$ are themselves connected. To this end, let us first fix a point $(a, b) \in X_1 \times X_2$. The reader may easily verify that $X_1 \times \{b\}$ is homeomorphic to X_1. Similarly, for every $x \in X_1$ it can be shown that $\{x\} \times X_2 \cong X_2$. Thus, by the previous theorem, both $X_1 \times \{b\}$ and $\{x\} \times X_2$ will be connected, for each $x \in X_1$. Noticing that

$$(x, b) \in (X_1 \times \{b\}) \cap (\{x\} \times X_2),$$

we apply Theorem 1.13 to deduce that the "slice"

$$\Gamma_x := (X_1 \times \{b\}) \cup (\{x\} \times X_2)$$

is also connected. Finally, by observing that $(a, b) \in \bigcap_{x \in X_1} \Gamma_x$, we see that this same theorem implies that $\bigcup_{x \in X_1} \Gamma_x = X_1 \times X_2$ is connected. □

[7] A continuous function between two metric spaces that preserves distance is called an *isometry*.

1.9 Completeness and Compactness in Metric Spaces

Let us recall that a metric space (X, d) is called *complete* when all of its Cauchy sequences converge. Furthermore, a *contraction* of a metric space is a mapping $f : X \to X$ such that there exists $c \in [0, 1)$ having the property that $d(f(x), f(y)) \leq cd(x, y)$ for all $x, y \in X$. In the lectures, the following major result was proven, via iterative means:

Theorem 1.19 (Banach Fixed Point Theorem)**.** *Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Then, there exists a unique fixed point of f in X. That is, there exists a unique point $\xi \in X$ such that $f(\xi) = \xi$.*

Example 1.9. Let (X, d) be a complete metric space and let $f : X \to X$ be given. Given $k \in \mathbb{N}$, let $f^{(k)}$ denote the k-fold composition

$$f^{(k)}(x) := \underbrace{(f \circ \cdots \circ f)}_{k-times}(x).$$

Suppose there exists $k \in \mathbb{N}$ such that $f^{(k)}$ is a contraction. Then, f still possesses a unique fixed point in X. Indeed, to see the existence of such a point, observe that $f^{(k)}$ has a fixed point $\xi \in X$ by virtue of the Banach Fixed Point Theorem. But then,

$$f^{(k)}(\xi) = \xi \implies f\left(f^{(k)}(\xi)\right) = f(\xi) \iff f^{(k)}(f(\xi)) = f(\xi),$$

meaning that $f(\xi)$ is a fixed point for $f^{(k)}$ whenever ξ is. By the uniqueness of the fixed point, we infer that $f(\xi) = \xi$. Hence, f has a fixed point. Let now $\eta \in X$ be any fixed point of f, i.e. $f(\eta) = \eta$. Clearly,

$$f^{(k)}(\eta) = f^{(k-1)}(f(\eta)) = f^{(k-1)}(\eta) = \cdots = f(\eta) = \eta$$

whence η is a fixed point of $f^{(k)}$. Since $f^{(k)}$ has a unique fixed point, it follows that f has a unique fixed point.

Complete metric spaces also satisfy the following intersection property, which is highly reminiscent of the finite intersection property we previously encountered (both within these notes and your assignments) for compact topological spaces.

Theorem 1.20 (Cantor's Intersection Theorem for Complete Metric Spaces)**.** *Let (X, d) be a complete metric space and let $\{F_n\}_{n\in\mathbb{N}}$ be a decreasing family of non-empty closed subsets of X. If, in addition,*

$$\lim_{n\to\infty} \operatorname{diam}(F_n) = 0,$$

where $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$*, then the intersection $\bigcap_{n\in\mathbb{N}} F_n$ is non-empty and consists of exactly one point.*

Proof. First, we show that $\bigcap_{n\in\mathbb{N}} F_n$ is non-empty via a constructive argument. Since each $F_n \neq \emptyset$ by assumption, we may extract some x_n from each F_n, thereby forming a sequence (x_n). Furthermore, since $F_n \supseteq F_{n+1}$ for each $n \in \mathbb{N}$, it follows that $x_m \in x_n$ if $m \geq n$. Thus, if $m \geq n$ there holds

$$d(x_n, x_m) \leq \operatorname{diam}(F_n).$$

Since $\operatorname{diam}(F_n) \longrightarrow 0$, it readily follows that the sequence (x_n) is Cauchy. Using that (X, d) is a complete space, there exists a point $x \in X$ such that $x_n \longrightarrow x$. Furthermore, x is a limit of every tail sequence $(x_m)_{m\geq n}$. Put otherwise, for each $n \in \mathbb{N}$ there is a sequence in F_n converging to x. Since F_n is closed, it must contain its limit points and so $x \in F_n$. As $n \in \mathbb{N}$ is arbitrary, this implies that $x \in \bigcap_{n\in\mathbb{N}} F_n$.

Next, we show that this intersection can only contain a single point. Let $x, y \in \bigcap_{n\in\mathbb{N}} F_n$ be given; we claim that $y = x$. Indeed, if $x, y \in \bigcap_{n\in\mathbb{N}} F_n$ then $x, y \in F_n$ for all $n \in \mathbb{N}$. Hence, $d(x, y) \leq \operatorname{diam}(F_n)$ for all $n \in \mathbb{N}$. Passing to the limit as $n \longrightarrow \infty$, we infer that $0 \leq d(x, y) \leq 0$ whence $x = y$. □

We now provide a practical example of a complete metric space.

Proposition 1.21. *Let (X, d) be a compact metric space and denote by $C(X)$ the real vector space of continuous functions $X \longrightarrow \mathbb{R}$. Then, $C(X)$ is a complete metric space when given the norm*

$$\|f\| := \sup_{x\in X} |f(x)|\,.$$

Proof. We leave it as an exercise to check that $(C(X), \|\cdot\|)$ is indeed a normed-space. To see that it is complete as a metric space, we must show that every Cauchy sequence is convergent. To this end, let (f_n) be a Cauchy sequence in $C(X)$. By definition, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|f_n - f_m\| = \sup_{x\in X} |f_n(x) - f_n(x)| < \varepsilon, \quad \forall n, m \geq N \tag{1.4}$$

In particular, for any $x \in X$ and every $n, m \geq N$ we find that $|f_n(x) - f_m(x)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows that $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, for each fixed $x \in X$. Since $\mathbb{R}$ is complete, every Cauchy sequence is convergent. Especially, there exists some number, which we denote $f(x)$, such that

$$f_n(x) \xrightarrow{n\to\infty} f(x).$$

This defines a function $f : X \longrightarrow \mathbb{R}$. Furthermore, given $\varepsilon > 0$ is N is as in equation (1.4) then

$$|f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in X \text{ and } n \geq N.$$

Taking the limit as $m \longrightarrow \infty$ and using that $f(x) := \lim_{m\to\infty} f_m(x)$, we find that

$$|f_n(x) - f(x)| \leq \varepsilon, \quad \forall x \in X \text{ and } n \geq N.$$

Put otherwise, we have shown that there exists a function $f : X \longrightarrow \mathbb{R}$ such that, given any $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\|f_n - f\| = \sup_{x \in X} |f_n(x) - f(x)| \leq \varepsilon. \tag{1.5}$$

The proof will now be compete if we can show that $f \in C(X)$, i.e. that f is continuous on X. To this end, fix a point $c \in X$ and let $\varepsilon > 0$. Using (1.5), we can find $N \in \mathbb{N}$ such that

$$\|f_n - f\| < \frac{\varepsilon}{3}, \quad \forall n \geq N.$$

Now, since f_N is continuous on X, there exists $\delta > 0$ such that $d(x, c) < \delta$ implies

$$|f_N(x) - f_N(c)| < \frac{\varepsilon}{3}.$$

Combining all of this, we see that for any $n \geq N$ and any $x \in X$ with $d(x, c) < \delta$ there holds

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &\leq 2\|f_N - f\| + |f_N(x) - f_N(c)| \\ &< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

Hence, $f \in C(X)$ and the proof is complete. □

Theorem 1.22. *Let X be compact metric space. Suppose $f : X \longrightarrow X$ is a function such that $d(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$.*[8] *Then, f is a surjective isometry.*

Proof. Let $x \in X$ and consider the sequence $x_1 = f(x)$ and $x_{n+1} = f(x_n)$. Put otherwise, if $f^{(n)}$ denotes the composition of f with itself n times, then $x_n = f^{(n)}(x)$. Now, suppose that $(x_n)_n$ has a convergent subsequence $(x_{n_k})_k$. In particular, since the latter sequence is Cauchy,

$$d\left(x_{n_{k+1}}, x_{n_k}\right) \xrightarrow{k \to \infty} 0.$$

On the other hand, observe that

$$\begin{aligned} d\left(x_{n_{k+1}}, x_{n_k}\right) = d\left(f^{(n_{k+1})}(x), f^{(n_k)}(x)\right) &\geq d\left(f^{(n_{k+1}-1)}(x), f^{(n_k-1)}(x)\right) \\ &\geq \cdots \\ &\geq d\left(f^{(n_{k+1}-n_k)}(x), x\right) \\ &= d\left(x_{n_{k+1}-n_k}, x\right). \end{aligned}$$

In particular, since $d\left(x_{n_{k+1}}, x_{n_k}\right) \to 0$, we see that $x_{n_{k+1}-n_k} \to x$ as $k \to \infty$. We summarize our work so far in the form of a claim.

[8]Such a mapping is called an *expansion* of X. Alternatively, f may be called an *expanding map*.

Claim: Given $x \in X$, let $(x_n)_n$ be the sequence defined by $x_1 = f(x)$ and $x_{n+1} = f(x_n)$, i.e. $x_n = f^{(n)}(x)$. If $(x_{n_k})_k$ is a convergent subsequence of $(x_n)_n$ then $(x_{n_{k+1}-n_k})_k$ converges to x.

The above claim has some immediate consequences. First, since X is compact, any sequence has a convergence subsequence. In particular, given $x \in X$ the associated sequences $(x_n)_n$ defined by $x_n = f^{(n)}(x)$ has a convergent subsequence $(x_{n_k})_k$. Next, let $y \in X$ be another element and consider also the sequence $(y_n)_n$ given by $y_n = f^{(n)}(y)$. Then, its subsequence $(y_{n_k})_k$ has a convergent subsequence $(y_{n_{k_j}})_j$. Then, using that subsequences of convergent sequences converge, we see that

$$(x_{n_{k_j}})_j, \qquad (y_{n_{k_j}})_j$$

are both convergent subsequences of, respectively, $(x_n)_n$ and $(y_n)_n$. For notational convenience, we re-label these sequences to

$$(x_{n_j})_j, \qquad (y_{n_j})_j$$

By our claim, $x_{n_{j+1}-n_j} \longrightarrow x$ and $y_{n_{j+1}-n_j} \longrightarrow y$ as $j \longrightarrow \infty$. Thus,

$$\begin{aligned} d(x,y) = \lim_{j\to\infty} d\left(x_{n_{j+1}-n_j}, y_{n_{j+1}-n_j}\right) &= \lim_{j\to\infty} d\left(f^{(n_{j+1}-n_j)}(x), f^{(n_{j+1}-n_j)}(y)\right) \\ &\geq d(f(x), f(y)) \end{aligned}$$

On the other hand, since we have assumed that $d(x,y) \leq d(f(x), f(y))$ we conclude that, in fact, equality holds. That is, $d(x,y) = d(f(x), f(y))$. Since $x, y \in X$ were arbitrary, this means that f is an isometry. In particular, f is (uniformly) continuous.

It remains only to verify that f is surjective. Put otherwise, we must show that $f(X) = X$. To see this, first observe that $f(X)$ is dense in X. Indeed, for any $x \in X$ we consider the sequence $(x_n)_n \subseteq f(X)$ given by $x_n = f^{(n)}(x)$. Then, since X is compact, we may extract a convergent subsequence $(x_{n_k})_k$. By our claim, $x_{n_{k+1}-n_k} \longrightarrow x$. That is, we found a sequence in $f(X)$ converging to x so, indeed, $f(X)$ is dense in X. Put otherwise, the closure of $f(X)$ is all of X

On the other hand, we have shown that f is continuous. Thus, since X is compact, its image $f(X)$ under f is also compact. Then, because X is Hausdorff, we see that $f(X) \subseteq X$ is closed. Thus, the closure of $f(X)$ is itself. But, by density, the closure of $f(X)$ is X. Ergo, $f(X) = X$ and we are done. □

1.10 Baire Category Theorem and Basic Applications

Theorem 1.23 (Baire Category Theorem)**.** *Let $(X, \mathfrak{T})$ be a completely metrizable space and suppose $\{U_n\}_n$ is a countable collection of open dense sets in X. Then $\bigcap_n U_n$ is dense in X.*

Proof. Let d be a complete metric on X inducing the topology $\mathfrak{T}$. Let $W \subseteq X$ be a non-empty open set; the statement amounts to showing that $W \cap \bigcap_n U_n$ is non-empty. To this end, notice

that $W \cap U_1$ is non-empty (since U_1 is dense in X) and thus contains a point, say x_1. There then exists $0 < r_1 < 2^{-1}$ such that
$$\overline{B(x_1, r_1)} \subseteq U_1 \cap W.$$
We now proceed inductively as follows:

Given x_k and r_k, we consider the intersection $B(x_k, r_k) \cap U_{k+1}$, which is non-empty by assumption. Thus, there exists a point
$$x_{k+1} \in B(x_k, r_k) \cap U_{k+1}$$
where this intersection is an open set. Once again, we may choose $r_{k+1} > 0$ such that
$$\overline{B(x_{k+1}, r_{k+1})} \subseteq B(x_k, r_k) \cap U_{k+1}.$$
Without harm, we may choose $r_{k+1} < 2^{-(k+1)}$.

Now, for every $n > 1$ we have found
$$\begin{aligned} x_n \in B(x_{n-1}, r_{n-1}) \cap U_n &\subseteq [B(x_{n-2}, r_{n-2}) \cap U_{n-1}] \cap U_n \\ &\subseteq B(x_1, r_1) \cap \bigcap_{k=2}^{n} U_k \\ &\subseteq W \cap \bigcap_{k=1}^{n} U_k. \end{aligned}$$
which means that we are done if the sequence of sets $\{U_n\}$ is finite. Otherwise, we have constructed a sequence $(x_n)_{n=1}^{\infty}$ of points in X. We now claim that this sequence is Cauchy. Indeed, let $\varepsilon > 0$ be given and let $N \in \mathbb{N}$ be such that $2^{-N} < \varepsilon/2$. If $n, m \geq N$ then
$$x_n, x_m \in B(x_N, r_N)$$
whence it follows that $d(x_n, x_m) < \varepsilon$. Since (X, d) is complete, there exists a point $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, for every $N \in \mathbb{N}$ our construction gives
$$x_n \in \overline{B(x_N, r_N)}, \quad \forall n \geq N.$$
Passing to the limit, we find that
$$x \in \overline{B(x_N, r_N)} \subseteq B(x_{N-1}, r_{N-1}) \cap W \subseteq U_N \cap W.$$
This implies that $x \in W \cap \bigcap_n U_n$ as was required. □

Remark 1.5. This result remains valid if $(X, \mathfrak{T})$ is instead a *locally compact Hausdorff* space (abbreviated LCH). That is, if $(X, \mathfrak{T})$ is a Hausdorff space such that, at each point $x \in X$, there exists an open set $U \ni x$ such that $\mathrm{cl}(U)$ is compact. The proof strategy is similar, but requires modifications (what are Cauchy sequences without a metric?). One cannot deduce the result for completely metrizable spaces from that for LCH-spaces.

1.10.1 Some Examples and Applications

Before we delve into some interesting applications of the Baire Category theorem, we require some terminology.

Definition 1.15. Let $(X, \mathfrak{T})$ be a topological space and let $E \subseteq X$. We say that E is *nowhere dense* in E if $\text{cl}(E)$ has empty interior. That is, if for each $x \in \text{cl}(E)$ and every neighbourhood $U \ni x$, one has $U \not\subseteq \text{cl}(E)$.

With this language, the Baire-Category theorem may be rephrased:

Corollary 1.24. *Let $(X, \mathfrak{T})$ be a completely metrizable space. Then, X is not the countable union of nowhere dense sets.*

Proof. We argue by contradiction. Suppose that one may write $X = \bigcup_n E_n$, where this union is countable and each $E_n \subseteq X$ is nowhere dense. Then, $\text{cl}(E_n)$ has empty interior which implies that $\text{cl}(E_n)^c$ is open and dense in X. Indeed, $\text{cl}(E_n)^c$ is open by definition. To see that this set is dense, let $U \neq \emptyset$ be an open subset of X and note that $U \not\subseteq \text{cl}(E_n)$ because $\text{cl}(E_n)$ has empty interior. Therefore, $U \cap \text{cl}(E_n)^c \neq \emptyset$. Put otherwise, $\text{cl}(E_n)^c$ intersects every non-empty open subset of X, i.e. $\text{cl}(E_n)^c$ is dense in X. Finally, appealing to the Baire Category Theorem ensures that the countable intersection of open dense sets $\bigcap_n \text{cl}(E_n)^c$ remains dense in X. Especially, $\bigcap_n \text{cl}(E_n)^c$ is non-empty whence

$$X = \bigcup_n E_n \subseteq \bigcup_n \text{cl}(E_n) = \left[\bigcap_n \text{cl}(E_n)^c\right]^c \neq X.$$

which is a contradiction. □

We obtain as a direct consequence of this an alternative proof that $\mathbb{R}$ is uncountable.

Corollary 1.25. *Let $(X, \mathfrak{T})$ be a completely metrizable space such that singletons are not open.*[9] *Then, X is uncountable. In particular, $\mathbb{R}$ is uncountable.*

Proof. Observe that if a singleton $\{x\}$ is not open then it must have empty interior. Otherwise, there would exist a non-empty open set $O \subseteq \{x\}$ whence $O = \{x\}$ and $\{x\}$ is open.

Now, we address the statement itself: we argue by contradiction and assume that X is countable. Then, we may enumerate its elements $\{x_n\}_n$. In particular, $X = \bigcup_n \{x_n\}$ where each $\{x_n\}$ is closed and with empty interior. Put otherwise, each $\{x_n\}$ is nowhere dense in X. Thus, we have written X as the countable union of nowhere dense sets which contradicts the Baire Category Theorem. □

Some surprising results can even be deduced about the rationals $\mathbb{Q}$ in $\mathbb{R}$.

[9] A topological space $(X, \mathfrak{T})$ such that no singleton $\{x\}$ is open is called *perfect*. That is, a perfect space is one with no isolated points. In some texts, these spaces are also referred to as *crowded* spaces.

Lemma 1.26. *There do not exist countably many open sets*[10] $\{U_n\}_n$ *in* $\mathbb{R}$ *such that* $\bigcap_n U_n = \mathbb{Q}$.

Proof. We proceed by way of contradiction: assume that there exists a countable family $\{U_n\}_n$ of open subsets of $\mathbb{R}$ such that $\mathbb{Q} = \bigcap_n U_n$. By allowing for repetitions in our collection, we may assume without loss of generality that the family $\{U_n\}_n = \{U_n\}_{n\in\mathbb{N}}$ is countably infinite. Furthermore, it is clear that each U_n contains $\mathbb{Q}$ whence each U_n is open and dense in $\mathbb{R}$. Let now $\{r_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$ (which is possible because $\mathbb{Q}$ is countably infinite) and define, for each $n \in \mathbb{N}$, the open set

$$V_n := U_n \setminus \{r_n\} = U_n \cap \{r_n\}^{\mathsf{c}}.$$

We note that V_n is also dense in $\mathbb{R}$. Indeed, given a non-empty open set $W \subseteq \mathbb{R}$, the set $W \setminus \{r_n\}$ is both open and non-empty. Since $U_n \supseteq \mathbb{Q}$ is dense, it follows that

$$(W \setminus \{r_n\}) \cap U_n \neq \varnothing$$

thereby implying that $W \cap V_n = W \cap (U_n \setminus \{r_n\})$ is non-empty. Hence, V_n is dense in $\mathbb{R}$ for each $n \in \mathbb{N}$. By virtue of the Baire Category Theorem, the countable intersection $\bigcap_{n\in\mathbb{N}} V_n$ is dense in $\mathbb{R}$. In particular, this intersection is non-empty. On the other hand,

$$\begin{aligned}\bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} (U_n \setminus \{r_n\}) &= \left(\bigcap_{n\in\mathbb{N}} U_n\right) \setminus \left(\bigcup_{n\in\mathbb{N}} \{r_n\}\right) \\ &= \left(\bigcap_{n\in\mathbb{N}} U_n\right) \setminus \mathbb{Q} \\ &= \mathbb{Q} \setminus \mathbb{Q} \\ &= \varnothing,\end{aligned}$$

which is an obvious contradiction. □

Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. We define the *continuity set* of f (labelled $\mathscr{C}(f)$) as the set of points in $\mathbb{R}$ at which the function f is continuous. Symbolically, we define

$$\mathscr{C}(f) := \{x \in \mathbb{R} : f \text{ is continuous at } x\}.$$

As it turns out, $\mathscr{C}(f)$ is *always* the countable intersection of open sets (i.e. a G_δ set). Indeed, it follows from the $\varepsilon - \delta$ definition of continuity that

$$\mathscr{C}(f) = \bigcap_{n\in\mathbb{N}} A_n \tag{1.6}$$

[10]Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. We say that A is a G_δ set if there exist countable many open sets $\{U_n\}_n$ such that $A = \bigcap_n U_n$. That is, a set is called G_δ if it can be written as the countable intersection of open sets. Analogously, A is called an F_σ set provided it is the countable union of closed sets. In short, this lemma states that $\mathbb{Q}$ is not G_δ in $\mathbb{R}$.

where

$$A_n := \left\{ x \in \mathbb{R} : \exists \delta > 0 \text{ s.t. } |f(s) - f(t)| < \frac{1}{n} \text{ for all } s, t \in \mathbb{R} \text{ with } (x - \delta, x + \delta) \right\}. \tag{1.7}$$

The proof of (1.6) is left as an exercise to the reader.[11] However, we shall prove that that each A_n is open in $\mathbb{R}$. Certainly, fix a point $x \in A_n$. By definition of the set A_n, there exists $\delta > 0$ such that

$$|f(s) - f(t)| < \frac{1}{n} \tag{1.8}$$

for all $s, t \in (x - \delta, x + \delta)$. Since we wish to show that A_n is open, we seek to exhibit some ε-neighbourhood $(x - \varepsilon, x + \varepsilon)$ of x such that $(x - \varepsilon, x + \varepsilon) \subseteq A_n$. To see that such an inclusion is possible, notice that, since $(x - \delta, x + \delta)$ is an open set, there exists for each $y \in (x - \delta, x + \delta)$ some $r > 0$ such that $(y - r, y + r) \subseteq (x - \delta, x + \delta)$. Furthermore, because (1.8) holds for all $s, t \in (x - \delta, x + \delta)$, this same identity must be valid in $(y - r, y + r) \subseteq (x - \delta, x + \delta)$ whence $y \in A_n$. Using that $y \in (x - \delta, x + \delta)$, we infer that $(x - \delta, x + \delta) \subseteq A_n$. In summary, we have shown that each A_n is an open set.

