Analysis 2 (Math 243) Tutorial Notes

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1 First Tutorial

Let's begin by recalling some important topological notions. A set $U \subseteq \mathbb{R}$ is said to be *open* if, for each $x \in U$, there exists $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq U$. Here, $V_\varepsilon(x) := (x - \varepsilon, x + \varepsilon)$ is the open interval of radius ε centered at x.[1] A set $F \subseteq \mathbb{R}$ is called *closed* if its complement is open, i.e. if $\mathbb{R} \setminus F$ is open in $\mathbb{R}$. Finally, a set $K \subseteq \mathbb{R}$ is called compact provided it is closed and bounded.[2] We also recall the following properties:

- Arbitrary unions of open sets are open. That is, if $\{U_\alpha\}_{\alpha\in I}$ is an indexed family of open sets, then $\bigcup_{\alpha\in I} U_\alpha$ is open.
- Finite intersections of open sets are open. More precisely, if $U_1, \ldots, U_n$ are open, then so is $\bigcup_{k=1}^n U_k$.
- Finite unions of closed sets are closed. Namely, given closed sets $F_1, \ldots, F_n$ in $\mathbb{R}$, their union $\bigcup_{k=1}^n F_k$ is also closed.
- Arbitrary intersections of closed sets remain closed, i.e. $\bigcap_{\alpha\in I} F_\alpha$ is closed if each F_α is closed.

We also point out that these topological notions will interact very nicely with our definitions of continuity and convergence.

Remark 1.1. It should be noted that the definitions of open and closed sets are in no way "opposites". That is, it is not true that a subset $A \subseteq \mathbb{R}$ is either open or closed. For example, the set of rational numbers $\mathbb{Q}$ is neither open nor closed. Therefore, if $A \subseteq \mathbb{R}$ is not open, it does *not* follow that A is closed. Similarly, A not being closed does not mean that A is open!

1.1 Uniform Continuity

Given a subset $A \subseteq \mathbb{R}$ and a function $f : A \to \mathbb{R}$, we say that f is *uniformly continuous* on A if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ satisfy $|x - y| < \delta$. In symbolic terms:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in A)(|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon).$$

[1] $V_\varepsilon(x)$ is often called an ε-neighbourhood of x. Therefore, the set U is called open if for every point $x \in U$, there exists a neighbourhood of x contained in U.

[2] Note that none of these sets are assumed to be intervals. For example, $(1, 2) \cup (3, 4)$ is open but not an interval. Similarly, $\{0, 1\}$ is closed and is also not an interval.

At a first glance this may seem to be the same as saying that f is continuous on A, but this is not the case. By way of clarification, let us compare our definition of *uniform continuity* to our $\varepsilon - \delta$ definition of *continuity*.

- Continuity is a *local property* and uniform continuity is a *global property*. We know what it means for the function f to be continuous at an individual point $a \in A$, but we have only defined uniform continuity for the entire set A. Namely, the continuity of f at a single point a depends only on the behaviour of f in a neighbourhood of a. On the other hand, uniform continuity depends on how f behaves on the entire set A.

- We say that f is continuous at a point $y \in A$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)(|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon).$$

  Thus, we say that f is continuous on the *entire set* A if

$$(\forall y \in A)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)(|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon).$$

  In this expression, the variable y appears before the "$\exists \delta$", which means that δ may depend on the point y. However, this is not the case in the definition of uniform continuity. For f to be uniformly continuous on A, the $\delta > 0$ cannot depend on x or y.

Let us also recall the following fundamental result from class.

Theorem 1.1 (Uniform Continuity Theorem)**.** *Let $K \subset \mathbb{R}$ be compact and let $f : K \to \mathbb{R}$ be continuous. Then, f is uniformly continuous on K.*

If K is not compact, this implication may fail. However, there are still many nice continuous functions that turn out to be uniformly continuous on unbounded intervals. One example is the function

$$f : [0, \infty) \to \mathbb{R}, \quad x \mapsto \sqrt{x}.$$

Example 1.1. We claim that the function f described above is uniformly continuous on $[0, \infty)$. First, let us establish the following useful inequality:

$$\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|}, \quad \forall x, y \geq 0. \tag{1.1}$$