Corollary 1.27. *There does not exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathscr{C}(f) = \mathbb{Q}$. That is, no function $\mathbb{R} \to \mathbb{R}$ can have $\mathbb{Q}$ as its continuity set.*

Proof. We proceed by way of contradiction. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $\mathscr{C}(f) = \mathbb{Q}$. By our discussion above, $\mathscr{C}(f)$ is the countable intersection of open sets (i.e. the A_n's from (1.7)) which contradicts Lemma 1.26. □

Definition 1.16. Let (X, d) be a metric space. The metric d is said to be *proper* if every closed ball

$$B_c(x, \varepsilon) := \{y \in X : d(x, y) \le \varepsilon\}$$

is compact in X.

Theorem 1.28. *A proper metric space is both complete and locally compact Hausdorff.*

Proof. Any metric space is Hausdorff so we need only verify that X is locally compact to establish that X is LCH. Indeed, given any point $x \in X$, the open ball $B(x, 1)$ is by definition a neighbourhood of x. Furthermore, since $B(x, 1) \subseteq B_c(x, 1)$, we find that $\operatorname{cl}(B(x, 1)) \subseteq B_c(x, 1)$, where $B_c(x, 1)$ is compact. Since closed subsets of compact sets remain compact, we have exhibited a neighbourhood of x with compact closure. Since x was arbitrary, we infer that X is LCH.

Next, we show that (X, d) is complete. To this end, let (x_n) be a Cauchy sequence in X. Since (x_n) is bounded, for any $a \in X$, there exists $M > 0$ such that $d(x_n, a) \le M$ for all $n \in \mathbb{N}$. That is, (x_n) is contained within the closed ball $B_c(a, M)$. But, since d is a proper metric, this set is (sequentially) compact. Hence, (x_n) has a convergent subsequence (x_{n_k}). By an assignment result, (see your assignment 4), a Cauchy sequence with a convergent subsequence is necessarily convergent. Thus, (x_n) is seen to convergence and (X, d) is complete. □

[11] Recall that, for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ with the property that $\frac{1}{n} < \varepsilon$.

1.11 Lebesgue Measure Zero in One-Dimension

In relevance to your fourth assignment, we now briefly turn our focus towards the very tip of the measure theory iceberg. That is, we introduce what is means for a subset of $\mathbb{R}$ to be "negligible" in a volumetric sense. We would also like to point out that, although we will be discussing sets of Lebesgue measure zero, we shall *not* be constructing the full Lebesgue measure or σ-algebra.

Let now $E \subseteq \mathbb{R}$ be an arbitrary set. Intuitively, we know how to measure the "length" of an interval (a, b). Indeed, we define its length (or measure) to be the value $b - a$. Now, let us imagine that we have some countable collection $\{I_k\}$ of open intervals whose union covers E, i.e. $E \subseteq \bigcup_k I_k$. Then, although we haven't yet given a definition for the measure or "length" of E, any reasonable definition would have to satisfy the following:

$$\text{measure of } E \ \leq \sum_k (b_k - a_k), \quad \text{where } I_k = (a_k, b_k).$$

That is, if $E \subseteq \bigcup_k I_k$, then the measure or length of E should be no-greater than the combined lengths of all the intervals covering it. With this, the following definition of having **measure zero** is better motivated:

Definition 1.17 (Lebesgue Measure Zero)**.** Let $E \subseteq \mathbb{R}$. We say that E has Lebesgue measure 0 (or, alternatively, a null set) provided, for each $\varepsilon > 0$, there exists a countable collection of open intervals $\{I_k\}$ such that

$$E \subseteq \bigcup_k I_k \quad \text{and} \quad \sum_k |I_k| \leq \varepsilon.$$

Here, $|I| = b - a$ denotes the length of any open interval $I = (a, b)$.

Loosely speaking, the set E is said to have measure 0 if it can be covered by countably many open intervals of arbitrarily small combined length.

Example 1.10. Every countable subset of $\mathbb{R}$ has Lebesgue measure 0.

Proof. Let $E = \{e_k\}_k$ be a countable set and let $\varepsilon > 0$ be given. For each index k, put

$$I_k := \left(e_k - \frac{\varepsilon}{2^{k+1}}, e_k + \frac{\varepsilon}{2^{k+1}}\right)$$

so that $\{I_k\}_k$ forms a countable collection open intervals whose union includes the set E, i.e. $\bigcup_k I_k \supseteq \bigcup_k \{e_k\} = E$. Furthermore,

$$\sum_k |I_k| = \sum_k \frac{\varepsilon}{2^k} = \varepsilon \sum_k \frac{1}{2^k} \leq \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = \varepsilon.$$

□

Remark 1.6 (Cantor Set has Measure Zero). Although it may be tempting at first, one cannot think of measure zero sets as being countable. Indeed, countable sets are necessarily measure zero sets but the converse need not hold. As a "simple" example, one can show that the Cantor set is a measure zero set, despite being uncountable.

Proposition 1.29. *The countable union of measure zero sets is again a measure zero set.*

Proof. Let $\{E_j\}_j$ be a countable family of (Lebesgue) measure zero sets in $\mathbb{R}$. That is, for each $\varepsilon > 0$ and every E_j, there exists a countable family of open intervals $\left\{I_k^{(j)}\right\}_k$ such that

$$E_j \subseteq \bigcup_k I_k^{(j)} \quad \text{and} \quad \sum_k \left|I_k^{(j)}\right| \leq \frac{\varepsilon}{2^j}. \tag{1.9}$$

Next, we consider the countable family of open intervals formed by collecting all of the aforementioned intervals: $\left\{I_k^{(j)}\right\}_{j,k}$. Clearly,

$$\bigcup_{j,k} I_k^{(j)} \supseteq \bigcup_j E_j$$

and, moreover,

$$\begin{aligned} \sum_{j,k} \left|I_k^{(j)}\right| = \sum_j \sum_k \left|I_k^{(j)}\right| \leq \sum_j \left(\frac{\varepsilon}{2^j}\right) &= \varepsilon \sum_j \frac{1}{2^j} \\ &\leq \varepsilon \sum_{j=1}^{\infty} \frac{1}{2^j} \\ &= \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was taken to be arbitrary, this concludes the proof. □

2 *Differentiability*

As your third assignment involves some questions with derivatives, it is likely beneficial for us to give a brief overview of the definition(s). In what follows, unless otherwise stated, we consider functions defined on non-trivial intervals $I \subset \mathbb{R}$.

Definition 2.1 (Differentiability)**.** Let I be an interval and $c \in I$. We say that a function $f : I \to \mathbb{R}$ is *differentiable* at the point c provided the the following limit exists and converges:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

In this case, we define

$$f'(c) := \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

and call it the *derivative of f at c*. If f is differentiable at all $c \in I$, we say that f is *differentiable* on I.

Much like with our treatment of spaces, it is nice to understand how different definitions relate to each other (e.g. normed spaces are metric spaces and metric spaces are topological spaces). In this case, it is an elementary fact that all differentiable functions are continuous. However, there are continuous functions that are nowhere differentiable (see the Weierstrass function).

Proposition 2.1. *Let $I \subseteq \mathbb{R}$ be an interval and $c \in I$. If $f : I \to \mathbb{R}$ is differentiable at c, then it is continuous at c.*

Proof. For all $x \neq c$ we may write

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c).$$

Thus, by the limit laws,

$$\begin{aligned} \lim_{x \to c} (f(x) - f(c)) &= \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \cdot (x - c) \right] \\ &= \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \right] \cdot \lim_{x \to c} (x - c) \\ &= f'(c) \cdot 0 \\ &= 0. \end{aligned}$$

Put otherwise, we have shown that $\lim_{x \to c} f(x) = f(c)$ which establishes continuity at c. □

Remark 2.1. Let $f : I \to \mathbb{R}$ be differentiable at a point $c \in I$. Then, by the sequential criterion, we have

$$f'(c) = \lim_{n\to\infty} \frac{f(c_n) - f(c)}{c_n - c},$$

for *any* sequence (c_n) in the domain I with $c_n \to c$ as $n \to \infty$. Thus, to evaluate the value of $f'(c)$, we need only evaluate the limit along a convenient chosen sequence of points. However, this approach is only valid when f is assumed to be differentiable a priori. Indeed, consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1 & \text{if } x \in \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \\ 0 & \text{otherwise.} \end{cases}$$

Then, f is fails the sequential criterion for continuity at $x = 0$ because although $\frac{1}{n} \to 0$ as $n \to \infty$, we have by inspection that

$$\lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} 1 = 1 \neq f(0) = 0.$$

Therefore, f cannot be differentiable at 0. Despite this, however, we can still come up with a sequence (c_n) converging to 0 such that

$$\lim_{n\to\infty} \frac{f(c_n) - f(c)}{c_n - c}$$

converges. Indeed, let us put $c_n := \frac{\sqrt{2}}{n}$ so that $c_n \notin \mathbb{Q}$ for all $n \in \mathbb{N}$. Clearly, $c_n \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{f(c_n) - f(c)}{c_n - c} = \lim_{n\to\infty} \frac{f\left(\frac{\sqrt{2}}{n}\right) - f(0)}{\frac{\sqrt{2}}{n} - 0} = \lim_{n\to\infty} 0 = 0.$$

Theorem 2.2. *Let $I \subseteq \mathbb{R}$ be an interval and $f, g : I \to \mathbb{R}$ be differentiable at a point $c \in I$. Then,*

1. *$f + g$ is differentiable at c and*

$$(f + g)'(c) = f'(c) + g'(c).$$

2. *fg is differentiable at c and*

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

3. *αf is differentiable at c, for every $\alpha \in \mathbb{R}$ and*

$$(\alpha f)'(c) = \alpha f'(c).$$

Proof.

1. Using that, by assumption, both f and g are differentiable at c, we have both limits below

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}, \tag{2.1}$$

$$g'(c) = \lim_{x \to c} \frac{g(x) - f(c)}{x - c} \tag{2.2}$$

are well defined and exist. Then, we observe that for all $x \in I$ with $x \neq c$ there holds

$$\begin{aligned} \frac{(f+g)(x) - (f+g)(c)}{x - c} &= \frac{f(x) + g(x) - (f(c) - g(c))}{x - c} \\ &= \frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c}. \end{aligned}$$

By the limit laws from Math 242, and the fact that the limits in (2.1) exist, we infer that

$$\lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x - c}$$

exists and is equal to

$$\begin{aligned} \lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x - c} &= \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c}. \right] \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= f'(c) + g'(c). \end{aligned}$$

This proves (1).

2. To establish the product rule, we employ the same approach but using a clever trick. Notice that, for all $x \neq c$ with $x \in I$ there holds

$$\begin{aligned} \frac{(fg)(x) - (fg)(c)}{x - c} &= \frac{f(x)g(x) - f(c)g(c)}{x - c} \\ &= \frac{f(x)g(x) \overbrace{- f(x)g(c) + f(x)g(c)}^{=0} - f(c)g(c)}{x - c} \\ &= \frac{f(x)g(x) - f(x)g(c)}{x - c} + \frac{f(x)g(c) - f(c)g(c)}{x - c} \\ &= f(x)\frac{g(x) - g(c)}{x - c} + g(c)\frac{f(x) - f(c)}{x - c}. \end{aligned}$$

Now, because f and g are differentiable at the point c, we know that the limits below

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

$$g'(c) = \lim_{x \to c} \frac{g(x) - f(c)}{x - c}$$

exist. Furthermore, since f is differentiable at c, it must be continuous at c. Namely, one has that $\lim_{x\to c} f(x) = f(c)$. Therefore, by applying the limit laws to the expression above, we see that the following limit exists and is equal to:

$$\begin{aligned}
\lim_{x\to c} \frac{(fg)(x) - (fg)(c)}{x - c} &= \lim_{x\to c} \left[f(x)\frac{g(x) - g(c)}{x - c} + g(c)\frac{f(x) - f(c)}{x - c} \right] \\
&= \lim_{x\to c} \left[f(x)\frac{g(x) - g(c)}{x - c} \right] + \lim_{x\to c} \left[+g(c)\frac{f(x) - f(c)}{x - c} \right] \\
&= \left[\lim_{x\to c} f(x) \right] \lim_{x\to c} \left[\frac{g(x) - g(c)}{x - c} \right] + g(c) \lim_{x\to c} \left[\frac{f(x) - f(c)}{x - c} \right] \\
&= f(c)g'(c) + g(c)f'(c).
\end{aligned}$$

3. This follows from the product rule with $g = \alpha$.

□

2.1 Examples and Elementary Results

For the sake of clarity and completeness, we now illustrate some explicit examples of common functions that are verifiably differentiable via the definition.

Example 2.1. Consider the function $f(x) := \sqrt{x}$ defined on $[0, \infty)$. We show that this function is differentiable on $(0, \infty)$ with derivative

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Fixing $c \in (0, \infty)$ and letting $x \neq c$ be positive we calculate

$$\begin{aligned}
\frac{f(x) - f(c)}{x - c} = \frac{\sqrt{x} - \sqrt{c}}{x - c} &= \frac{\sqrt{x} - \sqrt{c}}{x - c} \cdot \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} \\
&= \frac{x - c}{(x - c)(\sqrt{x} + \sqrt{c})} \\
&= \frac{1}{\sqrt{x} + \sqrt{c}}.
\end{aligned}$$

So, since $\sqrt{x} \to \sqrt{c}$ as $x \to c$, the limit laws yield

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \frac{1}{2\sqrt{c}}, \quad \forall c \in (0, \infty).$$

Therefore f is differentiable on $(0, \infty)$ with $f'(x) = \frac{1}{2\sqrt{x}}$.

Example 2.2. Consider the function $f(x) := \frac{1}{\sqrt{x}}$ on $(0, \infty)$. Then, f is differentiable on its entire domain with derivative given by

$$f'(x) = -\frac{1}{2x^{3/2}}.$$

Indeed, let $c > 0$ be fixed and let $x > 0$ with $x \neq c$. A straightforward calculation yields:

$$\begin{aligned}\frac{f(x) - f(c)}{x - c} &= \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{c}}}{x - c} \\ &= \frac{\sqrt{c} - \sqrt{x}}{(x - c)\sqrt{x}\sqrt{c}} \\ &= \frac{\sqrt{c} - \sqrt{x}}{(x - c)\sqrt{x}\sqrt{c}} \cdot \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} \\ &= \frac{c - x}{(x - c)\sqrt{x}\sqrt{c}(\sqrt{x} + \sqrt{c})} \\ &= -\frac{1}{\sqrt{x}\sqrt{c}(\sqrt{x} + \sqrt{c})}.\end{aligned}$$

Thus, applying the limit laws as above and using that $\sqrt{x} \to \sqrt{c}$ as $x \to c$, we deduce that

$$\begin{aligned}\lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \to c} \left[-\frac{1}{\sqrt{x}\sqrt{c}(\sqrt{x} + \sqrt{c})} \right] \\ &= -\frac{1}{\lim_{x \to c} \sqrt{x}\sqrt{c}(\sqrt{x} + \sqrt{c})} \\ &= -\frac{1}{\sqrt{c}\sqrt{c}\left(2\sqrt{c}\right)} \\ &= -\frac{1}{2c^{3/2}}.\end{aligned}$$

Example 2.3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by the following conditional rule:

$$f(x) := \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We claim that f is differentiable at $x = 0$ and *not* differentiable at all other points. To see that f is not differentiable away from 0, it suffices to show that f is discontinuous at all $c \neq 0$. Indeed, given $c \neq 0$, we may select (by density) a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to c$ as $n \to \infty$. Analogously, since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, there also exists a sequence (ξ_n) from $\mathbb{R} \setminus \mathbb{Q}$ such that $r_n \to c$ as $n \to \infty$. However,

$$\lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} r_n^2 = c^2, \quad \lim_{n \to \infty} f(\xi_n) = \lim_{n \to \infty} 0 = 0.$$

Since $c^2 \neq 0$, it follows that f fails the sequential criterion for continuity away from 0. As differentiability always implies continuity, we infer that f is not differentiable at $c \neq 0$.

Next, we show that $f'(0)$ exists and equals 0. Certainly, for any $x \in \mathbb{R}$, we have that $|f(x)| \leq x^2$. In particular, for every $x \neq 0$, we have

$$\left|\frac{f(x) - f(0)}{x - 0} - 0\right| = \frac{|f(x)|}{|x|} \leq \frac{x^2}{|x|} = |x|\,.$$

Next, let $\varepsilon > 0$ be given and set $\delta := \varepsilon$. If $x \in \mathbb{R}$ is such that $|x| < \delta$, then

$$\left|\frac{f(x) - f(0)}{x - 0} - 0\right| \leq |x| < \delta = \varepsilon.$$

It follows that

$$0 = \lim_{x \to c} \frac{f(x) - f(0)}{x - 0}$$

whence $f'(0) = 0$.

Proposition 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at a point $c \in \mathbb{R}$ with $f(c) = 0$. Then. $|f|$ is differentiable at c if and only if $f'(c) = 0$.*[12]

Proof. Put $g(x) := |f(x)|$ and note that $g(c) = 0$. First, suppose that $f'(c) = 0$. Then, g is differentiable at c with $g'(c) = 0$. Indeed, notice that (by the reverse triangle inequality)

$$\begin{aligned} 0 \leq \left|\frac{g(x) - g(c)}{x - c}\right| = \frac{||f(x)| - |f(c)||}{|x - c|} &\leq \frac{|f(x) - f(c)|}{|x - c|} \\ &= \left|\frac{f(x) - f(c)}{x - c} - 0\right| \xrightarrow{x \to c} 0. \end{aligned}$$

Thus, $g'(c)$ exists and equals 0 by definition. Conversely, let us assume that $g'(c)$ exists; we seek to show that $f'(c) = 0$. Indeed, it is obvious that $g \geq 0$ on $\mathbb{R}$. Thus, since $g(c) = 0$, we see that $c \in \mathbb{R}$ is a local minimum for the function g on $\mathbb{R}$. Since $g'(c)$ exists by assumption, we must have $g'(c) = 0$. This implies that

$$\lim_{x \to c} \frac{g(x)}{x - c} = 0.$$

But then,

$$\left|\frac{f(x) - f(c)}{x - c} - 0\right| = \left|\frac{f(x)}{x - c}\right| = \frac{|f(x)|}{|x - c|} = \frac{g(x)}{|x - c|} = \frac{|g(x)|}{|x - c|} = \left|\frac{g(x)}{x - c}\right| \xrightarrow{x \to c} 0.$$

Hence, f is differentiable at $x = c$ with $f'(0) = 0$. ◻

Lemma 2.4 (Straddle Lemma)**.** *Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$. For each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|f(x) - f(y) - (x - y)f'(c)| \leq \varepsilon(x - y)$$

for all $x, y \in I$ with $c - \delta < y \leq c \leq x < c + \delta$.

[12] Our proof shows that the " $\Longleftarrow$ " implication holds without $f(c) = 0$.

Proof. Let $\varepsilon > 0$ be given. Since $f'(c)$ exists (i.e. f is differentiable at c), the $\varepsilon - \delta$-definition of the limit ensures the existence of some $\delta > 0$ having the property that

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \varepsilon$$

whenever $x \in I$ satisfies $0 < |x - c| < \delta$. In fact, multiplying through by $|x - c| > 0$ shows that

$$|f(x) - f(c) - f'(c)(x - c)| < \varepsilon\,|x - c|\,.$$

Let now $x, y \in I$ be as in the statement and observe that $|f(x) - f(y) - (x - y)f'(c)|$ equals

$$\begin{aligned}
&|f(x) - f(y) + f(c) - cf'(c) - f(c) + cf'(c) - (x - y)f'(c)| \\
\leq\ & |f(x) - f(c) - (x - c)f'(c)| + |-f(y) + f(c) + (y - c)f'(c)| \\
\leq\ & |f(x) - f(c) - (x - c)f'(c)| + |f(y) - f(c) - (y - c)f'(c)| \\
\leq\ & \varepsilon(x - c) + \varepsilon(c - y) \\
=\ & \varepsilon(x - y).
\end{aligned}$$

□

2.2 The Mean Value Theorem

Lemma 2.5. *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \longrightarrow \mathbb{R}$ be differentiable at an interior point $c \in I$. If c is a relative extremum, then $f'(c) = 0$.*

Proof. Replacing f with $-f$, we may assume without loss of generality that c is an interior relative maximum. This means that $c \in \mathrm{Int}(I)$. Hence, there exists $\varepsilon_1 > 0$ such that $(c-\varepsilon_1, c+\varepsilon_1) \subseteq \mathrm{Int}(I) \subseteq I$. Furthermore, since c is a relative maximum, there exists $\varepsilon_2 > 0$ such that $f(x) \leq f(c)$ for all $x \in I \cap (x-\varepsilon_2, x+\varepsilon_2)$. Define $\delta := \min(\varepsilon_1, \varepsilon_2)$ and observe that $(c-\varepsilon, c+\varepsilon) \subseteq I$ and that $f(x) \leq f(c)$ on all of $(c - \delta, c + \delta)$. Furthermore, since f is differentiable at c,

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c^+} \frac{f(x) - f(c)}{x - c}$$

where both one-sided limits must agree because $f'(c)$ is well defined. Notice, however, that

$$\lim_{x\to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

since $f(x) \leq f(c)$ for all $x \in (c - \varepsilon, c + \varepsilon)$ and $x < c$ for all $x \in (c - \varepsilon, c)$. Similarly, one finds that

$$\lim_{x\to c^-} \frac{f(x) - f(c)}{x - c} \leq 0$$

whence $0 \leq f'(c) \leq 0$, thereby concluding our proof. □

Theorem 2.6 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous and assume that f is differentiable on (a, b). Suppose further that $f(a) = f(b)$. Then, there exists a point $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. By replacing f with $f - f(a)$ if necessary, we may assume without loss of generality that $f(a) = f(b) = 0$. Since f is continuous on the compact interval $[a, b]$, it achieves a global minimum m and a global maximum M on $[a, b]$. If $M = m = 0$, then $f \equiv 0$ on $[a, b]$ and so $f'(c) = 0$ for *all* points $c \in (a, b)$. Otherwise, one of M or m is non-trivial. In particular, f achieves a non-zero global extremum in $[a, b]$; let $c \in [a, b]$ be a point where this extremum is achieved. Since $f(a) = f(b) = 0$, we see that c cannot be the endpoints of the interval $[a, b]$, i.e. out extremum is achieved in the open interval (a, b). Thus, f has a local extremum at some point $c \in (a, b)$ whence $f'(c) = 0$, as was asserted. □

Theorem 2.7 (Cauchy Mean Value Theorem)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on the open interval (a, b). Assume in addition that $g'(x) \neq 0$ for all $x \in (a, b)$. Then, there exists a point $c \in (a, b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. By virtue of Rolle's theorem, we must have $g(b) \neq g(a)$. Now, let us consider the function

$$h(x) := \frac{f(b) - f(a)}{g(b) - g(a)} \left(g(x) - g(a)\right) - \left(f(x) - f(a)\right).$$

Clearly, h is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, by inspection, we have that $h(a) = 0$ and $h(b) = 0$. Thus, there exists a point $c \in (a, b)$ where

$$0 = h'(c) = \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) - f'(c)$$

whence a rearrangement of terms yields the assertion. □

Corollary 2.8 (Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on the open interval (a, b). Then, there exists a point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. This follows by taking $g(x) := x$ in the previous statement. □

2.3 A Word on the Lipschitz Condition

Let us recall an important definition. Let $A \subseteq \mathbb{R}$ be non-empty and let $f : A \to \mathbb{R}$ be a function. We say that f is *Lipschitz* continuous on A if there exists some constant $L > 0$ such that

$|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in A$. It is an easy exercise to check that every Lipschitz function is automatically (uniformly) continuous on its domain. The next theorem links the Lipschitz-property to the boundedness of the derivative f', provided your given function is a priori assumed to be differentiable.

Theorem 2.9. *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Then, f is Lipschitz continuous on I if and only if f' is bounded on I.*

Proof. Assume that f is Lipschitz continuous and fix a point $y \in I$. Since $f'(y)$ exists, we know that the limit

$$f'(y) = \lim_{x \to y} \frac{f(x) - f(y)}{x - y}$$

is defined. Therefore,

$$|f'(y)| = \lim_{x \to y} \left| \frac{f(x) - f(y)}{x - y} \right| \leq \lim_{x \to y} L = L.$$

Since $y \in I$ was arbitrary, the assertion follows.

We now establish the converse, i.e. we assume that f' is bounded on I and deduce that f is Lipschitz continuous there. Since f' is bounded on I, there exists $L > 0$ such that $|f'(x)| \leq L$ for all $x \in I$. Fix now two points $x, y \in I$; we want to show that

$$|f(x) - f(y)| \leq L|x - y|.$$

Note that this inequality is trivial when $x = y$. Therefore, we may assume without loss of generality that $x < y$. Clearly, we then have $[x, y] \subseteq I$. Thus, f is differentiable on (x, y) and continuous on $[x, y]$. By the mean value theorem, there exists a point $c \in (x, y) \subseteq I$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

whence

$$\left| \frac{f(y) - f(x)}{y - x} \right| = |f'(c)| \leq L.$$

This gives $|f(x) - f(y)| \leq L|x - y|$ which completes the proof. □

Remark 2.2. The differentiability assumption imposed on the function f can be marginally weakened if we are careful in how we craft our argument. Indeed, the argument used in our proof boils down to applying the mean value theorem on subintervals $[x, y] \subseteq I$, where one would only require that f' exist inside the open interval (x, y). Therefore, the last result remains valid if we assume merely that f is continuous on I and differentiable in the interior of I.

Corollary 2.10. *For all $x, y \in \mathbb{R}$ one has*

$$|\sin x - \sin y| \leq |x - y| \quad \text{and} \quad |\cos x - \cos y| \leq |x - y|.$$

Proof. Since $\sin x$ and $\cos x$ are differentiable on all of $\mathbb{R}$ with $(\sin x)' = \cos x$ and $(\cos x)' = -\sin(x)$, where $|\sin x| \leq 1$ and $|\cos x| \leq 1$, the two inequalities above readily follow from the previous theorem. □

Corollary 2.11. *Let $[a, b]$ be an interval and $f : [a, b] \to [a, b]$ be continuous on $[a, b]$ and differentiable on (a, b). Assume in addition that $|f'(x)| \leq c < 1$ for all $x \in (a, b)$. Then, f possesses a unique fixed point in $[a, b]$.*

Proof. Citing Theorem 2.9 and Remark 2.2, it is automatic that $|f(x) - f(y)| \leq c\,|x - y|$ for all $x, y \in I$. Because $0 \leq c < 1$, this implies that our mapping f is a contraction of I. Then, appealing to the Banach Fixed Point Theorem (Theorem 1.19) yields the existence of a unique fixed point. □

Remark 2.3. This theorem offers a nice characterization of Lipschitz continuous functions when the function in question is a priori assumed to be differentiable. Nonetheless, one should not make the assumption that every Lipschitz function is differentiable. Indeed, by the inequality

$$||x| - |y|| \leq |x - y|, \quad \forall x, y \in \mathbb{R},$$

the function $f(x) = |x|$ is Lipschitz on $\mathbb{R}$. However, f is not differentiable at 0.

2.4 Applications of the Mean Value Theorem and Exercises

Example 2.4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and assume that $f'(x) \to b$, with $b \in \mathbb{R}$, as $x \to \infty$.[13]

(i) Show that, for each $h > 0$, there holds

$$\lim_{x\to\infty} \frac{f(x + h) - f(x)}{h} = b.$$

(ii) Assume now that $\lim_{x\to\infty} f(x) = a \in \mathbb{R}$. Then, prove that $b = 0$.

Solution.

(i) Fix $h > 0$ and let $\varepsilon > 0$. Since $f'(x) \to b$ as $x \to \infty$, there exists some $N > 0$ so large that

$$|f'(x) - b| < \varepsilon, \quad \forall x \geq N. \tag{2.3}$$

Now, let $x \geq N$ be arbitrary and consider the subinterval $[x, x + h] \subset \mathbb{R}$. By assumption on f, we have that f is continuous on this compact interval and differentiable on the interior

[13]For the sake of clarity, we write $x \to \infty$ to mean $x \to +\infty$. When "approaching" $\pm\infty$, we write $x \to \pm\infty$ or $|x| \to \infty$.

of the interval. Consequently, f satisfies the requirements of the Mean Valu Theorem on $[x, x+h]$, Thus, there exists a point[14] $c_x \in (x, x+h)$ such that

$$f'(c_x) = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}. \tag{2.4}$$

Furthermore, since $c_x \in (x, x+h)$, we have $c_x > x \geq N$. But then, by combining (2.3)-(2.4) we infer that

$$\left| \frac{f(x+h) - f(x)}{h} - b \right| = |f'(c_x) - b| < \varepsilon.$$

Since $x \geq N$ was arbitrary, it follows by definition that

$$\lim_{x \to \infty} \frac{f(x+h) - f(x)}{h} = b.$$

(ii) With the additional assumption that $f(x) \longrightarrow a$ as $x \longrightarrow \infty$, we shall show that b is necessarily 0. For each $n \in \mathbb{N}$, let us consider the subinterval $(n, 2n) \subset \mathbb{R}$. Once again, the given function f satisfies the requirements of the Mean Value Theorem on $[n, 2n]$. Consequently, for each $n \in \mathbb{N}$, we may select a point $c_n \in (n, 2n)$ such that

$$f'(c_n) = \frac{f(2n) - f(n)}{2n - n} = \frac{f(2n) - f(n)}{n}.$$

Now, by the sequential criterion, we have

$$\lim_{n \to \infty} f'(x_n) = b \quad \text{and} \quad \lim_{n \to \infty} f(x_n) = a$$

for *all* sequences (x_n) in $\mathbb{R}$ with $x_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Since $c_n \in (n, 2n)$ for all $n \in \mathbb{N}$, it is obvious that $c_n \longrightarrow \infty$ (squeeze theorem!). In particular,

$$\lim_{n \to \infty} f'(c_n) = b, \quad \lim_{n \to \infty} f(n) = a, \quad \text{and} \lim_{n \to \infty} f(2n) = a.$$

But then, the limit laws tell us that

$$\begin{aligned} b = \lim_{n \to \infty} f'(c_n) &= \lim_{n \to \infty} \frac{f(2n) - f(n)}{n} \\ &= \lim_{n \to \infty} \left[(f(2n) - f(n)) \cdot \frac{1}{n} \right] \\ &= \lim_{n \to \infty} (f(2n) - f(n)) \cdot \lim_{n \to \infty} \frac{1}{n} \\ &= (a - a) \cdot 0 = 0. \end{aligned}$$

[14] We use the notation c_x to emphasize how c_x is selected in a way that depends on x.

□

Proposition 2.12. *Let $a < b$ be real numbers and assume that $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and continuous on $[a, b]$. If*

$$\lim_{x \to a^+} f'(x) = L_1$$

exists, then $f'(a) = L_1$. Similarly, if

$$\lim_{x \to b^-} f'(x) = L_2$$

then $f'(b) = L_2$.

Proof. Assume that $f'(x) \to L_1$ as $x \to a^+$. We want to show that

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = L_1.$$

Let $\varepsilon > 0$ be given; using that $\lim_{x \to a^+} f'(x) = L_1$, we can find $\delta > 0$ such that

$$|f'(x) - L_1| < \varepsilon$$

whenever $0 < x - a < \delta$.[15] For any such x, since f is continuous on $[a, x] \subseteq [a, b]$ and differentiable on (a, x), the mean value theorem guarantees the existence of a point $c_x \in (a, x)$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(c_x).$$

Since $0 < c_x - a < x - a < \delta$, we obtain

$$\left| \frac{f(x) - f(a)}{x - a} - L_1 \right| = |f'(c_x) - L_1| < \varepsilon.$$

We have therefore shown that

$$\left| \frac{f(x) - f(a)}{x - a} - L_1 \right| < \varepsilon$$

whenever $0 < x - a < \delta$. This proves that $f'(a) = L_1$. The second part can be verified by a symmetric argument. □

2.5 Uniform Differentiability

Definition 2.2. Let $I \subseteq \mathbb{R}$ be an interval. A differentiable function $f : I \to \mathbb{R}$ is said to be uniformly differentiable on I if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon$$

for all $x, y \in I$ with $0 < |x - y| < \delta$.[16]

[15] As we are taking the limit as $x \to a$ from "above", we are only considering points $x > a$. Therefore, no absolute values are needed here!

[16] Uniform differentiability has some nice applications in numerical analysis.

The definition of uniform differentiability can be compared to that of uniform continuity. Indeed, in the above, both x and y are allowed to "vary" when taking the limit. Namely, we are asking that the $\delta > 0$ obtained above be independent of the point y at which we are taking the derivative.

Proposition 2.13. *Let $f : I \to \mathbb{R}$ be uniformly differentiable on I. Then, f' is uniformly continuous on I. Especially, f is continuously differentiable on I.*[17]

Proof. We must show that f' is uniformly continuous on I. Let $\varepsilon > 0$ be given. By assumption, there exists $\delta > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| < \frac{\varepsilon}{2}. \tag{2.5}$$

for all $x, y \in I$ with $0 < |x - y| < \delta$. For all such x, y we have, by the triangle inequality,

$$\begin{aligned} |f'(x) - f'(y)| &= \left|f'(x) - \frac{f(x) - f(y)}{x - y} + \frac{f(x) - f(y)}{x - y} - f'(y)\right| \\ &\leq \left|\frac{f(x) - f(y)}{x - y} - f'(x)\right| + \left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| \\ &= \left|\frac{f(y) - f(x)}{y - x} - f'(x)\right| + \left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon; \end{aligned}$$

where we have used (2.5) in this last step. We note that this also trivially holds when $x = y$. Summarizing, we have shown that $|f'(x) - f'(y)| < \varepsilon$ whenever $x, y \in I$ are such that $|x - y| < \delta$. That is, we have that f' is uniformly continuous on the interval I. □

Remark 2.4. There exist uniformly continuous functions, differentiable at all points in the interior, whose derivative is unbounded. Indeed, let us consider the function $f(x) := \sqrt{x}$ on $[0, 1]$. Clearly, f is continuous on the compact set $[0, 1]$ and thus *uniformly* continuous there as well. However, by a previous example, we know that f is differentiable at all $x > 0$ with

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad \forall x > 0.$$

Hence, f' is unbounded on the interval $(0, 1)$. Since uniformly continuous functions on bounded intervals *are bounded*[18], we see that f' cannot be uniformly continuous on $(0, 1)$.

[17] A partial converse to this result is proven in your third assignment. Indeed, you are asked to prove that a continuously differentiable function $f : [a, b] \to \mathbb{R}$ is uniformly differentiable on $[a, b]$. The key difference is that we are assuming the interval in question is compact. If we remove the compactness assumption on $[a, b]$ (i.e. replace $[a, b]$ with half-open, open, or unbounded intervals) then we must also ask that f' be uniformly continuous on the aforementioned interval.

[18] This is because uniformly continuous functions map Cauchy sequences to Cauchy sequences! Use this in a proof by contradiction (with the Bolzano–Weierstrass theorem) to show that a uniformly continuous function on a bounded set is bounded. Alternatively, consult your Analysis 1 notes!

2.6 A Basic Form of l'Hôpital's Rule

The intuition behind l'Hôpital's Rule is straightforward and comes directly from the definition of the derivative. As a result, we begin with the following basic result.

Proposition 2.14 (Preliminary Form). *Let $I \subseteq \mathbb{R}$ be an interval and $f, g : I \to \mathbb{R}$ be differentiable at $c \in I$ with $f(c) = g(c) = 0$. Assume also that $g(x) \neq 0$ for all $x \in I \setminus \{c\}$ and that $g'(c) \neq 0$. Then,*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$

Proof. The proof is elementary: using that $f(c) = g(c) = 0$ we may write, for all $x \neq c$,

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}}. \tag{2.6}$$

Now, because

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \quad \text{and} \quad \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c) \neq 0,$$

the limit-laws applied to the expression in (2.6) yield

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} = \frac{\lim_{x \to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x \to c} \frac{g(x)-g(c)}{x-c}} = \frac{f'(c)}{g'(c)}.$$

This concludes the proof. □

However, this elementary form is often lackluster in terms of practical applications. For instance, it is hard to "successively" apply the result above to multiple derivatives. To counteract this, we refine the statement (and proof!) to obtain a more general result which uses the limit instead of requiring that f, g be differentiable at the point c (with $g'(x) \neq 0$). In fact, we developed the Cauchy Mean Value Theorem *for* this proof.

Theorem 2.15 (One-Sided l'Hôpital's Rule: Case 0/0 at a point). *Let $a < b$ be real numbers and f, g be differentiable functions on (a, b) and assume that $g(x), g'(x) \neq 0$ for all $x \in (a, b)$. Suppose in addition that*

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0. \tag{2.7}$$

If

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

Proof. Let $\varepsilon > 0$ be given. By assumption, we can find $\delta > 0$ such that $x \in (a, b)$ with $0 < x - a < \delta$ implies

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon.$$

Or, equivalently,

$$L - \varepsilon < \frac{f'(x)}{g'(x)} < L + \varepsilon,$$

for all $x \in (a, b)$ with $0 < x - a < \delta$. We may choose $\delta > 0$ so small that $a + \delta < b$. Now, let $a < \alpha < \beta < a + \delta$ and notice that the assumptions of the Cauchy Mean Value Theorem are satisfied on $[\alpha, \beta]$. Thus, there exists a point $c \in (\alpha, \beta)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)}.$$

Since $c \in (\alpha, \beta) \subseteq (a, a + \delta)$, we have that $0 < c - a < \delta$. Consequently,

$$L - \varepsilon < \frac{f'(c)}{g'(c)} < L + \varepsilon$$

whence

$$L - \varepsilon < \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} < L + \varepsilon$$

for all $a < \alpha < \beta < a + \delta$. Passing to the limit in the above as $\alpha \to a^+$ and using (2.8), it follows that

$$L - \varepsilon \leq \lim_{\alpha \to a^+} \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f(\beta)}{g(\beta)} \leq L + \varepsilon.$$

That is, we have

$$\left|\frac{f(\beta)}{g(\beta)} - L\right| \leq \varepsilon$$

for all $a < \beta < a + \delta$. This proves, by definition, that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

□

Remark 2.5. Using a symmetric argument, an analogous result holds for the "right endpoint". Indeed, by following the same proof-strategy as above, one can show that if

$$\lim_{x \to b^-} f(x) = \lim_{x \to b^-} g(x) = 0$$

and

$$\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to b^-}\frac{f(x)}{g(x)}=L.$$

In summary, l'Hôpital's rule also holds at the right end point.

Finally, by combining both the left and right hand versions of this rule, we obtain the following two-sided result:

Corollary 2.16 (l'Hôpital Rule: Case 0/0 at a Point)**.** *Let $a<c<b$ be real numbers and f,g be differentiable functions on $(a,b)\setminus\{c\}$. Assume that $g(x),g'(x)\neq 0$ for all $x\in(a,b)\setminus\{c\}$. Suppose in addition that*

$$\lim_{x\to c}f(x)=\lim_{x\to c}g(x)=0. \tag{2.8}$$

If

$$\lim_{x\to c}\frac{f'(x)}{g'(x)}=L,$$

then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=L.$$

We remark that there are many forms of l'Hôpital's rule, as well as several indeterminate forms that one can reduce to cases the l'Hôpital rules can handle. The multiple forms of these rules along with their detailed proofs can be found in §6.3 of Bartle-Sherbert's *Introducion to Real Analysis.* Rather than state and prove these, we supply the reader with an interesting example.

Example 2.5. Let $f:(0,\infty)\to\mathbb{R}$ be differentiable and assume that

$$\lim_{x\to\infty}(f(x)+f'(x))=L.$$

Prove that

$$\lim_{x\to\infty}f(x)=L\quad\text{and}\quad\lim_{x\to\infty}f'(x)=0.$$

Proof. Notice that we may write

$$f(x)=\frac{e^xf(x)}{e^x}=\frac{g(x)}{h(x)},\quad\forall x>0,$$

where we obviously set $g(x):=e^xf(x)$ and $h(x)=e^x$. Then, we find ourselves in the case of l'Hôpital's Rule II from Bartle-Sherbert (one can verify by inspection that the requirements of l'Hôpital's are satisfied). Consequently, we may evaluate

$$\begin{aligned}\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{g(x)}{h(x)}\overset{\text{l'Hôpital}}{=}\lim_{x\to\infty}\frac{g'(x)}{h'(x)}&=\lim_{x\to\infty}\frac{e^xf(x)+e^xf'(x)}{e^x}\\&=\lim_{x\to\infty}(f(x)+f'(x))\\&=L.\end{aligned}$$

Finally, observe that for each $x > 0$ one has

$$f'(x) = f'(x) - f(x) + f(x)$$

whence

$$\begin{aligned}\lim_{x\to\infty} f'(x) &= \lim_{x\to\infty} (f'(x) - f(x) + f(x))\\ &= \lim_{x\to\infty} (f'(x) - f(x)) + \lim_{x\to\infty} f(x)\\ &= \lim_{x\to\infty} f(x) - \lim_{x\to\infty} (f(x) - f'(x))\\ &= L - L = 0.\end{aligned}$$

□

2.7 The Intermediate Properties of Derivatives: Darboux's Theorem

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. We say that f has the *intermediate value property* if, for any $x, y \in I$ with $x \neq y$, any value between $f(x)$ and $f(y)$ is achieved at least once by f at some point between x and y.

Darboux's theorem asserts that derivatives satisfy the intermediate value property. More precisely, let us recall the following class result:

Theorem 2.17 (Darboux)**.** *Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be differentiable. If y is any point between $f'(a)$ and $f'(b)$, there exists a point x between a and b such that $f'(x) = y$.*

Darboux's theorem can sometimes make it easy to show that certain functions cannot be derivatives of any differentiable function. Put otherwise, Darboux's theorem sometimes implies that a given a function has no antiderivative. We provide such an example below.

Example 2.6. Consider the function

$$h(x) := \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

We claim that there does not exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f' = h$ on $\mathbb{R}$. Arguing by contradiction, suppose such a function f does indeed exist. Take $a = -1$ and $b = 1$. Then, $f'(a) = h(a) = 0$ and $f'(b) = h(b) = 1$. By Darboux's theorem, there must exist a point $x \in (-1, 1)$ such that

$$f'(x) = h(x) = \frac{1}{2}.$$

Clearly, no such point exists and we therefore have a contradiction. However, although h is not the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ that is everywhere differentiable, there exist uncountably many functions $f : \mathbb{R} \to \mathbb{R}$, differentiable away from 0, such that $f'(x) = h(x)$ for all $x \neq 0$. Indeed, for each $x_0 \in \mathbb{R}$,

$$f(x) := \begin{cases} x_0 & \text{if } x < 0, \\ x & \text{if } x \geq 0 \end{cases}$$

is differentiable at all points $c \neq 0$ and $f'(c) = h(c)$ at all such c.

Let us also provide a less trivial example utilizing the result of Darboux:

Example 2.7. Let $f : [0,2] \to \mathbb{R}$ be continuous on $[0,2]$ and differentiable on $(0,2)$. Suppose that the given function f satisfies the following:

- $f(0) = 0$;
- $f(1) = 1$;
- $f(2) = 1$.

Prove that there exists a point $c \in (0,2)$ such that $f'(c) = 1/3$.

Proof. First, consider the subinterval $[0,1]$ of $[0,2]$. Clearly, f is continuous on this compact interval and differentiable in the interior, i.e. on $(0,1) \subset (0,2)$. Thus, the requirements of the Mean Value Theorem are satisfied for f on $[0,1]$ whence there exists a point $c_1 \in (0,1)$ such that

$$f'(c_1) = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0) = 1 - 0 = 1. \tag{2.9}$$

In a similar vein, one can see by inspection that the conditions of the mean value theorem are *also* satisfied for f on $[1,2] \subset [0,2]$; therefore, there exists a point $c_2 \in (1,2)$ with

$$f'(c_2) = \frac{f(2) - f(1)}{2 - 1} = f(2) - f(1) = 1 - 1 = 0. \tag{2.10}$$

To summarize, we have found points $c_1, c_2 \in (0,2)$ such that $f'(c_1) = 1$ and $f'(c_2) = 0$. Appealing to Darboux's theorem, we see that f' has the intermediate value property. Furthermore, (2.9)-(2.10) yield

$$f'(c_2) < \frac{1}{3} < f'(c_1).$$

It then follows that there exists a point $c \in (c_1, c_2)$ such that $f'(c) = \frac{1}{3}$. □

3 The Riemann Integral

Let us now address the topic of integration, which begins with the following question:

> For which functions can one systematically assign a notion of "area" to the region bounded by its graph?