By symmetry, it is enough to show that the above holds whenever $0 \le y \le x$. For such x, y, the inequality in (1.1) reduces to proving that

$$\begin{aligned}
\sqrt{x} - \sqrt{y} \le \sqrt{x-y} &\iff \left(\sqrt{x} - \sqrt{y}\right)^2 \le (x-y) \\
&\iff x - 2\sqrt{xy} + y \le x - y \\
&\iff -2\sqrt{xy} + y \le -y \\
&\iff 2y \le 2\sqrt{xy} \\
&\iff y \le \sqrt{xy}.
\end{aligned}$$

This last inequality holds true because $y \le x$ implies

$$y = \sqrt{y^2} = \sqrt{y \cdot y} \le \sqrt{x \cdot y}.$$

Thus, we have proven (1.1). It now becomes trivial to show that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$. To see this, let $\varepsilon > 0$ be given and take $\delta := \varepsilon^2 > 0$. If $x, y \in [0, \infty)$ are such that $|x - y| < \delta$, then (1.1) tells us that

$$|f(x) - f(y)| = \left|\sqrt{x} - \sqrt{y}\right| \le \sqrt{|x-y|} < \sqrt{\delta} = \varepsilon.$$

Note our δ only depends on ε and **not** on x or y.

We now check that uniformly continuous functions behave as expected under composition. Namely, we claim that compositions of uniformly continuous functions are again uniformly continuous.

Proposition 1.2. *Let $A, B \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be uniformly continuous. Assume that $g : B \to \mathbb{R}$ is uniformly continuous and that $f(A) \subseteq B$. Then, the function*

$$g \circ f : A \to \mathbb{R}$$

is also uniformly continuous.

Proof. Let $\varepsilon > 0$ be given. Since g is uniformly continuous on B, there exists $\delta > 0$ such that $|g(x) - g(y)| < \varepsilon$ whenever $x, y \in B$ are such that $|x - y| < \delta$. Using instead that f is uniformly continuous, for this $\delta > 0$, we can find $\delta' > 0$ such that

$$|f(x) - f(y)| < \delta$$

whenever $x, y \in A$ satisfy $|x-y| < \delta'$. Therefore, given $x, y \in A$ with $|x-y| < \delta'$ we have

$$|f(x) - f(y)| < \delta$$

so that $|g(f(x)) - g(f(y))| < \varepsilon$. That is,

$$|(g \circ f)(x) - (g \circ f)(y)| < \varepsilon$$

for all $x, y \in A$ with $|x-y| < \delta'$. We infer that $g \circ f$ is uniformly continuous. □

1.1.1 A Return to Cauchy Sequences

Let $f : A \to \mathbb{R}$ be a uniformly continuous function and let (x_n) be a sequence in A. If (x_n) is Cauchy, then the induced sequence $(f(x_n))$ turns out to be *also* Cauchy. In this sense, we say that f takes Cauchy sequences to Cauchy sequences. This is **not** a property satisfied by functions that are merely continuous. Indeed, consider the function

$$f : (0,1) \to \mathbb{R}, \quad x \mapsto \frac{1}{x}.$$

Clearly, f is continuous on its entire domain. Now, the sequence (x_n) in $(0,1)$ defined by $x_n := \frac{1}{n}$ converges to 0 and must therefore be Cauchy. However, we have $f(x_n) = n$ for each $n \in \mathbb{N}$. Hence, the sequence $(f(x_n))$ is not Cauchy as it is unbounded.

Proposition 1.3. *Let $A \subseteq \mathbb{R}$ be non-empty and let $f : A \to \mathbb{R}$ be uniformly continuous. If (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is also Cauchy.*

Proof. Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in A$ satisfying $|x-y| < \delta$. Now, using that (x_n) is Cauchy in A, there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \delta$ whenever $n, m \geq N$. Consequently,

$$|f(x_n) - f(x_m)| < \varepsilon$$

for all $n, m \geq N$. As $\varepsilon > 0$ was arbitrary, it follows that $(f(x_n))$ is Cauchy. □

Next we ask whether the converse is true. Namely, if a function maps Cauchy sequences to Cauchy sequences, must it be uniformly continuous? To help answer this question, we consider a continuous function on an interval that is not uniformly continuous.