Although exhaustive methods have a rich history (dating back to the work of Archimedes), it wasn't until the times of Newton and Leibniz that integration was formally introduced. Furthermore, it wouldn't be until to the contributions of Riemann and Darboux for there to be a rigorous mathematical treatment of the theory.[19] Despite all this effort, the Riemann integral would nonetheless turn out to be "lacking" in certain areas. Namely, it only allows one to integrate over intervals (and by extension cubes in higher dimensions) and even so, places significant restriction on the function. To help rectify this, Lebesgue would later developed the Lebesgue integral.

Consider a closed and bounded interval $[a,b] \subseteq \mathbb{R}$ and let $f : [a,b] \to \mathbb{R}$ be an arbitrary function. A *partition* of the interval $[a,b]$ is a finite ordered set of points

$$\mathcal{P} = \{x_0, \dots, x_n\}$$

in $[a,b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$. Especially, $[a,b] = \bigcup_{j=1}^n [x_{j-1}, x_j]$. Informally, $\mathcal{P}$ describes a unique way of breaking the interval $[a,b]$ into non-overlapping (except at the endpoints) compact intervals $[x_{j-1}, x_j] \subseteq I$. A *tagged partition* is a partition $\mathcal{P}$ together with a set of points, called *tags*,

$$\{x_1^*, \dots, x_n^*\}$$

such that $x_j^* \in [x_{j-1}, x_j]$ for each index $j = 1, \dots, n$. To emphasize the fact that a partition $\mathcal{P}$ is equipped with a set of tags, we will write $\dot{\mathcal{P}}$.

Given a (possibly un-tagged) partition $\mathcal{P}$ of an interval $[a,b]$, the *mesh* of $\mathcal{P}$ is defined to be the length of the largest sub-interval defined by $\mathcal{P}$. More precisely, we define

$$\|\mathcal{P}\| := \max_{1 \le j \le n} \left(x_j - x_{j-1}\right) > 0.$$

Definition 3.1. Let $[a,b] \subset \mathbb{R}$ be a compact interval and $f : [a,b] \to \mathbb{R}$ a function. Given a tagged partition $\dot{\mathcal{P}}$ of $[a,b]$ as above, we define the Riemann sum of f over $\dot{\mathcal{P}}$ to be the sum

$$S(f; \dot{\mathcal{P}}) := \sum_{j=0}^{n} f(x_j^*)(x_j - x_{j-1}). \tag{3.1}$$

[19] It turns out that both treatments of the integral are equivalent, i.e. a bounded function is Riemann integrable if and only if it is integrable according to Darboux's theory. In our notes, we will mainly use Riemann's integral, whilst offering in the final subsection a brief overview of how the Darboux integral may be constructed.

Often times, we will write $\Delta x_j = (x_j - x_{j-1})$ so that $S(f;\dot{\mathcal{P}}) := \sum_{j=0}^{n} f(x_j^*)\Delta x_j$. The function f is said to be *Riemann integrable* on $[a,b]$ if there exists $\Lambda \in \mathbb{R}$ such that, for each $\varepsilon > 0$, there is $\delta > 0$ with the property that

$$\left|S(f;\dot{\mathcal{P}}) - \Lambda\right| < \varepsilon$$

for all tagged partitions $\dot{\mathcal{P}}$ of $[a,b]$ with $\|\dot{\mathcal{P}}\| < \delta$. In this case, we call Λ the Riemann integral of f on $[a,b]$ and denote this quantity by

$$\int_a^b f, \int_{[a,b]} f, \quad \text{or} \int_a^b f(x)\,\mathrm{d}x.$$

Let us take a moment to unpack the definition of Riemann integrability. The Riemann sum $S(f,\dot{\mathcal{P}})$ should be thought of as a rectangular approximation of the area under the "curve" of the function f. Indeed, the step function

$$\varphi(x) := \sum_{j=1}^{n} f(x_j^*)\mathbf{1}_{[x_{j-1},x_j)}(x)$$

is precisely an approximation of f by a function that takes the constant values $f(x_j^*)$ on each subinterval $[x_{j-1},x_j)$ of $[a,b]$. Here, $\mathbf{1}_A(x)$ denotes the *indicator function* (or characteristic function) of the set A.[20] Clearly, the classical area under the graph of φ is equal to the Riemann sum

$$S(f,\dot{\mathcal{P}}) = \sum_{j=0}^{n} f(x_j^*)(x_j - x_{j-1}).$$

Note that our choice of tags directly influences the approximation of f we obtain. Luckily, the definition of Riemann integrability guarantees that these Riemann sums $S(f,\dot{\mathcal{P}})$ converge to a meaningful real number, independently of our choice of tags[21], so long as we ensure that $\|\dot{\mathcal{P}}\|$ is sufficiently small, i.e. provided we refine our approximation of f sufficiently.

We now recall some basic properties of the Riemann integral that will/have been seen in the lectures.

Theorem 3.1. *Let $[a,b] \subset \mathbb{R}$ be a compact interval and denote by $\mathcal{R}([a,b])$ the set of all Riemann integrable functions on $[a,b]$. The following properties hold.*

[20]Given a set X and a subset $A \subseteq X$, we define $\mathbf{1}_A(x)$ to be the function given by the rule

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Some authors will instead write $\chi_A(x)$ to denote this function.

[21]The mesh function $\|\cdot\|$ does not take into account the tag points, and only thinks about the lengths of the subintervals of $[a,b]$ that $\mathcal{P}$ creates.

(i) If $f, g \in \mathcal{R}([a,b])$ then $f+g \in \mathcal{R}([a,b])$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

(ii) Given $f \in \mathcal{R}([a,b])$ and $\alpha \in \mathbb{R}$ we have $\alpha f \in \mathcal{R}([a,b])$ with

$$\int_a^b (\alpha f) = \alpha \int_a^b f.$$

Let us now give a detailed example in which we verify the Riemann integrability of an explicit function.

Example 3.1. Fix a number $c \in (0,1)$ and define $f : [0,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 0 & \text{if } 0 \le x < c, \\ 1 & \text{if } c \le x \le 1. \end{cases}$$

We claim that f is Riemann integrable on $[0,1]$ with integral equal to $(1-c)$. Let $\varepsilon > 0$ and take

$$\delta = \varepsilon.$$

If $\dot{\mathcal{P}}$ is a tagged partition of $[0,1]$ with $\|\dot{\mathcal{P}}\| < \delta$ then every subinterval of $[0,1]$ created by $\dot{\mathcal{P}}$ has length no larger than δ. Let us now enumerate the elements of $\dot{\mathcal{P}}$ as

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

with tags $x_j^* \in [x_{j-1}, x_j]$ for every $j = 1, \ldots, n$. Let $k \ge 1$ be the unique integer such that $x_{k-1} < c \le x_k$. By definition of the function f, we have $f(x_j^*) = 0$ for all $j < k$.[22] There are now two cases that we will distinguish:

(1) Assume that $x_{k-1} \le x_k^* < c \le x_k$. If $k = n$ then $f(x_j^*) = 0$ for all $j = 1, \ldots, n$ so that

$$\left|S(f; \dot{\mathcal{P}}) - (1-c)\right| = |1-c| = |x_k - c| \le \left\|\dot{\mathcal{P}}\right\| < \delta.$$

In this last step we have used that both c and x_k belong to the same subinterval of $\dot{\mathcal{P}}$. If

[22]Indeed, if $1 \le j < k$ then $x_j^* \le x_{k-1} < c$ so that $f(x_j^*) = 0$.

instead $k < n$ then

$$\begin{aligned}
\left|S(f;\dot{\mathcal{P}}) - (1-c)\right| &= \left|\sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) - (1-c)\right| \\
&= \left|\sum_{j=k+1}^{n} f(x_j^*)(x_j - x_{j-1}) - (1-c)\right| \\
&= \left|\sum_{j=k+1}^{n} (x_j - x_{j-1}) - (1-c)\right| \\
&= |(x_n - x_k - (1-c)| \\
&= |(1 - x_k) - (1-c)| \\
&= |x_k - c| \\
&< \delta,
\end{aligned}$$

where we have once again used the fact that x_k and c belong to the same subinterval of $\dot{\mathcal{P}}$

(2) Otherwise, we have $x_{k-1} < c \leq x_k^* \leq x_k$. Then, $f(x_k^*) = 1$ so that, as above,

$$\begin{aligned}
\left|S(f;\dot{\mathcal{P}}) - (1-c)\right| &= \left|\sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) - (1-c)\right| \\
&= \left|\sum_{j=k}^{n} f(x_j^*)(x_j - x_{j-1}) - (1-c)\right| \\
&= \left|\sum_{j=k}^{n}(x_j - x_{j-1}) - (1-c)\right| \\
&= |(x_n - x_{k-1} - (1-c)| \\
&= |x_{k-1} - c| \\
&< \delta.
\end{aligned}$$

In either case we have that

$$\left|S(f;\dot{\mathcal{P}}) - (1-c)\right| < \delta = \varepsilon.$$

Since $\dot{\mathcal{P}}$ was an arbitrary tagged partition of $[0,1]$ with $\|\dot{\mathcal{P}}\| < \delta$, we see that f is Riemann integrable on $[0,1]$ and that

$$\int_0^1 f = (1-c).$$

In addition to Example 3.1, let us also show that the "classical" function $f(x) = 1 - x$ is Riemann integrable on $[0, 1]$.

Example 3.2. Let $f : [0, 1] \to \mathbb{R}$ be given by $f(x) = 1 - x$. We claim that $f \in \mathcal{R}([0, 1])$ and that

$$\int_0^1 f = \frac{1}{2}.$$

Let $\varepsilon > 0$ be given and define $\delta := \varepsilon$. Let $\dot{\mathcal{P}}$ be a tagged partition of $[0, 1]$ with $\|\dot{\mathcal{P}}\| < \delta$. Denote the partition points of $\dot{\mathcal{P}}$ by

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

and let $\{x_1^*, \ldots, x_n^*\}$ be the tags of $\dot{\mathcal{P}}$. Let $\dot{\mathcal{Q}}$ be the tagged partition of $[0, 1]$ formed by taking the same partition points as $\dot{\mathcal{P}}$ with tags

$$y_j^* := \frac{x_j + x_{j-1}}{2}, \quad j = 1, \ldots, n.$$

An easy calculation then shows that

$$\begin{aligned} S(f; \dot{\mathcal{Q}}) = \sum_{j=1}^{n} f(y_j^*)(x_j - x_{j-1}) &= \sum_{j=1}^{n} \left[1 - \frac{x_j + x_{j-1}}{2}\right](x_j - x_{j-1}) \\ &= \sum_{j=1}^{n}(x_j - x_{j-1}) - \frac{1}{2}\sum_{j=1}^{n}\left(x_j^2 - x_{j-1}^2\right) \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

On the other hand, using the inequality $|a_1 + \cdots + a_n| \le |a_1| + \cdots + |a_n|$,

$$\begin{aligned} \left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| &= \left|\sum_{j=1}^{n}\left(f(x_j^*) - f(y_j^*)\right)(x_j - x_{j-1})\right| && (3.2) \\ &\le \sum_{j=1}^{n}\left|f(x_j^*) - f(y_j^*)\right|(x_j - x_{j-1}) && (3.3) \\ &= \sum_{j=1}^{n}\left|x_j^* - \frac{x_j + x_{j-1}}{2}\right|(x_j - x_{j-1}). && (3.4) \end{aligned}$$

Since $x_j^* \in \left[x_{j-1}, x_j\right]$ and $\frac{x_j + x_{j-1}}{2}$ is the midpoint of this same interval, we find that

$$\left|x_j^* - \frac{x_j + x_{j-1}}{2}\right| \le (x_j - x_{j-1}) \le \left\|\dot{\mathcal{P}}\right\| < \delta,$$

for each index $j = 1, \ldots, n$. Returning to (3.4) gives

$$\left|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})\right| < \delta \sum_{j=1}^{n} (x_j - x_{j-1}) = \delta = \varepsilon.$$

Finally, recalling that $S(f, \dot{\mathcal{Q}}) = \frac{1}{2}$, we see that

$$\left|S(f;\dot{\mathcal{P}}) - \frac{1}{2}\right| = \left|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})\right| < \varepsilon$$

for all tagged partitions $\dot{\mathcal{P}}$ of $[0,1]$ with $\|\dot{\mathcal{P}}\| < \delta$.

We finalize this section by providing a direct example of a function that will turn out to *not* be Riemann integrable. We will provide a direct proof of this fact later, when we treat the Cauchy Criterion for Riemann Integrability.

Example 3.3. Consider the Dirichlet function $\mathfrak{d} : \mathbb{R} \to \mathbb{R}$ given by the rule

$$\mathfrak{d}(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

In other words, $\mathfrak{d}(x) = \mathbf{1}_{\mathbb{Q}}(x)$. Then, as previously stated, we will later prove that $\mathfrak{d}$ is *not* integrable on any compact interval $[a,b] \subset \mathbb{R}$ despite being bounded.

3.1 Approximating by Riemann Sums

Given the striking similarities between the $\varepsilon - \delta$ definition of the limit and our definition of Riemann integrability (see Definition 3.1), it is reasonable to hope that one can approximate $\int_a^b f$ by simply taking a limit along a sequence of Riemann sums with the mesh of the partitions tending to zero. Informally, by sufficiently refining our tagged partitions, we should be able to approximate the area under the graph of f. This is confirmed by the following:

Lemma 3.2. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. If $(\dot{\mathcal{P}}_n)$ is a sequence of tagged partitions of $[a,b]$ such that*

$$\lim \|\dot{\mathcal{P}}_n\| = 0,$$

then

$$\lim S(f;\dot{\mathcal{P}}_n) = \int_a^b f.$$

Proof. Let $\varepsilon > 0$ be given. Since f is Riemann integrable on $[a,b]$, there exists $\delta > 0$ such that

$$\left|S(f;\dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon$$

whenever $\dot{\mathcal{P}}$ is a tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \delta$. Now, because $\|\dot{\mathcal{P}}_n\| \to 0$ as $n \to \infty$, we can find $N \in \mathbb{N}$ such that

$$\|\dot{\mathcal{P}}_n\| < \delta$$

for all $n \geq N$. It follows that

$$\left| S(f; \dot{\mathcal{P}}_n) - \int_a^b f \right| < \varepsilon$$

for all $n \geq N$. This shows that $S(f; \dot{\mathcal{P}}_n) \to \int_a^b f$ as $n \to \infty$. □

Remark 3.1. This lemma can sometimes make it relatively easy to extend properties of Riemann sums to the Riemann integral. Informally speaking, if a certain property holds for Riemann sums and this property is preserved by limits, one might use the lemma above to extend this property to the integral. See the next example for an application of such an argument.

Example 3.4. Let $f, g \in \mathcal{R}([a, b])$ be such that $f \leq g$ on $[a, b]$. Let $(\mathcal{P}_n)$ be a sequence of tagged partitions of $[a, b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ as $n \to \infty$. Since we have $f \leq g$ on all of $[a, b]$, one clearly has $S(f; \dot{\mathcal{P}}_n) \leq S\left(g, \dot{\mathcal{P}}_n\right)$ for each $n \in \mathbb{N}$. By virtue of Lemma 3.2 applied to both f and g, we have that $S(f; \dot{\mathcal{P}}_n) \to \int_a^b f$ and $S\left(g, \dot{\mathcal{P}}_n\right) \to \int_a^b g$ as $n \to \infty$. Since non-strict inequalities are preserved by limits, we infer that

$$\int_a^b f \leq \int_a^b g.$$

Note that this argument will fail if we do not know *a priori* that $f, g \in \mathcal{R}([a, b])$.

Example 3.5. Let $f \in \mathcal{R}([a, b])$ be such that $|f|$ is also Riemann integrable on $[a, b]$.[23] Then,

$$\left| \int_a^b f \right| \leq \int_a^b |f| \, .$$

Indeed, this follows from the previous example by observing that $-|f| \leq f \leq |f|$ on all of $[a, b]$. Thus,

$$-\int_a^b |f| = \int_a^b (-|f|) \leq \int_a^b f \leq \int_a^b |f|$$

which gives that $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Example 3.6. Consider the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) := \begin{cases} \frac{1}{x} & \text{if } x \in \mathbb{Q} \setminus \{0\}, \\ 0 & \text{if } x \notin \mathbb{Q} \text{ or } x = 0. \end{cases}$$

[23] The integrability assumption on $|f|$ turns out to be superfluous. Indeed, $|f|$ is Riemann integrable whenever f is. However, we do not yet possess the tools required to show that $f \in \mathcal{R}(a, b])$ implies that $|f| \in \mathcal{R}([a, b])$. We therefore have no qualms about making this assumption in our example.

Let (x_n) be the sequence in $[0,1]$ defined by $x_n := \frac{1}{n}$. Clearly, $f(x_n) = n$ for each $n \in \mathbb{N}$ whence f is *unbounded* on the interval $[0,1]$. Consequently, f cannot be Riemann integrable on $[0,1]$. Nonetheless, one can find a sequence $(\dot{\mathcal{P}}_n)$ of tagged partitions of $[0,1]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ and $\lim S(f;\dot{\mathcal{P}}_n)$ exists. Indeed, given $n \in \mathbb{N}$, divide $[0,1]$ into n-subintervals of equal length and choose from each subinterval an irrational tag (this can be done because $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$). The resulting tagged partition $\dot{\mathcal{P}}_n$ will be such that $\|\dot{\mathcal{P}}_n\| \leq \frac{1}{n}$ and $S(f;\dot{\mathcal{P}}_n) = 0$.

We also have the following sequential condition for non-integrability:

Corollary 3.3. *Let $f : [a,b] \to \mathbb{R}$ be a function. Let $(\dot{\mathcal{P}}_n)$ and $(\dot{\mathcal{Q}}_n)$ two sequences of tagged partitions of $[a,b]$ such that*

$$\lim \left\|\dot{\mathcal{P}}_n\right\| = \lim \left\|\dot{\mathcal{Q}}_n\right\| = 0.$$

*If $\lim S(f;\dot{\mathcal{P}}_n) \neq \lim S(f;\dot{\mathcal{Q}}_n)$, then f is **not** Riemann integrable on $[a,b]$. Moreover, if one of these limits does not exist, then $f \notin \mathcal{R}([a,b])$.*

Proof. By way of contradiction, let us assume that $f \in \mathcal{R}([a,b])$. Citing Lemma 3.2, we must have

$$\lim S(f;\dot{\mathcal{P}}_n) = \int_a^b f = \lim S(f;\dot{\mathcal{Q}}_n)$$

which is a contradiction. □

Consider once more the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ given by (3.5). We have already seen that f is discontinuous at every point $c \in \mathbb{R}$. Using Corollary 3.3, we will show that f is not Riemann integrable on any interval $[a,b] \subseteq \mathbb{R}$. Indeed, let $(\mathcal{P}_n)$ be any sequence of partitions of $[a,b]$ with

$$\|\mathcal{P}_n\| \leq \frac{b-a}{n}$$

for each n.[24] From each subinterval of $\mathcal{P}_n$, we choose (by density) a rational tag and denote the resulting tagged partition by $\dot{\mathcal{P}}_n$. Similarly, from every subinterval of $\mathcal{P}_n$ we choose an irrational tag (again by density) and let $\dot{\mathcal{Q}}_n$ be the corresponding tagged partition of $[a,b]$. Since $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ have the same partition points as $\mathcal{P}_n$, it is easy to see that

$$\lim \left\|\dot{\mathcal{P}}_n\right\| = \lim \left\|\dot{\mathcal{Q}}_n\right\| = 0.$$

On the other hand, because every tag of $\dot{\mathcal{P}}_n$ is rational, we have $S(f;\dot{\mathcal{P}}_n) = (b-a)$ for each $n \in \mathbb{N}$. But, as every tag of $\dot{\mathcal{Q}}_n$ is irrational, we instead have $S(f;\dot{\mathcal{Q}}_n) = 0$ for each $n \in \mathbb{N}$. Citing Corollary 3.3, we infer that $f \notin \mathcal{R}([a,b])$.

[24] Such a sequence $(\mathcal{P}_n)$ can be obtained by dividing $[a,b]$ into n-subintervals of equal length.

3.2 *Cauchy's Criterion for Riemann Integrability*

Let us recall the Cauchy Criterion for Riemann integrability, which characterizes the Riemann integrability of a function without relying upon the "value" of its integral. Much like the case of the real line, it is often far easier to practically deduce integrability through the Cauchy criterion rather than the definition; this is especially true when there is no obvious "guess" for the integral.

Theorem 3.4 (Cauchy's Criterion). *Let $f : [a,b] \to \mathbb{R}$ be a function. Then, f is Riemann integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})\right| < \varepsilon$$

for all tagged partitions $\dot{\mathcal{P}}$, $\dot{\mathcal{Q}}$ of $[a,b]$ such that $\|\dot{\mathcal{P}}\| < \delta$ and $\|\dot{\mathcal{Q}}\| < \delta$.