Lemma 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := x^2$. The function f is not uniformly continuous on $\mathbb{R}$, but is uniformly continuous on any compact interval $[-a, a]$.*

Proof. The only non-trivial claim is that f is not uniformly continuous on all of $\mathbb{R}$. To see this, we argue by contradiction. If f is uniformly continuous on $\mathbb{R}$, then for $\varepsilon := 1$ we can find $\delta > 0$ such that

$$\left|x^2 - y^2\right| < 1$$

whenever $|x - y| < \delta$. Given $x \in \mathbb{R}$ we consider the point

$$y := x + \frac{\delta}{2}.$$

It is clear that $|x - y| < \delta$, and we must therefore have

$$\left|x^2 - \left(x + \frac{\delta}{2}\right)^2\right| < 1.$$

However, the expression above reduces to the following:

$$\left|x\delta + \frac{\delta^2}{4}\right| < 1.$$

Since $x \in \mathbb{R}$ was arbitrary, the above implies that the function

$$g(x) := x\delta + \frac{\delta^2}{4}$$

is bounded on $\mathbb{R}$. Clearly, this is a contradiction. □

Having exhibited a continuous function $\mathbb{R} \to \mathbb{R}$ that is not uniformly continuous, we can answer our original question.

Example 1.2. We claim that the function $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$ takes Cauchy sequences in $\mathbb{R}$ to Cauchy sequences. To see this, let (x_n) be a Cauchy sequence. Since Cauchy sequences are bounded, the sequence (x_n) is contained within some compact interval $[-a, a]$. That is, (x_n) is a Cauchy sequence in $[-a, a]$. By continuity on all of $\mathbb{R}$, the function f is uniformly continuous on $[-a, a]$. Since uniformly continuous functions take Cauchy sequences to Cauchy sequences, we see that $(f(x_n))$ is Cauchy in $\mathbb{R}$.

Thus, we have found a continuous map $\mathbb{R} \to \mathbb{R}$ that takes Cauchy sequences to Cauchy sequences, even though it is *not* uniformly continuous on $\mathbb{R}$. Moving away from examples, let us now prove some more powerful/general statements.

Proposition 1.5. *Let $A \subseteq \mathbb{R}$ be non-empty and let $f, g : A \to \mathbb{R}$ be uniformly continuous on A.*

(1) $f + g$ is uniformly continuous as a function $A \to \mathbb{R}$.

(2) The product fg may not be uniformly continuous on A.

Proof. We begin with the first claim. Let $\varepsilon > 0$ be given and let $\delta_1 > 0$ be such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2}$$

whenever $x, y \in A$ satisfy $|x - y| < \delta_1$. Similarly, because g is uniformly continuous on A, we can find $\delta_2 > 0$ with the property that $|x - y| < \delta_2$ (with $x, y \in A$) implies

$$|g(x) - g(y)| < \frac{\varepsilon}{2}.$$

Taking $\delta := \min\{\delta_1, \delta_2\} > 0$, we see that for any $x, y \in A$ with $|x - y| < \delta$ we have:

$$\begin{aligned} |(f+g)(x) - (f+g)(y)| &= |f(x) + g(x) - f(y) - g(y)| \\ &\leq |f(x) - f(y)| + |f(y) - g(y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

This means that $f+g$ is uniformly continuous on A. For the second part, consider the functions $f(x) = g(x) = x$ on all of $\mathbb{R}$. Clearly, both f and g are uniformly continuous but their product $fg = x^2$ is *not* (by our previous lemma). ◻

1.2 Uniformly Continuous Functions Are Bounded

Given a non-compact *bounded* subset A of $\mathbb{R}$, a continuous function $f : A \to \mathbb{R}$ need not be bounded. Indeed, this is the case for the continuous map

$$f : (0, 1] \to \mathbb{R}, \quad x \mapsto \frac{1}{x}.$$

Thus, continuous functions need not (in general) be bounded. However, it is true that uniformly continuous functions on bounded domains are bounded. We prove this below.