Negating the condition given above, we see that a function $f : [a,b] \to \mathbb{R}$ is **not** Riemann integrable on $[a,b]$ if and only if there exists $\varepsilon_0 > 0$ such that, for every $\delta > 0$, one can find tagged partitions $\dot{\mathcal{P}}_\delta$ and $\dot{\mathcal{Q}}_\delta$ of $[a,b]$, each having mesh strictly less than δ, such that

$$\left|S(f;\dot{\mathcal{P}}_\delta) - S(f;\dot{\mathcal{Q}}_\delta)\right| \geq \varepsilon_0.$$

In this case, taking $\delta := \frac{1}{n}$ gives two tagged partitions $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ such that

$$\left\|\dot{\mathcal{P}}_n\right\| < \frac{1}{n} \quad \text{and} \quad \left\|\dot{\mathcal{Q}}_n\right\| < \frac{1}{n}$$

with $\left|S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{Q}}_n)\right| \geq \varepsilon_0$. Conversely, assume that we are given two sequences of tagged partitions $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ such that

$$\lim \left\|\dot{\mathcal{P}}_n\right\| = \lim \left\|\dot{\mathcal{Q}}_n\right\| = 0$$

but $\left|S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{Q}}_n)\right| \geq \varepsilon_0 > 0$ for each $n \in \mathbb{N}$. Can we conclude from this that $f \notin \mathcal{R}([a,b])$? Indeed, for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$\left\|\dot{\mathcal{P}}_n\right\| < \delta \quad \text{and} \quad \left\|\dot{\mathcal{Q}}_n\right\| < \delta$$

for all $n \geq N$. In particular, for $n = N$ we have

$$\left\|\dot{\mathcal{P}}_N\right\| < \delta \quad \text{and} \quad \left\|\dot{\mathcal{Q}}_N\right\| < \delta.$$

On the other hand,

$$\left|S(f;\dot{\mathcal{P}}_N) - S(f;\dot{\mathcal{Q}}_N)\right| \geq \varepsilon_0.$$

Taking $\dot{\mathcal{P}}_\delta := \dot{\mathcal{P}}_N$ and $\dot{\mathcal{Q}}_\delta := \dot{\mathcal{Q}}_N$, we see that the Cauchy criterion fails and f is *not* Riemann integrable on $[a,b]$. To summarize, we have proven the following:

Corollary 3.5. *Let $f : [a,b] \to \mathbb{R}$ be a function. Then, f is not Riemann integrable on $[a,b]$ if and only if there exists $\varepsilon_0 > 0$ and two sequences $(\dot{\mathcal{P}}_n)$, $(\dot{\mathcal{Q}}_n)$ of tagged partitions of $[a,b]$ such that* *(i)* $\|\dot{\mathcal{P}}_n\| \to 0$ *and* $\|\dot{\mathcal{Q}}_n\| \to 0$ *as* $n \to \infty$*;*

(ii) $\left|S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{Q}}_n)\right| \geq \varepsilon_0$ *for all* $n \in \mathbb{N}$*.*

Example 3.7. This criterion makes it possible to show that a large class of functions are not Riemann integrable. Let $g : \mathbb{Q} \to \mathbb{R}$ be a function such that $g(x) \geq \varepsilon_0 > 0$ for all $x \in \mathbb{Q}$ and some $\varepsilon_0 > 0$. Define $f : [a,b] \to \mathbb{R}$ by

$$f(x) := \begin{cases} g(x) & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Note that by taking $g = 1$ we recover the Dirichlet function in (3.5). We now claim that f is *not* Riemann integrable on $[a,b]$. To prove this, we will make use of the criterion proven in Corollary 3.5. For each $n \in \mathbb{N}$ we can create an *untagged* partition $\mathcal{P}_n$ of $[a,b]$ by dividing $[a,b]$ into n-subintervals of equal length. Clearly, this gives us a sequence of partitions of $[a,b]$ such that

$$\|\mathcal{P}_n\| = \frac{b-a}{n} \to 0$$

as $n \to \infty$. For fixed $n \in \mathbb{N}$ we can choose by density a rational tag from each subinterval of $\mathcal{P}_n$; doing so gives us a tagged partition $\dot{\mathcal{P}}_n$ of $[a,b]$ having only rational tags and the same partition points as $\mathcal{P}_n$. Similarly, we build from $\mathcal{P}_n$ a tagged partition $\dot{\mathcal{Q}}_n$ of $[a,b]$ having only irrational tags and the same partition points as $\mathcal{P}_n$. Since $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ have the same partition points as $\mathcal{P}_n$, it is obvious that

$$\lim \left\|\dot{\mathcal{P}}_n\right\| = \lim \left\|\dot{\mathcal{Q}}_n\right\| = 0.$$

On the other hand, because f vanishes at every irrational number, $S(f;\dot{\mathcal{Q}}_n) = 0$ for all $n \in \mathbb{N}$. Consequently, letting $x_0, \ldots, x_k$ and $\left\{x_j^*\right\}_{j=1}^{k}$ be the partition points and tags (respectively) of $\dot{\mathcal{P}}_n$ for fixed n, we infer that

$$\begin{aligned} \left|S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{Q}}_n)\right| = \left|S(f;\dot{\mathcal{P}}_n)\right| &= \left|\sum_{j=1}^{k} f(x_j^*)(x_j - x_{j-1})\right| \\ &= \left|\sum_{j=1}^{k} g(x_j^*)(x_j - x_{j-1})\right| \\ &= \sum_{j=1}^{k} g(x_j^*)(x_j - x_{j-1}) \\ &\geq \varepsilon_0 \sum_{j=1}^{k} (x_j - x_{j-1}) \\ &= \varepsilon_0(b-a). \end{aligned}$$

Since $n \in \mathbb{N}$ was arbitrary, we see from Corollary 3.5 that f cannot be Riemann integrable on $[a,b]$.

3.3 The Squeeze Theorem

In addition to the Cauchy Criterion in Theorem 3.4, we possess the following criterion for Riemann integrability:

Theorem 3.6 (Squeeze Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a function. Then, f is Riemann integrable on $[a,b]$ if and only if, for each $\varepsilon > 0$, there exists two Riemann integrable functions $\alpha_\varepsilon, \omega_\varepsilon : [a,b] \to \mathbb{R}$ such that*

$$\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x), \quad \forall x \in [a,b],$$

and

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon.$$

Remark 3.2. In practice, one will often use step-functions to "build" these $\alpha_\varepsilon, \omega_\varepsilon$. This has the added benefit that the integrals $\int_a^b \alpha_\varepsilon$ and $\int_a^b \omega_\varepsilon$ are easy to evaluate, for each $\varepsilon > 0$. Then, since the Squeeze Theorem produces functions $\alpha_\varepsilon, \omega_\varepsilon$ with

$$\alpha_\varepsilon \leq f \leq \omega_\varepsilon \quad \text{on } [a,b], \quad \text{and} \quad \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon.$$

we have that

$$\int_a^b \alpha_\varepsilon \leq \int_a^b f \leq \int_a^b \omega_\varepsilon$$

by monotonicity (Example 3.4). Furthermore,

$$0 \leq \max\left\{ \int_a^b f - \int_a^b \alpha_\varepsilon, \int_a^b \omega_\varepsilon - \int_a^b f \right\} < \varepsilon.$$

It therefore follows that

$$\int_a^b f = \lim_{\varepsilon \searrow 0} \int_a^b \omega_\varepsilon = \lim_{\varepsilon \searrow 0} \int_a^b \alpha_\varepsilon.$$

To see how Theorem 3.6 can be used in practice, let us consider the following example.

Example 3.8. Consider the function $f : [0,1] \to \mathbb{R}$ given by $x \mapsto x$. Using the Squeeze Theorem, we will prove that f is Riemann integrable on $[0,1]$ and that $\int_0^1 x = \frac{1}{2}$. Given a $n \in \mathbb{N}$, let $\mathcal{P}_n$ be the partition of $[0,1]$ obtained by dividing $[0,1]$ into n-subintervals of equal length. Clearly,

$$\lim \|\mathcal{P}_n\| = \lim \frac{1}{n} = 0.$$

Fix *any* $n \in \mathbb{N}$. The partition points of $\mathcal{P}_n$ are given by

$$x_j = \frac{j}{n}, \quad j = 0, \dots, n$$

Define two functions α_n, ω_n on $[0,1]$ by

$$\alpha_n(x) := \begin{cases} \frac{j-1}{n} & \text{if } \frac{j-1}{n} \le x < \frac{j}{n}, \\ 1 & \text{if } x = 1, \end{cases} \quad \text{and} \quad \omega_n(x) := \begin{cases} \frac{j}{n} & \text{if } \frac{j-1}{n} \le x < \frac{j}{n}, \\ 1 & \text{if } x = 1. \end{cases}$$

and note that $\alpha_n \le f \le \omega_n$ on $[0,1]$. Since step functions are always integrable, α_n and ω_n are both Riemann integrable on $[0,1]$. Now, a direct calculation shows that

$$(\omega_n - \alpha_n)(x) = \begin{cases} \frac{1}{n} & \text{if } \frac{j-1}{n} \le x < \frac{j}{n}, \\ 0 & \text{if } x = 1 \end{cases}$$

whence

$$\int_0^1 (\omega_n - \alpha_n) = \sum_{j=1}^n \frac{1}{n}\left(\frac{j}{n} - \frac{j-1}{n}\right) = \frac{1}{n^2}\sum_{j=1}^n 1 = \frac{1}{n}.$$

In particular, for any n such that $\frac{1}{n} < \varepsilon$, one has

$$\alpha_n \le f \le \omega_n \quad \text{and} \quad \int_0^1 (\omega_n - \alpha_n) < \varepsilon.$$

It follows from the Squeeze theorem that $f \in \mathcal{R}([0,1])$ and that $\int_0^1 f$ exists. Now, note that

$$\int_0^1 \alpha_n = \sum_{j=1}^n \frac{j-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}\sum_{j=1}^n (j-1) = \frac{1}{n^2}\left[\frac{n(n+1)}{2} - n\right]$$
$$= \frac{1}{2} - \frac{1}{n}.$$

Similarly,

$$\int_0^1 \omega_n = \sum_{j=1}^n \frac{j}{n} \cdot \frac{1}{n} = \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{1}{2} + \frac{1}{2n}.$$

By monotonicity of the integral,

$$\frac{1}{2} - \frac{1}{n} = \int_0^1 \alpha_n \le \int_0^1 f \le \int_0^1 \omega_n = \frac{1}{2} + \frac{1}{2n}.$$

Since $n \in \mathbb{N}$ was arbitrary, we can take the limit as $n \to \infty$ and deduce from the Squeeze Theorem (for sequences) that $\int_0^1 f = \frac{1}{2}$.

3.4 Additivity

You will have seen in the lectures (alternatively, see Bartle-Sherbert §7.2) that the Riemann integral is additive in the following sense:

Theorem 3.7 (Additivity of the Integral)**.** *Let $f : [a, b] \to \mathbb{R}$ be a function. Then, f is Riemann integrable on $[a, b]$ if and only if, for each $a < c < b$, f is Riemann integrable on $[a, c]$ and on $[c, b]$. In this case, one has*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

The additivity theorem makes it much easier to construct suitable functions $\alpha_\varepsilon, \omega_\varepsilon$ for use in the application of the Squeeze Theorem; we illustrate this below by way of an explicit example.

Example 3.9. We prove (using the Squeeze Theroem) that

$$f(x) := \begin{cases} \left|\sin\left(\frac{1}{x}\right)\right| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is Riemann integrable on $[0, 1]$. Fix an arbitrary $\varepsilon > 0$ and let $N \geq 2$ be such that $\frac{1}{N} < \frac{\varepsilon}{2}$.[25] Since f is continuous on $\left[\frac{1}{N}, 1\right]$, it must be integrable there. Now, it is clear that $0 \leq f(x) \leq 1$ on $\left[0, \frac{1}{N}\right]$. Furthermore,

$$\int_0^{\frac{1}{N}} (1 - 0) = \frac{1}{N} < \varepsilon$$

Consider the functions

$$\alpha_\varepsilon(x) := \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{N}, \\ f(x) & \text{if } \frac{1}{N} \leq x \leq 1 \end{cases} \quad \text{and} \quad \omega_\varepsilon(x) := \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{N}, \\ f(x) & \text{if } \frac{1}{N} \leq x \leq 1. \end{cases}$$

Clearly, $\alpha_\varepsilon \leq f \leq \omega_\varepsilon$ on all of $[0, 1]$. Since f is Riemann integrable on $\left[\frac{1}{N}, 1\right]$ and $\alpha_\varepsilon = f$ there, α_ε is also Riemann integrable on $\left[\frac{1}{N}, 1\right]$. On the other hand, α_ε is equal to a constant function on $\left[0, \frac{1}{N}\right]$ except at the one point $x = \frac{1}{N}$. Thus, α_ε is integrable on the interval $\left[0, \frac{1}{N}\right]$. By the Additivity Theorem, we infer that $\alpha_\varepsilon \in \mathcal{R}([0, 1])$. Similarly, $\omega_\varepsilon \in \mathcal{R}([0, 1])$. Finally, another application of the Additivity Theorem gives

$$\begin{aligned} \int_0^1 (\omega_\varepsilon - \alpha_\varepsilon) &= \int_0^{1/N} (\omega_\varepsilon - \alpha_\varepsilon) + \int_{1/N}^1 (\omega_\varepsilon - \alpha_\varepsilon) \\ &= \int_0^{1/N} (1 - 0) + \int_{1/N}^1 0 \\ &= \frac{1}{N} \\ &< \varepsilon. \end{aligned}$$

By the Squeeze Theorem, we infer that $f \in \mathcal{R}([0, 1])$.

[25] I only want $N \geq 2$ to guarantee that $\frac{1}{N} < 1$. This way, both $[0, 1/N]$ and $[1/N, 1]$ are non-trivial intervals.

Remark 3.3. Let $f : [a, b] \to \mathbb{R}$ be a function and assume that $f \in \mathcal{R}([c, b])$ for all $c \in (a, b)$. Does it follow that $f \in \mathcal{R}([a, b])$? Unfortunately, one cannot conclude that f is Riemann integrable on $[a, b]$. Consider the function

$$f(x) := \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Since f is continuous on $[c, 1]$ for every $0 < c < 1$, we see that f is Riemann integrable on $[c, 1]$ for each such c. On the other hand, because f is unbounded on $[0, 1]$, it cannot be Riemann integrable there.

Despite this, one can recover the phenomenon from Example 3.9 if we assume that f is bounded *and* belongs to $\mathcal{R}([c, b])$ for all $c \in (a, b)$. Indeed, this is confirmed by the following integral analogue to Proposition 2.12

Proposition 3.8. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function such that, for each $c \in (a, b)$, there holds $f \in \mathcal{R}([c, b])$. Then, $f \in \mathcal{R}([a, b])$.*

Proof. We adapt the argument used in the solution of Example 3.9. Let $\varepsilon > 0$ be given; since f is bounded, there exists some $M > 0$ such that $-M \leq f(x) \leq M$ for all $x \in [a, b]$. Next, let $\delta > 0$ be so small that

$$\delta < \frac{\varepsilon}{2M} \quad \text{and} \quad a + \delta < b.$$

We construct functions α_ε and ω_ε on $[a, b]$ as follows:

$$\alpha_\varepsilon(x) := \begin{cases} -M & \text{if } a \leq x < a + \delta, \\ f(x) & \text{if } a + \delta \leq x \leq b, \end{cases} \qquad \omega_\varepsilon(x) := \begin{cases} M & \text{if } a \leq x < a + \delta, \\ f(x) & \text{if } a + \delta \leq x \leq b, \end{cases}$$

By assumption, $\alpha_\varepsilon, \omega_\varepsilon$ is integrable on $[a + \delta, b]$ because f is assumed to be integrable there. Furthermore, $\alpha_\varepsilon, \omega_\varepsilon$ are equal to a constant function on all of $[a, a+\delta)$. Hence, $\alpha_\varepsilon, \omega_\varepsilon$ are integrable on $[a, a+\delta]$. By the additivity theorem, we see that $\alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}([a, b])$. Furthermore, it is obvious that

$$\alpha_\varepsilon \leq f \leq \omega_\varepsilon, \quad \text{on } [a, b].$$

Finally, notice that, by the additivity theorem,

$$\begin{aligned} \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) &= \int_a^{a+\delta} (\omega_\varepsilon - \alpha_\varepsilon) + \int_{a+\delta}^b (\omega_\varepsilon - \alpha_\varepsilon) \\ &= \int_a^{a+\delta} (M - (-M)) + \int_{a+\delta}^b 0 \\ &= 2M\delta \\ &< \varepsilon. \end{aligned}$$

Thus, it follows from the Squeeze Theorem that $f \in \mathcal{R}([a, b])$. □

3.5 Examples with Riemann Sums: Even and Odd Functions

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and let $(\dot{\mathcal{P}}_n)$ be a sequence of tagged partitions of $[a, b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$, as $n \to \infty$. Lemma 3.2 states that the Riemann sums $S(f; \dot{\mathcal{P}}_n)$ converge to $\int_a^b f$, i.e.

$$\lim S(f; \dot{\mathcal{P}}_n) = \int_a^b f.$$

As demonstrated in Example 3.5, this can sometimes help us extend properties of Riemann *sums* to the Riemann *integral*. This type of argument is very common in analysis and is extremely useful. We provide another application of this argument below:

Proposition 3.9. *Let $a > 0$ and $f : [-a, a] \to \mathbb{R}$ be Riemann Integrable. In particular, f is Riemann integrable on every closed subinterval of $[-a, a]$. If f is even, i.e. if $f(x) = f(-x)$ for all $x \in [-a, a]$, then*

$$\int_{-a}^{a} f = 2\int_0^a f.$$

Proof. For each $n \in \mathbb{N}$ we construct a partition $\mathcal{P}_n$ of $[0, a]$ by dividing this interval in to n-subintervals of equal length. Clearly, $\|\mathcal{P}_n\| = \frac{a}{n}$ for each $n \in \mathbb{N}$. Let $x_1^*, \ldots, x_k^*$ be any set of tags for $\mathcal{P}_n$ and let $\dot{\mathcal{P}}_n$ denote the resulting tagged partition of $[a, b]$. Let

$$0 = x_0 < \cdots < x_n = a$$

be the partition points of $\dot{\mathcal{P}}_n$. Let $\mathcal{Q}_n$ be the tagged partition of $[-a, a]$ with partition points

$$-a = -x_n < -x_{n-1} < \cdots < -x_1 < x_0 < \cdots < x_n = a.$$

Finally, we give $\mathcal{Q}_n$ the tags

$$\begin{cases} x_j^* \in \left[x_{j-1}, x_j\right] & 1 \leq j \leq n, \\ -x_j^* \in \left[-x_j, -x_{j-1}\right] & 1 \leq j \leq n. \end{cases}$$

Then, $\dot{\mathcal{Q}}_n$ is tagged partition of $[-a, a]$ with $\|\dot{\mathcal{Q}}_n\| = \|\dot{\mathcal{P}}_n\| = \frac{a}{n}$. Since f is an even function,

$$\begin{aligned} S(f; \dot{\mathcal{Q}}_n) &= \sum_{j=1}^{n} f(-x_j^*)\left(-x_{j-1} - (-x_j)\right) + \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) + \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) \\ &= 2S(f; \dot{\mathcal{P}}_n). \end{aligned}$$

As $n \in \mathbb{N}$ was arbitrary, this gives us a sequence of tagged partitions $\dot{\mathcal{P}}_n$ of $[0, a]$, and a related sequence $\dot{\mathcal{Q}}_n$ of tagged partitions of $[-a, a]$, such that

$$\lim \|\dot{\mathcal{Q}}_n\| = \lim \|\dot{\mathcal{P}}_n\| = \lim \frac{a}{n} = 0.$$

Consequently, two applications of Lemma 3.2 implies

$$\int_{-a}^{a} f = \lim S(f; \dot{\mathcal{Q}}_n) = 2 \lim S(f; \dot{\mathcal{P}}_n) = 2 \int_{0}^{a} f.$$

□

A similar argument will allow us to obtain the following:

Proposition 3.10. *Let $a > 0$ and $f : [-a, a] \to \mathbb{R}$ be Riemann integrable. In particular, f is Riemann integrable on every closed subinterval of $[-a, a]$. If f is odd, i.e. if $-f(x) = f(-x)$ for all $x \in [-a, a]$, then*

$$\int_{-a}^{a} f = 0.$$

Proof. For each $n \in \mathbb{N}$ we defined tagged partitions $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ as in the proof of the Proposition 3.9. However, in this case, we obtain

$$\begin{aligned} S(f; \dot{\mathcal{Q}}_n) &= \sum_{j=1}^{n} f(-x_j^*) \left(-x_{j-1} - (-x_j)\right) + \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) \\ &= -\sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) + \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) \\ &= 0. \end{aligned}$$

By the same argument as before, Lemma 3.2 yields

$$\int_{-a}^{a} f = \lim S(f; \dot{\mathcal{Q}}_n) = 0.$$

□

3.6 Integrating Continuous Functions

We know that all monotone increasing functions $[a, b] \to \mathbb{R}$ are Riemann integrable. Perhaps even more importantly, it has been proven that continuous functions are always Riemann integrable. The Riemann integral enjoys much nicer properties when restricted to continuous functions. Namely, much more can be said about $\int_a^b f$ when f is assumed to be continuous on $[a, b]$.

Proposition 3.11. *Let $f : [a,b] \to \mathbb{R}$ be continuous and assume that $f \geq 0$ on $[a,b]$. Then $\int_a^b f = 0$ if and only if $f = 0$ on $[a,b]$.*

Proof. Clearly, if $f = 0$ on $[a,b]$ then $\int_a^b f = 0$. Conversely, we show that if $f(c) > 0$ at some point $c \in [a,b]$ then $\int_a^b f > 0$. Indeed, by continuity, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \frac{f(c)}{2}$$

for all $x \in [a,b]$ with $|x - c| < \delta$. Therefore, for all such x, the above implies that

$$\frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}.$$

Let now $I = [c,d] \subseteq [a,b]$ be the interval (try drawing a picture) given by

$$[a,b] \cap \left[c - \frac{\delta}{2}, c + \frac{\delta}{2}\right].$$

If $x \in [c,d]$ then $|x - c| < \delta$ so that $f(x) > \frac{f(c)}{2}$ by the above. By the additivity theorem, we know that

$$\int_a^b f = \int_a^c f + \int_c^d f + \int_b^d f \geq \int_c^d f.$$

Here we have used that $f(x) \geq 0$ on $[a,b]$ and that the integral is monotone (by your assignment problem). It follows that

$$\int_a^b f \geq \int_c^d f \geq \int_c^d \frac{f(c)}{2} = \frac{f(c)}{2}(d-c) > 0.$$

This completes the proof. □

Remark 3.4. It is easy to see that one cannot drop the continuity assumption on f. Indeed, the function $f : [0,1] \to \mathbb{R}$ given by $f(x) = 0$ for all $0 \leq x < 1$ and $f(1) = 1$. Then, as f differs from the constant map $x \mapsto 0$ at only a single point, we have that $f \in \mathcal{R}([0,1])$ with $\int_a^b f = 0$. However, $f \not\equiv 0$.

Since compositions of continuous functions are continuous, $|f|$ is Riemann integrable on $[a,b]$ whenever f is a continuous function on $[a,b]$. Recalling Example 3.5, we obtain the following:

Proposition 3.12. *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then $|f|$ is Riemann integrable on $[a,b]$ and*

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Let us now discuss the mean value theorem for integrals. As in the case of differentiable functions, there is a generalized "Cauchy" Mean Value Theorem to explore.

Theorem 3.13. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous and assume that $g(x) > 0$ on all of $[a, b]$. Then, $\int_a^b g > 0$ and there exists a point $c \in [a, b]$ such that*

$$\int_a^b fg = f(c) \int_a^b g. \tag{3.6}$$

Proof. First, we argue that $\int_a^b g > 0$. Indeed, since g is continuous on the interval $[a, b]$, it achieves an absolute minimum μ on $[a, b]$. In fact, since $g(x) > 0$ on $[a, b]$, we must have $\mu > 0$. Consequently, by monotonicity,

$$\int_a^b g \geq \int_a^b \mu = \mu(b - a) > 0.$$

We now establish the equality in (3.6). Using that f is continuous on $[a, b]$, the extreme value theorem implies that f achieves an absolute minimum m and an absolute maximum M on $[a, b]$. Then, since

$$m = \frac{\int_a^b (mg)}{\int_a^b g} \leq \frac{\int_a^b fg}{\int_a^b g} \leq \frac{\int_a^b (Mg)}{\int_a^b g} = M.$$

Then, the Intermediate Value Theorem ensures the existence of a point $c \in [a, b]$ such that

$$f(c) = \frac{\int_a^b fg}{\int_a^b g}$$

which completes the proof. □

Remark 3.5. Observe that one cannot drop the requirement that $g(x)$ be positive for all $x \in [a, b]$. Indeed, consider the functions $f(x) = g(x) = x$ on $[-1, 1]$. Clearly, each function is continuous (even Lipschitz) on $[-1, 1]$. Furthermore, we will later be able to prove (using the fundamental theorem of calculus – see §3.8) that

$$\int_{-1}^1 x^2 = \frac{x^3}{3}\Big|_{x=-1}^{x=1} = \frac{2}{3}.$$

Similarly[26],

$$\int_{-1}^1 x = 0.$$

Clearly, there cannot exist a point $c \in [-1, 1]$ such that $\frac{2}{3} = f(c) \cdot 0$.