Theorem 1.6. *Let $A \subset \mathbb{R}$ be bounded and suppose that $f : A \to \mathbb{R}$ is uniformly continuous on A. Then, f is bounded on A.*

Proof. We argue by contradiction. If f were unbounded on A, we could find a sequence (x_n) in A such that $|f(x_n)| > n$ for each $n \in \mathbb{N}$. Since A is bounded, the sequence (x_n) is bounded. Thus, we can find a subsequence (x_{n_k}) that converges in $\mathbb{R}$. Equivalently, (x_{n_k}) is a Cauchy sequence in A such that

$$\left|f(x_{n_k})\right| > n_k \geq k$$

for all $k \in \mathbb{N}$. However, since f is uniformly continuous on A, we see that $(f(x_{n_k}))$ is Cauchy in $\mathbb{R}$. In particular, the sequence $(f(x_{n_k}))_k$ is bounded. This contradicts the fact that

$$\left|f(x_{n_k})\right| > n_k \geq k.$$

Hence, we conclude that f is bounded on A. □

Remark 1.2. Note that the boundedness assumption on the domain of f cannot be dropped. That is, a uniformly continuous function $f : A \to \mathbb{R}$ may be unbounded if A is unbounded. This is the case for $f(x) = x$ and $A = \mathbb{R}$.

1.3 Lipschitz Continuity

Lipschitz continuity is an additional continuity condition one can impose on a function that is stronger than uniform continuity.

Definition 1.1. Let A be a non-empty subset of $\mathbb{R}$. A function $f : A \to \mathbb{R}$ is said to be *Lipschitz continuous* on A if there exists $L > 0$ such that

$$|f(x) - f(y)| \leq L\,|x - y|$$

for all $x, y \in A$. The set of all Lipschitz continuous functions $A \to \mathbb{R}$ is often denoted by $\mathrm{Lip}(A;\mathbb{R})$ or $\mathrm{Lip}(A)$.

It is automatic that Lipschitz continuous functions are uniformly continuous on their domains. Indeed, given $f \in \mathrm{Lip}(A)$ and $\varepsilon > 0$, we define $\delta = \frac{\varepsilon}{L}$ and notice that if $x, y \in A$ satisfy $|x - y| < \delta$ then

$$|f(x) - f(y)| \leq L\,|x - y| < L\delta = \varepsilon.$$

One can extend the concept of a Lipschitz function to that of a Hölder continuous function. We make this precise below.

Definition 1.2. Let $A \subseteq \mathbb{R}$ be non-empty and let $f : A \to \mathbb{R}$ be a function. We say that f is Hölder continuous on A with exponent $\alpha > 0$ if there exists $M > 0$ such that

$$|f(x) - f(y)| \leq M |x - y|^{\alpha}$$

for all $x, y \in A$.

Note that $f \in \mathrm{Lip}(A; \mathbb{R})$ if and only if f is Hölder continuous on A with exponent $\alpha = 1$.

Exercise 1.1. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be Hölder continuous on A with exponent $\alpha > 0$. Prove that f is uniformly continuous on A.

Let us now give some examples of Lipschitz/Hölder continuous functions.

(i) By the reverse triangle inequality, the function $f(x) := |x|$ is Lipschitz on $\mathbb{R}$. Indeed, note that

$$||x| - |y|| \leq |x - y|$$

for all $x, y \in \mathbb{R}$.

(ii) In (1.1) we showed that

$$\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|} = |x - y|^{1/2}, \quad \forall x, y \in [0, \infty).$$

Thus, we were really proving that $f(x) := \sqrt{x}$ is Hölder continuous on $[0, \infty)$ with exponent $\alpha = \frac{1}{2}$.

(iii) The function $f(x) = \sqrt{x}$ is *not* Lipschitz on $[0, 1]$. To see this, we argue by contradiction. If it were Lipschitz on $[0, 1]$, we could find $L > 0$ such that

$$\left|\sqrt{x} - \sqrt{y}\right| \leq L |x - y|, \quad \forall x, y \in [0, 1].$$

Thus,

$$\sup_{\substack{x, y \in [0,1] \\ x \neq y}} \frac{\left|\sqrt{x} - \sqrt{y}\right|}{|x - y|} \leq L < \infty.$$

In particular (taking $y = 0$),

$$\sup_{\substack{x\in[0,1]\\x\neq 0}} \frac{1}{\sqrt{x}} = \sup_{\substack{x\in[0,1]\\x\neq 0}} \frac{\sqrt{x}}{x} \leq \sup_{\substack{x,y\in[0,1]\\x\neq y}} \frac{\left|\sqrt{x}-\sqrt{y}\right|}{|x-y|} \leq L < \infty.$$

Since $\frac{1}{\sqrt{x}}$ is unbounded on $(0, 1]$, we have a contradiction.