[26] This also follows from Proposition 3.10.

Theorem 3.14 (Mean Value Theorem for Integrals)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, there exists a point $c \in [a, b]$ such that*

$$\frac{1}{b-a} \int_a^b f = f(c).$$

Proof Using Theorem 3.13. Apply Theorem 3.13 with $g \equiv 1$. □

Direct Proof. Since $[a, b]$ is compact and f is continuous, f achieves an absolute minimum m and an absolute maximum M on $[a, b]$. By the monotonicity of the integral, we must have

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Consequently,

$$m \leq \frac{1}{b-a} \int_a^b f \leq M.$$

By the intermediate value theorem, f must achieve every value in $[m, M]$. In particular, there exists a point $c \in [a, b]$ such that $f(c) = \frac{1}{b-a} \int_a^b f$. This completes the proof. □

Corollary 3.15. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous. If $\int_a^b f = \int_a^b g$, there exists a point $c \in [a, b]$ such that $f(c) = g(c)$.*

Proof. Applying the Mean Value Theorem for Integrals to $f - g$ implies the existence of a point $c \in [a, b]$ such that

$$(f-g)(c) = f(c) - g(c) = \frac{1}{b-a} \int_a^b (f-g) = \frac{\int_a^b f - \int_a^b g}{b-a} = 0.$$

That is, $f(c) = g(c)$. □

In Example 3.8, we proved that the function $f(x) = x$ is Riemann integrable on $[0, 1]$ using the Squeeze Theorem. There, we approximated f above and below by conveniently chosen step functions. More precisely, we divided $[0, 1]$ into n-subintervals $\{I_j\}_{j=1}^n$ of equal length and defined two step functions α_n and ω_n by taking α_n to be the minimum of f over I_j and ω_n to be the maximum. In this sense, we approximating f from above and below by step functions taking on the extrema of f (on these I_j). This is precisely the idea behind the *Darboux integral*, which we briefly discuss.

3.7 Darboux Integration

Let $I = [a, b]$ be a compact interval and let $\mathcal{P}$ be a partition of $[a, b]$. Let

$$a = x_0 < \cdots < x_n = b$$

be the partition points of $\mathcal{P}$. Given a bounded function $f : [a, b] \to \mathbb{R}$, we can define the *upper* and *lower* Darboux sums of f on $\mathcal{P}$, respectively, by

$$L(f;\mathcal{P}) := \sum_{j=1}^{n} m_j(x_j - x_{j-1}), \quad m_j := \inf_{x \in [x_{j-1}, x_j]} f(x), \tag{3.7}$$

$$U(f;\mathcal{P}) := \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \quad M_j := \sup_{x \in [x_{j-1}, x_j]} f(x). \tag{3.8}$$

Since $m_j \leq M_j$ for all $j = 1, \ldots, n$, we see that $L(f;\mathcal{P}) \leq U(f;\mathcal{P})$. Furthermore, let $M > 0$ be such that $-M \leq f(x) \leq M$ on $[a, b]$. Then, $L(f;\mathcal{P})$ is bounded from above independently of $\mathcal{P}$. Indeed,

$$L(f;\mathcal{P}) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}) \leq \sum_{j=1}^{n} M(x_j - x_{j-1}) = M(b - a).$$

Similarly,

$$U(f;\mathcal{P}) \geq \sum_{j=1}^{n} -M(x_j - x_{j-1}) = -M(b - a).$$

Hence, $U(f;\mathcal{P})$ is bounded from below independently of $\mathcal{P}$.

Definition 3.2. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function. Denote by $\mathscr{P}(I)$ the collection of all partitions $\mathcal{P}$ of $[a, b]$. The lower and upper Darboux integrals of f are defined, respectively, by

$$L(f) := \sup_{\mathcal{P} \in \mathscr{P}(I)} L(f;\mathcal{P}) \quad \text{and} \quad U(f) := \inf_{\mathcal{P} \in \mathscr{P}(I)} U(f;\mathcal{P}).$$

Note that these quantities exist by our previous argument. We say that f is Darboux integrable on $[a, b]$ if $U(f) = L(f)$. In this case the Darboux integral of f on $[a, b]$ is defined as:

$$\int_a^b f := U(f) = L(f).$$

One can show (see Bartle §7.4) that a function is Darboux integrable on $[a, b]$ if and only if it is Riemann integrable.

Unlike the Riemann integral, the Darboux integral declares a function f to be integrable if it can be approximated from above and below by step functions. Put otherwise, a bounded function f is considered integrable if and only if if can be "squeezed" between two step functions. This bears a notable resemblance to the statement of the squeeze theorem 3.6.

Although the Riemann and Darboux integrals are equivalent, the Riemann one is "better" in the sense that it more easily generalizes. By making a minor modification to the definition of the Riemann integral (replacing δ with something called a *gauge*), one obtains the so-called **gauge integral**. As with the Riemann integral, the gauge integral is defined on intervals. In fact, the gauge integral can be defined on *unbounded* intervals. On an interval I, it turns out that the gauge integral is *more* general than the Lebesgue integral. However, the Lebesgue integral can be defined in *much* more general settings than the gauge integral. For instance, the Lebesgue integral can be defined on subsets of $\mathbb{R}$ that are not intervals.

3.8 Lebesgue Criterion & The Fundamental Theorem of Calculus

Before presenting both parts of Fundamental Theorem of Calculus, we briefly review the Lebesgue integrability criterion. A proof of this result may be found in the Appendix of Bartle-Sherbert, as this proof is relatively technical. Regardless, it is an important result which provides a necessary and sufficient condition for a bounded function to be Riemann integrable, purely in terms of its continuity points. We reiterate the formal statement below.

Theorem 3.16 (Lebesgue Integrability Criterion)**.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then, f is Riemann integrable on $[a, b]$ if and only if the set*

$$\{x \in [a, b] : f \text{ is discontinuous at } x\}$$

has Lebesgue measure zero.

This result is very useful as it shows that continuous functions map Riemann integrable functions to Riemann integrable functions. In particular, it enables us to show that $|f|$ is Riemann integrable wherever f is.

Corollary 3.17. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and φ a bounded continuous real-valued function whose domain contains the image of f. Then, $\varphi \circ f$ is Riemann integrable on $[a, b]$.*

Proof. Put $g(x) := (\varphi \circ f)(x)$. Observe that $f \in \mathcal{R}([a, b])$ implies that f is bounded; since φ is bounded, so is g. Furthermore, if f is continuous at $x \in [a, b]$, then so is g by composition. Consequently, defining

$$\mathcal{Z}_h := \{x \in [a, b] : h \text{ is discontinuous at } x\}$$

for any function $h : [a, b] \to \mathbb{R}$, it follows from the Lebesgue criterion that $\mathcal{Z}_f$ has Lebesgue measure 0. By our argument above, we find that $\mathcal{Z}_g \subseteq \mathcal{Z}_f$ whence $\mathcal{Z}_g$ has Lebesgue measure zero. Applying the Lebsegue criterion for integrability, we infer that g is also Riemann integrable on $[a, b]$. ∎

As stated previously, the afore Corollary allows us to strengthen the conclusions of Example 3.5. Indeed, we need no longer assume that f is continuous in the context of Proposition 3.12

Proposition 3.18. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then $|f|$ is Riemann integrable on $[a, b]$ and*

$$\left| \int_a^b f \right| \leq \int_a^b |f| \,.$$

3.8.1 The Fundamental Theorem of Calculus

There are two forms of the Fundamental Theorem of Calculus, both of which are proven in §7.3 of Bartle-Sherbert. For the sake of completeness, we reiterate their statements below. For the proofs, we of course refer the reader to our aforementioned reference.

Theorem 3.19 (Fundamental Theorem of Calculus – Form 1)**.** *Let $[a, b] \subset \mathbb{R}$ be a compact interval and $E \subset [a, b]$ a finite set. Let $f, F : [a, b] \to \mathbb{R}$ be functions such that the following hold:*

(a) F is continuous on $[a, b]$;

(b) $F'(x) = f(x)$ for $x \in [a, b] \setminus E$;

(c) f is Riemann integrable on $[a, b]$.

Then,

$$\int_a^b f = F(b) - F(a). \tag{3.9}$$

Typically, students are quite familiar with a basic form of this result (when $E = \emptyset$ most often). Indeed, this is main result one uses in a standard Calculus course to calculate integrals using antiderivatives!

Theorem 3.20 (Fundamental Theorem of Calculus – Form 2)**.** *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and suppose that f is continuous at a point $c \in [a, b]$. Then, the function*

$$F : [a, b] \to \mathbb{R}, \quad x \mapsto \int_a^x f \tag{3.10}$$

is differentiable at c. Moreover, $F'(c) = f(c)$. The mapping F is called the indefinite integral *of f with basepoint a.*

Example 3.10. Suppose $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function. Then, the function F defined by (3.10) is continuous. In fact F is even Lipschitz continuous.

Solution. To see this, first let $M > 0$ be such that

$$|f(x)| \leq M \qquad \forall x \in [a,b].$$

Given $x, y \in [a,b]$ with $x < y$ there holds

$$|F(y) - F(x)| = \left| \int_a^y f - \int_a^x f \right| = \left| \int_x^y f \right|.$$

By Proposition 3.18,

$$|F(y) - F(x)| \leq \int_x^y |f|.$$

Using that $|f| \leq M$, we conclude that

$$|F(y) - F(x)| \leq \int_x^y M = M(y-x) = M|y-x|.$$

In other words, F is Lipschitz continuous. □

Example 3.11. Suppose $f : [a,b] \to \mathbb{R}$ is a continuous function satisfying

$$\int_a^x f = \int_x^b f \qquad \forall x \in [a,b].$$

Then, define $F : [a,b] \to \mathbb{R}$ by $F(x) := \int_a^x f$ and observe that

$$F(x) = \int_a^x f = \int_x^b f = \int_a^b f - \int_a^x f = F(b) - F(x).$$

By the fundamental theorem of calculus, taking the derivative on either side of the above equation yields

$$f(x) = 0 - f(x) = -f(x).$$

Thus, $f(x) = 0$ for all $x \in [a,b]$.

4 *Sequences of Functions*

We now turn towards the study of function sequences. More precisely, rather than simply study the convergence of sequences consisting of real (or complex) numbers, we analyze the convergence of sequences of *functions*. Formally, given a set X, let us denote by $\mathbb{R}^X$ the set of all functions $X \to \mathbb{R}$. A *sequence of functions* defined on X is a function $f : \mathbb{N} \to \mathbb{R}^X$. That is, we have for each $n \in \mathbb{N}$ an associated function $f(n) : X \to \mathbb{R}$.[27] Typically, since each $f(n)$ is itself a function $X \to \mathbb{R}$, we write f_n to denote the function $f(n)$ so that $f_n(x) = (f(n))(x)$ for all points $x \in X$.

Definition 4.1 (Pointwise Convergence)**.** Let $A \subseteq \mathbb{R}$ and (f_n) be a sequence of functions defined on A. That is, for each $n \in \mathbb{N}$ we have an associated function $f_n : A \to \mathbb{R}$. We say that (f_n) converges *pointwise* to a function $f : A \to \mathbb{R}$ (as $n \to \infty$) if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for every *fixed* $x \in A$. Here, as x is fixed, the limit above is understood as the limit of a sequence of real numbers.

For example, let us consider the sequence of functions on $[0, 1]$ defined by

$$f_n(x) := x^n.$$

At $x = 0$, one has $f_n(x) = f_n(0) = 0$ for each $n \in \mathbb{N}$. Therefore, $f_n(0)$ is the constant sequence $f_n(0) = 0$ and so $\lim_{n\to\infty} f_n(0) = 0$. Similarly, one has $\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1 = 1$. Now, if $0 < x < 1$, we have seen in analysis 1 that $x^n \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0.$$

Combining all possible cases for x, we see that

$$\lim_{n\to\infty} f_n(x) = f(x) := \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

This means that $f_n \to f$ *pointwise* on $[0, 1]$.

Definition 4.2. Let $A \subseteq \mathbb{R}$ and (f_n) a sequence of functions defined on A. We say that f_n converges *uniformly* to a function $f : A \to \mathbb{R}$ (as $n \to \infty$) if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in A$ and all $n \geq N$ there holds

$$|f_n(x) - f(x)| < \varepsilon.$$

[27] Notice that this is similar to the definition employed for sequences of real numbers. Indeed, a sequence of real numbers (x_n) is simply a function $\mathbb{N} \to \mathbb{R}$ where we write $x_n := x(n)$.

Note that here the natural number N is *not* allowed to depend on the point x. Equivalently, we say that $f_n \to f$ uniformly on A provided, for each $\varepsilon > 0$, one can find $N \in \mathbb{N}$ such that

$$\sup_{x \in A} |f_n(x) - f(x)| \leq \varepsilon$$

for all $n \geq N$.

Example 4.1. Define on $\mathbb{R}$ the sequence of functions

$$f_n(x) := \sqrt{x^2 + \frac{1}{n^2}}.$$

Clearly, each f_n is continuous (even differentiable) on all of $\mathbb{R}$. Now, for each fixed $x \in \mathbb{R}$, the standard limit laws show that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n^2}} = \sqrt{x^2} = |x|.$$

Letting $f(x) := |x|$, we see that $f_n \to f$ pointwise on $\mathbb{R}$. We now inspect the possible uniform convergence of this sequence. First, we observe that

$$\begin{aligned} |x| = \sqrt{x^2} \leq f_n(x) = \sqrt{x^2 + \frac{1}{n^2}} &\leq \sqrt{|x|^2 + \frac{2|x|}{n} + \frac{1}{n^2}} \\ &= \sqrt{\left(|x| + \frac{1}{n}\right)^2} \\ &= |x| + \frac{1}{n}. \end{aligned}$$

In particular, for each $x \in \mathbb{R}$ and all $n \in \mathbb{N}$,

$$|x| - \frac{1}{n} \leq f_n(x) \leq |x| + \frac{1}{n}$$

whence

$$|f_n(x) - f(x)| \leq \frac{1}{n}, \quad \forall x \in \mathbb{R} \text{ and } \forall n \in \mathbb{N}.$$

Given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then, for all $n \geq N$ and all $x \in \mathbb{R}$,

$$|f_n(x) - f(x)| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

This shows that $f_n \to f$ uniformly on $\mathbb{R}$.

4.1 Technical Examples of Non-Uniform Convergence

Despite the simplicity the last example might suggest, you will frequently encounter sequences of functions that converge pointwise but not uniformly. Nevertheless, establishing non-uniform convergence can be just as tricky. For this reason, our next several examples will include cases in which the sequence will not converge uniformly.

Example 4.2. Consider the sequence of functions f_n defined by

$$f_n(x) := \frac{nx}{1 + x^2n^2}$$

on $\mathbb{R}$. Clearly, for every fixed $x \neq 0$, we have

$$0 \leq |f_n(x)| = \frac{n\,|x|}{1 + x^2n^2} \leq \frac{n\,|x|}{x^2n^2} = \frac{1}{|x|} \cdot \frac{1}{n}$$

where $\frac{1}{n} \to 0$ as $n \to \infty$. By the Squeeze Theorem for sequences, we see that $f_n(x) \to 0$ as $n \to \infty$. This is simply the statement that $f_n \to 0$ pointwise on $\mathbb{R} \setminus \{0\}$. Since $f_n(0) = 0$ for all n and $f_n(0) \to 0$ as $n \to \infty$, we conclude that $f_n \to 0$ pointwise on all of $\mathbb{R}$.

Let $a > 0$ be given; we claim that $f_n \to 0$ uniformly on $[a, \infty)$. Indeed, for all $x \in [a, \infty)$ and every $n \geq 1$ one has

$$|f_n(x) - 0| = f_n(x) = \frac{nx}{1 + x^2n^2} \leq \frac{nx}{n^2x^2} = \frac{1}{nx} \leq \frac{1}{na}.$$

By the Archimedean property, there exists $N \in \mathbb{N}$ such that $\frac{1}{Na} < \varepsilon$, where $\varepsilon > 0$ is arbitrary but fixed. Then, for all $n \geq N$ and all $x \in [a, \infty)$,

$$|f_n(x) - 0| \leq \frac{1}{na} \leq \frac{1}{Na} < \varepsilon.$$

It follows that $f_n \to 0$ uniformly on $[a, \infty)$ for all $a > 0$. Nonetheless, f_n does not converge uniformly to 0 on $[0, \infty)$. To see this, assume by contradiction that $f_n \to 0$ uniformly on $[0, \infty)$. By definition, one could find $N \in \mathbb{N}$ such that

$$|f_n(x) - 0| = \frac{nx}{1 + x^2n^2} < \frac{1}{2}$$

for all $n \geq N$ and every $x \in [0, \infty)$. In particular, for each $n \geq N$,

$$\frac{1}{2} = \frac{n(1/n)}{1 + (1/n)^2n^2} = \left|f_n\left(\frac{1}{n}\right) - 0\right| < \frac{1}{2}$$

which is a contradiction. This shows that f_n does not converge to 0 uniformly on $[0, \infty)$.

Example 4.3. Consider once more the sequence of functions (f_n) given in Example 4.1. We note that each

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$$

converged uniformly to $f(x) = |x|$ on $\mathbb{R}$. By the chain rule and quotient rule, each f_n is continuously differentiable with derivative given by

$$f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}}.$$

Clearly,

$$f_n'(x) \longrightarrow g(x) := \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Despite uniform convergence of (f_n) and the pointwise convergence of (f_n'), the sequence of derivatives (f_n') does *not* converge uniformly on any interval of the form $[-a.a]$! Indeed, proceeding by way of contradiction, let us assume that $f_n' \longrightarrow g$ uniformly on $[-a, a]$. Then, taking $\varepsilon := \frac{1}{2}$ we can find $N \in \mathbb{N}$ such that

$$\left|f_n'(x) - g(x)\right| < \frac{1}{2}, \quad \forall n \geq N,$$

and all $x \in [-a, a]$. In particular, for every $x \in (0, a]$,

$$\left|f_N'(x) - g(x)\right| < \frac{1}{2}.$$

However, this implies that

$$\left|\frac{x}{\sqrt{x^2 + \frac{1}{N^2}}} - 1\right| < \frac{1}{2}, \quad \forall x \in (0, a].$$

Consider now the sequence (x_k) given by $x_k := \frac{1}{k}$. Since $x_k \longrightarrow 0$ as $k \longrightarrow \infty$, there exists $K \in \mathbb{N}$ such that $x_k \in (0, a] \subseteq [-a, a]$ for all $k \geq K$.[28] Therefore,

$$\left|\frac{x_k}{\sqrt{x_k^2 + \frac{1}{N^2}}} - 1\right| = \left|f_N'(x_k) - g(x_k)\right| < \frac{1}{2}$$

[28]By the $\varepsilon - N$ definition of convergence for sequences, there exists $K \in \mathbb{N}$ such that $|x_k| < a$ for all $k \geq K$. Thus, $x_k \in (0, a)$ for all $k \geq K$.

for all $k \geq K$. Taking the limit as $k \to \infty$, we find that

$$|0-1| = \left| \lim_{k\to\infty} \frac{x_k}{\sqrt{x_k^2 + \frac{1}{N^2}}} - 1 \right| = \lim_{k\to\infty} \left| \frac{x_k}{\sqrt{x_k^2 + \frac{1}{N^2}}} - 1 \right| \leq \frac{1}{2}$$

which is a contradiction. Therefore, f_n' does not converge to g uniformly.

4.2 A Squeeze type Theorem for Uniform Convergence

Recall that a given sequence of functions $f_n : A \to \mathbb{R}$ converges uniformly to a function $f : A \to \mathbb{R}$, as $n \to \infty$, if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in A$ and all $n \geq N$. In practice, to show that (f_n) converges uniformly to f on A, we try to find a sequence (a_n) *converging to* 0 such that

$$|f_n(x) - f(x)| \leq a_n$$

for all $x \in A$ and all $n \in \mathbb{N}$. This is sufficient because, for each $\varepsilon > 0$, one can find $N \in \mathbb{N}$ such that $a_n < \varepsilon$ for all $n \geq N$. Consequently, for all $n \geq N$ and all $x \in A$ there holds

$$|f_n(x) - f(x)| \leq a_n < \varepsilon$$

whence $f_n \to f$ uniformly on A. We restate this below in the form of a lemma:

Lemma 4.1. *Let $A \subseteq \mathbb{R}$ and (f_n) be a sequence of functions on A and fix a function $f : A \to \mathbb{R}$. Assume that there exists a sequence (a_n) such that*

$$|f_n(x) - f(x)| \leq a_n$$

for all $n \in \mathbb{N}$ and every $x \in A$. If $a_n \to 0$ as $n \to \infty$, then $f_n \to f$ uniformly as $n \to \infty$.

Example 4.4. Consider the sequence of functions (f_n) on $\mathbb{R}$ defined by

$$f_n(x) := \frac{x}{1 + nx^2}.$$

To identify the uniform limit of these f_n's (if it exists), we should first try and determine their pointwise limit. Clearly, for every fixed $x \neq 0$ we have

$$|f_n(x)| \leq \frac{|x|}{1 + nx^2} \leq \frac{|x|}{nx^2} = \frac{1}{n\,|x|} \xrightarrow{n\to\infty} 0.$$

Thus, if $x \neq 0$ is fixed, then the Squeeze Theorem for sequences implies that

$$\lim_{n\to\infty} f_n(x) = 0.$$

Moreover,

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0.$$

We infer that $f_n(x) \longrightarrow 0$, as $n \longrightarrow \infty$, for each $x \in \mathbb{R}$. This is precisely the statement that $f_n \longrightarrow 0$ pointwise on $\mathbb{R}$. We now assert that $f_n \longrightarrow 0$ uniformly on $\mathbb{R}$. For this, we will require the following identity:

$$\frac{|t|}{1+t^2} \leq \frac{1}{2}, \quad \forall t \in \mathbb{R}. \tag{4.1}$$

To prove (4.1), it suffices to check that

$$t^2 - 2|t| + 1 \geq 0, \quad \forall t \in \mathbb{R}.$$

However, $t^2 - 2|t| + 1 = (|t| - 1)^2 \geq 0$ for all $t \in \mathbb{R}$. This verifies (4.1). Returning to the example, we can apply (4.1) with $t = \sqrt{n}x$ to obtain the following uniform estimate:

$$\begin{aligned}
|f_n(x) - 0| = \frac{|x|}{1+nx^2} &= \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{n}\,|x|}{1+nx^2} \\
&= \frac{1}{\sqrt{n}} \cdot \frac{\left|\sqrt{n}x\right|}{1+\left(\sqrt{n}x\right)^2} \\
&\overset{(4.1)}{\leq} \frac{1}{2\sqrt{n}}
\end{aligned}$$

for all $x \in \mathbb{R}$ and every $n \in \mathbb{N}$. Applying Lemma 4.1 with $a_n := \frac{1}{2\sqrt{n}}$ then shows that $f_n \longrightarrow 0$ uniformly.

Remark 4.1. We have seen in the lectures and tutorials that uniform convergence does *not* imply the uniform convergence of the derivatives. As is turns out, counter examples to such a statement are typically neither rare nor contrived. In fact, the example above is yet another instance of such a sequence. Indeed, by the chain rule and quotient rule, each f_n is continuously differentiable with derivative given by

$$f_n'(x) = \frac{(1+nx^2) - 2nx^2}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}.$$

Clearly,

$$\lim_{n\to\infty} f_n'(0) = \lim_{n\to\infty} 1 = 1.$$

On the other hand, if $x \neq 0$ is fixed,

$$\left|\frac{1-nx^2}{(1+nx^2)^2}\right| \leq \frac{1+nx^2}{(1+nx^2)^2} = \frac{1}{1+nx^2} \leq \frac{1}{nx^2}$$

which tends to 0 as $n \to \infty$. Hence, $f_n'(x) \to 0$, as $n \to \infty$, for all $x \neq 0$. Put otherwise,

$$f_n'(x) \to g(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

However, defining $f(x) := 0$ on $\mathbb{R}$, we have shown above that $f_n \to f$ uniformly. Despite this, $f_n'(0) \not\to f'(0)$. Thus, $f_n' \not\to f'$ pointwise on $\mathbb{R}$ even though f_n' *does* converge pointwise. Furthermore, (f_n') does not converge uniformly to g on $\mathbb{R}$ since g is discontinuous at 0.

It should now be clear that uniform convergence does not imply the uniform convergence of the derivatives. In fact, if (f_n) is a sequence of differentiable functions converging uniformly to a differentiable function f, the sequence (f_n') may not even converge pointwise to f'. Therefore, deducing properties about the convergence of (f_n') using the uniform convergence of (f_n) can in general be quite tricky.

4.3 Continuity of the Uniform Limit

We also recall the following theorem. In general, this result makes it much easier to show that certain pointwise convergences cannot be uniform.

Theorem 4.2. *Let $A \subseteq \mathbb{R}$ and let (f_n) be a sequence of functions defined on A converging uniformly to a function $f : A \to \mathbb{R}$ on the set A. If each f_n is continuous at a point $c \in A$, then so is f.*

Proof. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly on A, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$$

for all $x \in A$ and every $n \geq N$. Now, f_N is continuous at the point $c \in A$. Thus, there exists $\delta > 0$ such that

$$|f_N(x) - f_N(c)| < \frac{\varepsilon}{3}$$

for all $x \in A$ with $|x - c| < \delta$. Finally, note that the triangle inequality implies

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \frac{2\varepsilon}{3} + |f_N(x) - f_N(c)|. \end{aligned}$$

If in addition $|x - c| < \delta$, then

$$|f(x) - f(c)| < \frac{2\varepsilon}{3} + |f_N(x) - f_N(c)| < \varepsilon.$$

This completes the proof. ◻

Corollary 4.3. *Let $A \subseteq \mathbb{R}$ and let (f_n) be a sequence of functions defined on A converging uniformly to a function $f : A \to \mathbb{R}$ on the set A. If each f_n is continuous on A, then so is f.*

Equipped with this result, we provide an alternative approach to Example 4.3. Recall Example 4.1 where we showed that the sequence

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$$

converges uniformly to $f(x) = |x|$ on $\mathbb{R}$. Then, in Example 4.3 we showed however that (f_n') does not converge uniformly.

Example 4.5. By the chain rule, every function f_n defined above is differentiable on $\mathbb{R}$ with derivative given by

$$f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}}.$$

However, the uniform limit of these f_n's is not differentiable at 0. Indeed, the function $f(x) = |x|$ does not have a derivative at $x = 0$. Despite this, the functions f_n' still converge pointwise on $\mathbb{R}$. More precisely,

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}} = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

But, is it true that f_n' converges uniformly to this function described above? To answer this, denote by g the pointwise limit of (f_n'), i.e.

$$g(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

We claim that f_n' does **not** converge uniformly to g on any compact interval $[-a, a]$. To see this, we argue by way of contradiction. Assume that $f_n' \to g$ *uniformly* on $[-a, a]$. Since every f_n' is continuous on $[-a, a]$, the previous Corollary implies that the uniform limit g must be continuous on $[-a, a]$ as well. However, g is clearly discontinuous at 0. Hence, the convergence cannot be uniform.

We have proven that uniform limits of continuous functions are continuous. More precisely, we proved in Theorem 4.2 that if $f_n \to f$ uniformly and every f_n is continuous at a point c, then so is the limit f. It is therefore reasonable to ask whether uniform limits also preserve points of *discontinuitiy*. This answer is negative and this is justified by the following example:

Example 4.6. For each $n \in \mathbb{N}$ consider the function

$$f_n : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} \frac{1}{n} & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We claim that every f_n is discontinuous on all of $\mathbb{R}$. Indeed, if f_n were continuous at a point $c \in \mathbb{R}$, then so would be the function nf_n. However, nf_n is precisely the Dirichlet function $1_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$, which is everywhere discontinuous. Hence, f_n is discontinuous at every point $c \in \mathbb{R}$.

Despite this, $f_n \to 0$ uniformly on $\mathbb{R}$, where the constant function $x \mapsto 0$ is everywhere continuous (even infinitely differentiable). To see this, note that for each $n \in \mathbb{N}$ one has

$$|f_n(x)| = f_n(x) \leq \frac{1}{n}$$

for all $x \in \mathbb{R}$. By Lemma 4.1, it follows that $f_n \to 0$ uniformly on $\mathbb{R}$.

4.4 Uniform Limits and Bounded functions

At this point it is quite convincing that uniform limits enjoy nicer properties than simple pointwise limits. This section explores more of these nice properties, the first of which is an elegant boundedness result.

Proposition 4.4. *Let $A \subseteq \mathbb{R}$ and (f_n) be a sequence of functions converging uniformly to a function $f : A \to \mathbb{R}$ on the set A. If each f_n is bounded, then so is f.*[29] *That is, uniform limits of bounded functions are bounded.*

Proof. Let $\varepsilon = 1$. Since $f_n \to f$ uniformly on A, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1$$

for all $n \geq N$ and every $x \in A$. Since f_N is bounded, one has

$$\begin{aligned} |f(x)| = |f(x) - f_N(x) + f_N(x)| &\leq |f(x) - f_N(x)| + |f_N(x)| \\ &< 1 + M_N \end{aligned}$$

for any $x \in A$. This is precisely the statement that f is bounded on A. □

We also offer a slight improvement of our previous statement:

[29] Here we assume that, for each $n \in \mathbb{N}$, there exists $M_n > 0$ such that $|f_n(x)| \leq M_n$ for all $x \in A$. It is allowed that these M_n depend on the index n of the sequence!

Corollary 4.5. *Let (f_n) be a sequence of functions, each defined on a set $A \subseteq \mathbb{R}$, converging uniformly to a function $f : A \to \mathbb{R}$ on A. If each f_n is bounded, then the sequence (f_n) is uniformly bounded. More precisely, if there exists for each $n \in \mathbb{N}$ some $M_n > 0$ such that*

$$|f_n(x)| \le M_n, \quad \forall x \in A,$$

then there exists $M > 0$ such that

$$|f_n(x)| \le M$$

for all $x \in A$ and all $n \in \mathbb{N}$. Moreover,

$$|f(x)| \le M$$

for every $x \in A$.

Proof. Citing the previous proposition, the uniform limit f is bounded on A. Thus, there exists a constant $C > 0$ such that $|f(x)| \le C$ for all $x \in A$. Now, as $f_n \to f$ uniformly on A, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1$$

for all $n \ge N$ and each $x \in A$. In particular, we have

$$|f_n(x)| \le |f_n(x) - f(x)| + |f(x)| < 1 + C$$

for every $n \ge N$ and any $x \in A$. Since $|f_n| \le M_n$ on A for every $n \in \mathbb{N}$, it follows from the above that, given any $n \in \mathbb{N}$,

$$|f_n(x)| \le \max\{M_1, M_2, \ldots, M_{N-1}, 1 + C\} =: M.$$

for every point $x \in A$. It follows that $|f_n(x)| \le M$ for every $x \in A$ and all $n \in \mathbb{N}$. Since $|f(x)| \le C < 1 + C \le M$ on A, we are done. □

Example 4.7. Consider the sequence of functions (f_n) given by

$$f_n : (0, \infty) \to \mathbb{R}, \quad f_n(x) = \frac{n}{nx + 1} = \frac{1}{x + \frac{1}{n}}.$$

Observe that this is a sequence of *bounded* functions. Indeed, each function f_n is bounded since

$$0 \le f_n(x) = \frac{n}{nx + 1} \le \frac{n}{1} = n.$$

for any $x > 0$. Moreover, $f_n \to f$ pointwise where $f : (0, \infty) \to \mathbb{R}$ is the function $f(x) = 1/x$. However, f_n does **not** converge to f uniformly on $(0, \infty)$. Indeed, if this were the case then f would be bounded. However, $f(x) = 1/x$ is an unbounded function.

By a similar argument to the proof of Proposition 4.4, we also obtain the following:

Proposition 4.6. *Let $A \subseteq \mathbb{R}$ and (f_n) be a sequence of functions converging uniformly to a function $f : A \to \mathbb{R}$ on the set A. If every f_n is unbounded, then f is also unbounded.*

Proof. We argue by contradiction. Assuming that f is bounded on A, there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$. For $\varepsilon := 1$, we can also find $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1$$

for all $n \geq N$ and every $x \in A$. In particular,

$$|f_N(x)| = |f_N(x) - f(x) + f(x)| \leq |f_N(x) - f(x)| + |f(x)| < 1 + M$$

for every $x \in A$. This implies that f_N is bounded on A, which is a contradiction. □

4.5 More on Uniform Convergence

We recall from class that uniform limits preserve Riemann integrability:

Theorem 4.7. *Let (f_n) be a sequence of Riemann integrable functions on $[a, b]$ converging uniformly to a function $f : [a, b] \to \mathbb{R}$ on $[a, b]$. Then, f is Riemann integrable on $[a, b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

In fact, one can say slightly more:

Corollary 4.8. *If (f_n) is a sequence of Riemann integrable functions on $[a, b]$ converging uniformly to a function $f : [a, b] \to \mathbb{R}$ on $[a, b]$, then f is Riemann integrable on $[a, b]$ and*

$$\lim_{n\to\infty} \int_a^b |f_n - f| = 0.$$

Proof. Since f_n is Riemann integrable for every $n \in \mathbb{N}$ and $f \in \mathcal{R}([a, b])$ by the previous theorem, $(f_n - f)$ is a sequence of Riemann integrable functions on $[a, b]$. By Proposition **??**, $|f_n - f|$ is also Riemann integrable on $[a, b]$. Because $f_n \to f$ uniformly if and only if $|f_n - f| \to 0$ uniformly, another application of the previous theorem ensures that

$$\lim_{n\to\infty} \int_a^b |f_n - f| = \int_a^b 0 = 0.$$

□

We have seen that uniform convergence preserves many properties of functions. But what operations preserve uniform convergence? One example, is composition through continuous functions. More precisely, we have the following result:

Proposition 4.9 (Composition Theorem for Uniform Convergence)**.** *Let $A \subseteq \mathbb{R}$ and let (f_n) be a sequence of functions converging uniformly to a function f on A. Assume that every f_n is bounded on A. If $g : \mathbb{R} \to \mathbb{R}$ is continuous, then*

$$g \circ f_n \to g \circ f$$

uniformly on A.

Proof. By virtue of Corollary 4.5, there exists $M > 0$ such that $|f_n(x)| \leq M$ and $|f(x)| \leq M$ for every $n \in \mathbb{N}$ and each $x \in A$. Let $\varepsilon > 0$ be given. Since $[-M, M]$ is compact, g is uniformly continuous on $[-M, M]$. Thus, there exists $\delta > 0$ such that

$$|g(y) - g(v)| < \varepsilon \tag{4.2}$$

for all $y, v \in [-M, M]$ with $|y - v| < \delta$. On the other hand, because $f_n \to f$ uniformly on A, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \delta$$

for all $n \geq N$. For any such n and all $x \in A$, we see from (4.2) (with $y := f_n(x)$ and $v := f(x)$) that

$$|(g \circ f_n)(x) - (g \circ f)(x)| = |g(f_n(x)) - g(f(x))| < \varepsilon.$$

Since this holds for all $n \geq N$ and every $x \in A$, it follows that $g \circ f_n \to g \circ f$ uniformly on A. □

Remark 4.2. Note that the result above continues to hold if g is merely assumed to be continuous on the interval $[-M, M]$.

4.6 Uniform Limits of Uniformly Continuous Functions

Having shown that uniform limits preserve certain local properties (e.g. continuity), we now ask which global properties survive limits. Having already seen that boundedness is preserved by uniform limits, we now check that uniform limits of *uniformly* continuous functions are still uniformly continuous.

Proposition 4.10. *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Let (f_n) be a sequence of uniformly continuous functions defined on A and assume that $f_n \to f$ uniformly on A. Then, f is uniformly continuous on A.*

Proof. We can use the same proof as in Theorem 4.2. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ and each $t \in A$,

$$|f_n(t) - f(t)| < \frac{\varepsilon}{3}.$$

For any $x, u \in A$ there holds

$$\begin{aligned} |f(x) - f(u)| &= |f(x) - f_N(x) + f_N(x) - f_N(u) + f_N(u) - f(u)| \\ &\le |f(x) - f_N(x)| + |f_N(x) - f_N(u)| + |f_N(u) - f(u)| \\ &\le \frac{2\varepsilon}{3} + |f_N(x) - f_N(u)|\,. \end{aligned}$$

Now, f_N is uniformly continuous on A. Thus, there exists $\delta > 0$ such that

$$|f(x) - f(u)| < \frac{\varepsilon}{3}$$

whenever $|x - u| < \delta$. So, if $|x - u| < \delta$, our calculations above indicate that

$$|f(x) - f(u)| \le \frac{2\varepsilon}{3} + |f_N(x) - f_N(u)| < \varepsilon$$

whence f is uniformly continuous on A. □

5 Infinite Series

Since we will spend a significant portion of this tutorial reviewing the midterm exam solutions, these tutorial notes will be significantly shorter than usual. However, the midterm solutions will be posted (shortly) within a self contained document on MyCourses. These notes shall only contain the tutorial content related to the convergence of series.

Definition 5.1. Let (a_n) be a sequence of real numbers and define for each $N \in \mathbb{N}$ the partial sum

$$S_N := \sum_{n=1}^{N} a_n$$

is a well defined real number. Furthermore, the partial sums themselves form a sequence $(S_N)_{N\in\mathbb{N}}$. With this in mind, we say that the *series* $\sum_{n=1}^{\infty} a_N$ converges provided the *sequence* of partial sums $(S_N)_{N\in\mathbb{N}}$ is convergent in $\mathbb{R}$. In this case, we define

$$\sum_{n=1}^{\infty} a_n := \lim_{N\to\infty} S_N.$$

Although we do not yet possess many tools related to the convergence of series, we have already seen a few useful results in class, the first of which is the straightforward divergence test:

Theorem 5.1 (Divergence Test). *Let (a_n) be a sequence of real numbers. If $\sum_{n=1}^{\infty} a_n$ converges, then we must have*

$$\lim_{n\to\infty} a_n = 0.$$

Equivalently, if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Using this as our only series-test, it is already possible for us to prove that the series

$$\sum_{n=1}^{\infty} \sin(n)$$

is divergent. We justify this in the following example:

Example 5.1. Using the divergence test stated above, we shall prove that the series $\sum_{n=1}^{\infty} \sin n$ is divergent. Thus, we are reduced to proving that

$$\lim_{n\to\infty} \sin n \neq 0,$$

which requires some work. Recall the sin addition formula which states that

$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x), \quad \forall x, y \in \mathbb{R}. \tag{5.1}$$

In particular, since $\sin(-y) = -\sin(y)$, we obtain

$$\sin(x - y) = \sin(x)\cos(y) - \sin(y)\cos(x), \quad \forall x, y \in \mathbb{R}. \tag{5.2}$$

By way of contradiction, let us assume that

$$\lim_{n\to\infty} \sin n = 0.$$

Then,

$$\lim_{n\to\infty} \sin(n+1) = \lim_{n\to\infty} \sin n = \lim_{n\to\infty} \sin(n-1) = 0.$$

Now, observe that by combining (5.1)-(5.2) we obtain the following expression:

$$\begin{aligned} \sin(n+1) - \sin(n-1) &= [\sin(n)\cos(1) + \sin(1)\cos(n)] - [\sin(n)\cos(1) - \sin(1)\cos(n)] \\ &= 2\sin(1)\cos(n). \end{aligned}$$

But, by limit laws, this implies that

$$\begin{aligned} \lim_{n\to\infty} \cos(n) &= \lim_{n\to\infty} \frac{1}{2\sin(1)} \lim_{n\to\infty} (\sin(n+1) - \sin(n-1)) \\ &= \frac{1}{2\sin(1)} \left(\lim_{n\to\infty} \sin(n+1) - \lim_{n\to\infty} \sin(n-1) \right) \\ &= \frac{1}{2\sin(1)} (0 - 0) \\ &= 0. \end{aligned}$$

Consequently, $\sin(n) \to 0$ and $\cos(n) \to 0$ as $n \to \infty$. However, this implies that $\sin^2(n), \cos^2(n) \to 0$ as $n \to \infty$, which is a contradiction because

$$1 = \sin^2(n) + \cos^2(n), \quad \forall n \in \mathbb{N}.$$

It follows that $\sin(n) \not\to 0$ as $n \to \infty$ whence we see from the divergence test that $\sum_{n=1}^{\infty} \sin(n)$ diverges.

A second example involving *convergence* is also in order:

Example 5.2. Given $\alpha > 0$, we will show that the series

$$\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)}$$

converges and determine its value. In this case, the "trick" is to realize that the summand term

$$\frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha+n} - \frac{1}{\alpha+n+1}$$

admits a convenient partial fractions decomposition. Consequently, the partial sums for the series $\sum_{n=1}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)}$ can also be written in the following way:

$$\begin{aligned}\sum_{n=0}^{N} \frac{1}{(\alpha+n)(\alpha+n+1)} &= \sum_{n=0}^{N} \left(\frac{1}{\alpha+n} - \frac{1}{\alpha+(n+1)} \right) \\ &= \sum_{n=0}^{N} \frac{1}{\alpha+n} - \sum_{n=0}^{N} \frac{1}{\alpha+(n+1)} \\ &= \sum_{n=0}^{N} \frac{1}{\alpha+n} - \sum_{n=1}^{N+1} \frac{1}{\alpha+n} \\ &= \frac{1}{\alpha} - \frac{1}{\alpha+N+1}.\end{aligned}$$

Therefore,

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{(\alpha+n)(\alpha+n+1)} = \lim_{N\to\infty} \left[\frac{1}{\alpha} - \frac{1}{\alpha+N+1} \right] = \frac{1}{\alpha}.$$

By definition, this means that

$$\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha}.$$

5.1 *The Cauchy Condensation Test & Examples*

Before providing a few worked out examples, let us recall the precise statement of the Cauchy condensation test:

Theorem 5.2 (Cauchy Condensation Test)**.** *Let (a_n) be a decreasing sequence of non-negative real numbers. Then, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the* condensed *series $\sum_{n=1}^{\infty} a_{2^n} 2^n$ converges.*

Remark 5.1. Although this almost certainly has/will be addressed in class, I would like to point out that the Cauchy condensation test provides a simple way to show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Example 5.3. Using the Cauchy condensation test, we shall show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

diverges. First, note that if $a_n := \frac{1}{n \ln n}$ then (a_n) is non-negative and decreasing (since both n and $\ln(n)$ are non-negative and increasing with respect to n). Furthermore, the terms of our condensed series are given by

$$a_{2^n} = \frac{1}{2^n \ln(2^n)} = \frac{1}{2^n n \ln(2)} \implies 2^n a_{2^n} = \frac{1}{n \ln(2)}.$$

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows from the above that our condensed series

$$\sum_{n=1}^{\infty} \frac{1}{n \ln(2)}$$

diverges and hence so does our original series.

Example 5.4. Let $c > 1$ be given an consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)^c}.$$

Using the Cauchy condensation test, we will show that this series is actually convergent. Indeed, as above, if we set $a_n := \frac{1}{n \ln(n)^c}$ we find that

$$2^n a_{2^n} = 2^n \cdot \frac{1}{2^n (\ln(2^n))^c} = \frac{1}{(n \ln(2))^c} = \frac{1}{\ln(2)^c} \cdot \frac{1}{n^c}.$$

Now, since $c > 1$ and the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$, it follows that the series

$$\sum_{n=2} 2^n a_{2^n} = \sum_{n=2}^{\infty} \frac{1}{\ln(2)^c n^c} = \frac{1}{\ln(2)^c} \sum_{n=2}^{\infty} \frac{1}{n^c}$$

converges. Hence, by the condensation test, our original series must also converge.

5.2 The Comparison & Limit-Comparison Tests

We devote this tutorial to the proofs of the limit comparison test and the ratio test (both the limit-free and limit versions). Since the proofs of these respective results will fundamentally boil down to a clever application of this comparison test, we will reiterate this result for completeness.

Theorem 5.3 (Comparison Test). *Let $N \in \mathbb{N}$ and suppose that $(a_n), (b_n)$ are non-negative sequences such that $a_n \leq b_n$ for all $n \geq N$.*

1. *If $\sum b_n$ converges then $\sum a_n$ converges.*
2. *If $\sum a_n$ diverges then $\sum b_n$ diverges.*

Equipped with this, the following limit comparison test is within reach:

Proposition 5.4 (Limit Comparison Test)**.** *Let (a_n) be a non-negative sequence and (b_n) a strictly positive sequence such that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Then,

1. *If $0 < L < \infty$ then $\sum a_n$ converges if and only if $\sum b_n$ converges.*
2. *If $L = 0$ then $\sum a_n$ converges whenever $\sum b_n$ converges.*
3. *If $L = \infty$, then $\sum a_n$ diverges whenever $\sum b_n$ diverges.*

Proof.

1. Choose $\varepsilon > 0$ such that $L - \varepsilon > 0$ and observe that there exists $N \in \mathbb{N}$ such that

   $$\left|\frac{a_n}{b_n} - L\right| < \varepsilon, \quad \forall n \geq N.$$

   Equivalently, one has

   $$L - \varepsilon < \frac{a_n}{b_n} < L + \varepsilon, \quad \forall n \geq N$$

   whence it follows that

   $$(L - \varepsilon)b_n < a_n < (L + \varepsilon)b_n, \quad \forall n \geq N.$$

   It therefore follows from the comparison test that $\sum a_n$ converges if and only if $\sum b_n$ converges.

2. In the event that $L = 0$, for $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that

   $$0 \leq \frac{a_n}{b_n} < 1, \quad \forall n \geq N.$$

   Thus, $a_n < b_n$ for all n large whence by comparison we see that $\sum a_n$ converges whenever $\sum b_n$ does.

3. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, there exists $N \in \mathbb{N}$ such that $a_n/b_n > 1$ for all $n \geq N$. By comparison, the assertion follows.

□

We now come to one of the most famous tests used in the analysis of series: the ratio test. We begin with a statement/version of this test that does not involve limits. Although the statement of the limit-free version is slightly more technical that its counterpart, it has the added benefit of being stronger.

Theorem 5.5 (Limit-Free Ratio Test). *Let (a_n) be a sequence of positive numbers. Then*

1. *If there exists $r \in [0, 1), N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \leq r$ for all $n \geq N$ then $\sum a_n$ converges.*
2. *If there exists $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \geq 1$ for all $n \geq N$ then $\sum a_n$ diverges.*

Proof. In the first case, we infer via induction that $0 < a_{N+k} \leq r^k a_N$ for all $k \geq 1$. Therefore, one has $0 < a_n \leq r^{n-N} a_N$ for all $n \geq N$ whence the series $\sum a_n$ converges by comparison with a geometric series.

In the second case, we have $a_n \geq a_N > 0$ for all $n \geq N$ whence, by the divergence test, the series diverges. □

As a consequence, we recover the ratio test in a slightly more familiar form from calculus:

Corollary 5.6 (Ratio Test). *Let (a_n) be a sequence of positive numbers such that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

1. *If $L < 1$ then $\sum a_n$ converges.*
2. *If $L > 1$ then $\sum a_n$ diverges.*

Proof. If $L < \infty$ then, given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$L - \varepsilon < \frac{a_{n+1}}{a_n} < L + \varepsilon, \quad \forall n \geq N.$$

If $L < 1$, we can choose $\varepsilon > 0$ so small that $L + \varepsilon < 1$ whence we have $a_{n+1} \leq (L + \varepsilon)a_n$ for all $n \geq N$. It then follows from the limit free ratio test that the series $\sum a_n$ is convergent. If $L > 1$, we choose $\varepsilon > 0$ such that $L - \varepsilon > 1$ whence it follows by the limit free ratio test that the given series is divergent.

Similarly if $L = \infty$, there exists $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} > 1 \quad \forall n \geq N.$$

Once, again, it follows from the ratio-free limit test that $\sum a_n$ diverges. □

5.3 Examples of Convergence/Divergence of Series.

Having dispensed with the proofs of the limit comparison test and the ratio tests, we are ready to tackle some more examples of series. Note that although we will not have the time to cover all of these in the tutorial, the full solutions to each example is included nonetheless.

Example 5.5. Let $a_1 = 1$ and, given a_n, define

$$a_{n+1} = \frac{2 + (-1)^n}{4} a_n.$$

By induction, observe that (a_n) is a sequence of positive numbers. To determine whether $\sum a_n$ converges, we could try to apply the ratio test. However, the limit of

$$\frac{a_{n+1}}{a_n} = \frac{2 + (-1)^n}{4}$$

does not exist. Despite this, observe that

$$\frac{a_{n+1}}{a_n} = \frac{2 + (-1)^n}{4} \leq \frac{1 + 2}{4} = \frac{3}{4}.$$

Therefore, by the limit-free ratio test $\sum a_n$ converges.

Example 5.6. Consider the series $\sum_{n=6}^{\infty} \frac{n^2+n}{n^4-n}$.

Solution. Define $a_n := \frac{n^2+n}{n^4-n}$ and put $b_n := \frac{1}{n^2}$. Then, for each $n \in \mathbb{N}$, one has

$$\frac{a_n}{b_n} = \frac{n^2 + n}{n^4 - n} \cdot n^2 = \frac{n^4 + n^3}{n^4 - n}.$$

It follows that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1$$

whence by the limit comparison test the series $\sum a_n$ is convergent. □

Example 5.7. Consider the series $\sum \frac{(2n)!}{2^n}$.

Solution. Define

$$a_n := \frac{(2n)!}{2^n}$$

so that

$$a_{n+1} = \frac{(2n+2)!}{2^{n+1}}.$$

Then, for each $n \in \mathbb{N}$, we have

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{2^{n+1}} \cdot \frac{2^n}{(2n)!} = \frac{(2n+2)(2n+1)}{2} \xrightarrow{n \to \infty} \infty.$$

By the ratio test, the series $\sum a_n$ is divergent. □

Example 5.8. Consider the series $\sum_{n=8}^{\infty} \frac{(2n)!}{[(n-1)!]^3}$.

Solution. Let us define (a_n) by the rule

$$a_n := \frac{(2n)!}{[(n-1)!]^3}, \quad \forall n \in \mathbb{N}$$

and observe that

$$a_{n+1} := \frac{(2n+2)!}{[n!]^3}, \quad \forall n \in \mathbb{N}.$$

Clearly,

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{[n!]^3} \cdot \frac{[(n-1)!]^3}{(2n)!} = \frac{(2n+2)(2n+1)}{n^3}.$$

Thus,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$$

whence our series $\sum a_n$ is convergent by the ratio test. □

Example 5.9. Consider the series $\sum_{n=1}^{\infty} \frac{n^{5+n}}{(5n+1)^n}$.

Solution. Let us define $a_n := \frac{n^{5+n}}{(5n+1)^n}$ so that

$$|a_n|^{1/n} = a_n^{1/n} = \frac{n^{5/n} \cdot n}{5n+1} = (n^{1/n})^5 \cdot \frac{n}{5n+1}$$
$$\to \frac{1}{5}.$$

□

Example 5.10. Consider the series $\sum_{n=1}^{\infty} (1 - \cos(1/n))$.

Solution. Set

$$a_n = (1 - \cos(1/n)), \quad b_n = 1 - \cos^2\left(\frac{1}{n}\right)$$

Observe that

$$b_n = (1 - \cos(1/n))(1 + \cos(1/n)) = (1 + \cos(1/n))\, a_n.$$

Hence,

$$\frac{a_n}{b_n} = \frac{1}{1 + \cos(1/n)} \to \frac{1}{1 + \cos(0)} = \frac{1}{2}.$$

By the limit comparison test, $\sum a_n$ converges if and only $\sum b_n$ converges.

On the other hand,

$$b_n = 1 - \cos^2\left(\frac{1}{n}\right) = \sin^2\left(\frac{1}{n}\right)$$

We have seen in the second tutorial that $\sin(x) \leq |x|$. Therefore,

$$b_n = \left(\sin\left(\frac{1}{n}\right)\right)^2 \leq \left|\frac{1}{n}\right|^2 = \frac{1}{n^2}.$$

By the p-test, $\sum b_n$ converges. We conclude that $\sum_{n=1}^{\infty} (1 - \cos(1/n))$ is a convergent series. $\square$

Example 5.11. Consider the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$.

Solution. Set $a_n = \ln(n)/n$. When a series contains logarithms, we have previously used the Cauchy condensation test. However, this test only applies if (a_n) is a decreasing sequence. Since it is not clear whether (a_n) is decreasing, let us try a different method. Consider $b_n = \frac{1}{n}$. Observe that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \ln(n) = \infty.$$

Therefore, by the limit comparison test, since $\sum b_n$ diverges $\sum a_n$ also diverges. $\square$

Example 5.12. Consider the series $\sum \left(\sqrt[n]{e} - 1\right)^n$.

Solution. By the root test, this series converges. Indeed, if we define $a_n := \left(\sqrt[n]{e} - 1\right)^n$ we see that $a_n \geq 0$ and that $a_n^{1/n} = \sqrt[n]{e} - 1$. Since $\sqrt[n]{e} \longrightarrow 1$ as $n \longrightarrow \infty$, it follows that $\lim_{n\to\infty} a_n^{1/n} = 1 - 1 = 0$. $\square$

Example 5.13. Consider $\sum_{n=1}^{\infty} \left(\sqrt[n]{e} - 1\right)$. To determine if the series converges or diverges, you can use that the derivative of $f(x) = e^x$ is $f'(x) = e^x$.

Solution. First, we claim that $e^x \geq x + 1$ for all $x \geq 0$. To this end, consider $h : [0, \infty) \longrightarrow \mathbb{R}$ given by $h(x) = e^x - x - 1$. We have $h'(x) = e^x - 1 \geq 0$ so h is an increasing function. Since, $h(0) = 0$, we obtain $h(x) \geq 0$ or, equivalently, $e^x \geq x + 1$ for all $x \geq 0$.

Using this inequality, we may write

$$\sqrt[n]{e} - 1 = e^{1/n} - 1 \geq \left(\frac{1}{n} + 1\right) - 1 = \frac{1}{n}.$$

Since $\sum \frac{1}{n}$ diverges, the comparison test implies that $\sum_{n=1}^{\infty} \left(\sqrt[n]{e} - 1\right)$ diverges. $\square$

Example 5.14. Consider the series $\sum_{n=1}^{\infty} \left(\frac{2^n + n^2}{n^2 \cdot 2^n}\right)$.

Solution. Observe that

$$\frac{2^n + n^2}{n^2 \cdot 2^n} = \frac{1}{n^2} + \frac{1}{2^n}.$$

Since $\sum \frac{1}{2^n}$ and $\sum \frac{1}{n^2}$ converge, the limit laws imply that $\sum_{n=1}^{\infty} \left(\frac{n^2 + 2^n}{n^2 \cdot 2^n}\right)$ converges. $\square$

Example 5.15. Consider the series $\sum_{n=1}^{\infty} \left(\frac{2^n + n}{n \cdot 2^n}\right)$.

Solution. We can write

$$\frac{n + 2^n}{n \cdot 2^n} = \frac{1}{n} + \frac{1}{2^n}.$$

Since $\sum \frac{1}{n}$ diverges, the comparison test implies that $\sum_{n=1}^{\infty} \left(\frac{2^n+n}{n \cdot 2^n}\right)$ diverges. □

Example 5.16. Consider the series $\sum_{n=1}^{\infty} \frac{2^n n^7}{3^n}$.

Solution. Setting $a_n = \frac{2^n n^7}{3^n}$ we see that

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)^7}{3^{(}n+1)} \cdot \frac{3^n}{n^7 \cdot 2^n} = \frac{2}{3}\left(\frac{n+1}{n}\right)^7.$$

By the limit laws,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2}{3} < 1.$$

By the ratio test, the series $\sum_{n=1}^{\infty} \frac{2^n n^7}{3^n}$ converges. □

Example 5.17. Consider $\sum_{n=1}^{\infty} \frac{n!+(-1)^n}{n+5}$.

Solution. By the divergence test, this series diverges. Indeed,

$$\frac{n! + (-1)^n}{n+5} = \frac{n!}{n+5} + \frac{(-1)^n}{n+5} = \frac{n}{n+5}(n-1)! + \frac{(-1)^n}{n+5}$$

does not converge to 0. □

Example 5.18. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$.

(a) Show that the ratio test is inconclusive when applied to this series.

(b) By establishing the following inequality:

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \geq \frac{1}{2n}, \quad \forall n \in \mathbb{N}.$$

Solution.

Letting (a_n) be the sequence of summands given, we see that for all $n \geq 1$

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n) \cdot (2n+2)} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \\ &= \frac{2n+1}{2n+2}. \end{aligned}$$

Therefore, $a_{n+1}/a_n \to 1$ which is inconclusive.

We establish the stated inequality via induction. Plugging in $n = 1$, the identity reduces to

$$\frac{2(1)-1}{2(1)} = \frac{1}{2} \geq \frac{1}{2}$$

which holds true. Assuming the identity holds for $n \in \mathbb{N}$, observe that by our inductive hypothesis

$$\begin{aligned}\frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdots(2n)(2n+2)} &= \left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \geq \frac{1}{2n}\right]\cdot\frac{2n+1}{2n+2}\\ &\geq \frac{1}{2n}\cdot\frac{2n+1}{2n+2}\\ &\geq \frac{1}{2n}.\end{aligned}$$

Hence, we need only compare with the harmonic series. □

5.4 Abel's Test

We now treat examples relating to Abel's test, which we recall below from the class notes.

Theorem 5.7 (Abel's Test)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and let (b_n) be a bounded monotone sequence. Then, the series*

$$\sum_{n=1}^{\infty} a_n b_n$$

is convergent.

Example 5.19. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\cdot\left(1+\frac{1}{n}\right)^n.$$

This can be written as

$$\sum_{n=1}^{\infty} a_n b_n$$

where

$$a_n := \frac{(-1)^n}{n} \quad \text{and} \quad b_n := \left(1+\frac{1}{n}\right)^n.$$

Now, the sequence

$$|a_n| = \frac{1}{n}$$

is monotone decreasing and converges to 0. Therefore, the Alternating Series Test ensures that $\sum_{n=1}^{\infty} a_n$ is convergent. If we can show that (b_n) is bounded and monotone, it will follow from

Abel's test that the original series $\sum_{n=1}^{\infty} a_n b_n$ converges. Luckily for us, this is already a known fact! Certainly, it was proven Analysis 1 that

$$b_n = \left(1 + \frac{1}{n}\right)^n$$

is monotone increasing with $\lim b_n = e$.

Example 5.20. We prove that the series

$$\sum_{n=1}^{\infty} \left(\frac{2n^2 - 3n + 1}{4n^5 - 3} \sum_{k=1}^{n} \frac{1}{k^2} \right)$$

is convergent. With the hope of applying Abel's test, we will write this series as $\sum_{n=1}^{\infty} a_n b_n$ where

$$a_n := \frac{2n^2 - 3n + 1}{4n^5 - 3} \quad \text{and} \quad b_n := \sum_{k=1}^{n} \frac{1}{k^2}.$$

To successfully apply Abel's test, the following must be verified:

(i) $\sum_{n=1}^{\infty} a_n$ converges;

(ii) (b_n) is monotone;

(iii) (b_n) is bounded.

Let us first verify (ii)-(iii). Clearly, for each $n \in \mathbb{N}$,

$$b_n = \sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{n+1} \frac{1}{k^2} = b_{n+1}$$

whence (b_n) is monotone increasing. Now, every b_n is a partial sum for the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ which was shown to be convergent in the previous section (or tutorial). Consequently, (b_n) is convergent and hence bounded. Therefore, it only remains to establish (i). Here we will make use of the Limit Comparison Test. By the p-test, we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is convergent. Since

$$\lim \frac{\frac{2n^2-3n+1}{4n^5-3}}{1/n^3} = \lim \frac{n^3\left(2n^2 - 3n + 1\right)}{4n^5 - 3} = \lim \frac{2n^5 - 3n^4 + 1}{4n^5 - 3}$$
$$= \frac{1}{2} > 0$$

it follows from the Limit Comparison Test that $\sum_{n=1}^{\infty} a_n$ converges. By our previous remarks and Abel's test, we infer that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

5.5 Dirichlet's Test

Theorem 5.8 (Dirichlet's Test). *Let (a_n) be a decreasing sequence of real numbers such that* $\lim a_n = 0$. *Let (b_n) be a sequence such that there exists $M > 0$ with the property that*

$$\left|\sum_{n=1}^{N} b_n\right| \leq M$$

for all $N \geq 1$. Then,

$$\sum_{n=1}^{\infty} a_n b_n$$

is convergent.

Example 5.21. We will prove that the series

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\ln n}$$

is convergent using Dirichlet's test. For this, we define

$$a_n := \frac{1}{\ln n} \quad \text{and} \quad b_n := \cos(\pi n).$$

Clearly, (a_n) is monotone decreasing and converges to 0 as $n \to \infty$. Therefore, to apply Dirichlet's test, we need only check that there exists $M > 0$ such that

$$\left|\sum_{n=1}^{N} b_n\right| = \left|\sum_{n=1}^{N} \cos(\pi n)\right| \leq M$$

for all $N \in \mathbb{N}$. To see this, we note that

- $\sum_{n=1}^{1} \cos(\pi n) = \cos(\pi) = -1$;
- $\sum_{n=1}^{2} \cos(\pi n) = \cos(\pi) + \cos(2\pi) = -1 + 1 = 0$;
- $\sum_{n=1}^{3} \cos(\pi n) = \cos(\pi) + \cos(2\pi) + \cos(3\pi) = -1 + 1 - 1 = -1$;
- $\sum_{n=1}^{4} \cos(\pi n) = \cos(\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) = -1 + 1 - 1 + 1 = 0$.

By induction, it follows that

$$\left|\sum_{n=1}^{N} b_n\right| = \left|\sum_{n=1}^{N} \cos(\pi n)\right| \leq 1$$

for all $N \in \mathbb{N}$. Hence, Dirichlet's test applies.

Example 5.22. Consider the series

$$\sum_{n=1}^{\infty} \frac{c_n}{\sqrt{n}}$$

where $(c_n) = (1, -4, 1, 2, 1, -4, 1, 2, 1, \ldots)$. Using Dirichlet's test, we will prove that this series is convergent. Clearly, $\frac{1}{\sqrt{n}}$ is a decreasing sequence of real numbers converging to 0. It remains to show that the partial sums $\sum_{n=1}^{N} c_n$ are bounded in N. As before, we try to notice a pattern in these partial sums:

- For $N = 1$ we have $\sum_{n=1}^{N} c_n = 1$;
- For $N = 2$ we have $\sum_{n=1}^{N} c_n = 1 - 4 = -3$;
- For $N = 3$ we have $\sum_{n=1}^{N} c_n = 1 - 4 + 1 = -2$;
- For $N = 4$ we have $\sum_{n=1}^{N} c_n = 1 - 4 + 1 + 2 = 0$;
- For $N = 5$ we have $\sum_{n=1}^{N} c_n = 1 - 4 + 1 + 2 + 1 = 1$;

and so forth. Therefore,

$$\left| \sum_{n=1}^{N} c_n \right| \leq 3$$

for all $N \in \mathbb{N}$. Dirichlet's test thus yields the convergence of our series.

Example 5.23. Consider the series

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$$

Namely, we consider the harmonic series where each 3^{rd} term is negative. By way of contradiction, let us assume that this series converges. Let (S_N) denote the sequence of partial sums. By assumption, (S_N), and hence (S_{3N}), is convergent. Now, each S_{3N} is given by

$$S_{3N} = \sum_{n=1}^{N} \left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n} \right).$$

Since $\lim S_{3N}$ exists by assumption, we see that the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n} \right) = \sum_{n=1}^{\infty} \frac{9n^2 - 2}{3n(3n-2)(3n-1)}.$$

must be convergent. Note that every term of the series

$$\sum_{n=1}^{\infty} \frac{9n^2 - 2}{3n(3n-2)(3n-1)}$$

is non-negative. On the other hand,

$$\lim \frac{\frac{9n^2-2}{3n(3n-2)(3n-1)}}{\frac{1}{n}} = \lim \frac{9n^3-2n}{3n(3n-2)(3n-1)} = \frac{1}{3} > 0.$$

Using the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, we infer that the series

$$\sum_{n=1}^{\infty} \frac{9n^2-2}{3n(3n-2)(3n-1)}$$

is divergent. This contradiction shows that (S_N) cannot be convergent.

5.6 Weierstass M-Test

Theorem 5.9. *Let (f_n) be a sequence of real valued functions defined on a set $A \subseteq \mathbb{R}$ and assume that for each $n \in \mathbb{N}$ there exists $M_n > 0$ with the property that $|f_n(x)| \leq M_n$ for all $x \in A$. Then, if*

$$\sum_{n=1}^{\infty} M_n$$

converges, the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly and absolutely on A as $n \to \infty$.

Example 5.24. Prove that the following series

$$f(x) := \sum_{n=1}^{\infty} \frac{n^2+x^4}{n^4+x^2}$$

converges to a continuous function $f : \mathbb{R} \to \mathbb{R}$.

Proof. Let $R > 0$ be given and consider the interval $I = [-R, R]$. Given any $x \in I$, observe that

$$\begin{aligned} \left|\frac{n^2+x^4}{n^4+x^2}\right| \leq \frac{n^2+x^4}{n^4} &\leq \frac{n^2+R^4}{n^4} \\ &= \frac{1}{n^2} + \frac{R^4}{n^4}. \end{aligned}$$

If we define

$$M_n := \frac{1}{n^2} + \frac{R^4}{n^4}$$

, then it automatically follows from the p-test that

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2} + R^4 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

is convergent. Consequently, it follows from the Weierstass M-test that the series defining f converges absolutely and uniformly on I. Since the partial sums are continuous and uniform limits preserve continuity, we infer that f is continuous on $I = [-R, R]$. Since continuity is a local property and $R > 0$ was arbitrary, we see that f is continuous $\mathbb{R} \to \mathbb{R}$. □

Example 5.25. We claim that the series

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos(n)}{n}$$

are convergent.

Solution. We only handle the first series, as a similar argument applies to the latter. Observe that $2\sin(1) \neq 0$ and so, for each $N \in \mathbb{N}$,

$$\begin{aligned} 2\sin(1) \sum_{n=1}^{N} \sin(n) &= \sum_{n=1}^{N} 2\sin(1)\sin(n) \\ &= \sum_{n=1}^{N} \left[\cos(1-n) - \cos(1+n)\right] \\ &= \sum_{n=1}^{N} \left[\cos(n-1) - \cos(1+n)\right], \end{aligned}$$

where we have used the identity

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta).$$

Consequently,

$$2\sin(1) \sum_{n=1}^{N} \sin(n) = \cos(0) - \cos(1+N)$$

whence

$$\left| \sum_{n=1}^{N} \sin(n) \right| = \frac{|\cos(0) - \cos(1+N)|}{2\sin(1)} \leq \frac{2}{2\sin(1)} = \frac{1}{\sin(1)}.$$

is bounded independently of N. Consequently, the series converges by Dirichlet's test. Similarly, one can handle the series with cos. □