1.4 *Differentiation*

We now introduce the notion of a derivative. Let $I \subseteq \mathbb{R}$ be an interval in $\mathbb{R}$ and assume that $f : I \to \mathbb{R}$ is a function.

Definition 1.3. The function f is said to be *differentiable* at a point $c \in I$ provided the limit

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

exists. If it does, we denote this limit by $f'(c)$ and call $f'(c)$ the *derivative* of f at c. If $f'(c)$ exists for all $c \in I$ we say that f is differentiable on I. In this case, the map $x \mapsto f'(x)$ is a well defined function on I that we call the *derivative* of f.[3]

Let's also take a moment to recall the following observation from the lectures:

Proposition 1.7. *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a function. If f is differentiable at $c \in I$, then f is continuous at $c \in I$.*

Proof. For all $x \in I$ with $x \neq c$ one may write

$$f(x) - f(c) = \frac{f(x)-f(c)}{x-c} \cdot (x - c)$$

whence

$$\lim_{x\to c} (f(x) - f(c)) = \lim_{x\to c} \left(\frac{f(x)-f(c)}{x-c}\right) \lim_{x\to c}(x - c) = f'(c) \cdot 0 = 0.$$

That is, $\lim_{x\to c} f(x) = f(c)$ and therefore f is continuous at c. □

[3]In the case where c is an endpoint of I, the limit

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

should be interpreted as a one-sided limit.

A few simple examples are in order:

Example 1.3. Consider the function $f(x) = x^2$ on $\mathbb{R}$ and fix $c \in \mathbb{R}$. For each $x \in \mathbb{R}$ different from c, we obtain

$$\frac{f(x) - f(c)}{x - c} = \frac{x^2 - c^2}{x - c} = \frac{(x - c)(x + c)}{x - c} = x + c$$

so that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c}(x + c) = 2c.$$

Therefore, $f'(c) = 2c$ for all $c \in \mathbb{R}$. We infer that f is differentiable on $\mathbb{R}$ with $f'(x) = 2x$.

Example 1.4. Consider the function $f(x) := \sqrt{x}$ defined on $(0, \infty)$ with

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Fixing $c \in (0, \infty)$ and letting $x \neq c$ be positive we calculate

$$\begin{aligned} \frac{f(x) - f(c)}{x - c} = \frac{\sqrt{x} - \sqrt{c}}{x - c} &= \frac{\sqrt{x} - \sqrt{c}}{x - c} \cdot \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} \\ &= \frac{x - c}{(x - c)(\sqrt{x} + \sqrt{c})} \\ &= \frac{1}{\sqrt{x} + \sqrt{c}}. \end{aligned}$$

So, since $\sqrt{x} \to \sqrt{c}$ as $x \to c$, the limit laws yield

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \frac{1}{2\sqrt{c}}, \quad \forall c \in (0, \infty).$$

Therefore f is differentiable on $(0, \infty)$ with $f'(x) = \frac{1}{2\sqrt{x}}$.

Example 1.5. We claim that the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = 1/x$ is also differentiable. Indeed, for any $c > 0$ and $x > 0$ different from c, we see that

$$\frac{f(x) - f(c)}{x - c} = \frac{\frac{1}{x} - \frac{1}{c}}{x - c} = \frac{c - x}{xc} \frac{1}{x - c} = -\frac{1}{xc}.$$

Therefore,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = -\frac{1}{c^2}.$$

We conclude that f is indeed differentiable with derivative given by $f'(x) = -\frac{1}{x^2}$.

If $f : I \to \mathbb{R}$ is differentiable and Lipschitz continuous on the interval I, it is possible to bound the derivative f' on the entire interval I. Indeed, we confirm this below.

Proposition 1.8. *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be differentiable on I. If f is Lipschitz continuous on I, then f' is bounded on I. More precisely, if there exists a constant $L > 0$ such that*

$$|f(x) - f(y)| \leq L\,|x - y|, \quad \forall x, y \in I,$$

then one has $|f'(x)| \leq L$ for all $x \in I$.

Remark 1.3. We will soon see that the converse is also true. That is, if $f : I \to \mathbb{R}$ is differentiable on I and f' is bounded on I, then f must be Lipschitz continuous there too.

Proof. Fix a point $y \in I$. Since $f'(y)$ exists, we know that the limit

$$f'(y) = \lim_{x \to y} \frac{f(x) - f(y)}{x - y}$$

is defined. Therefore,

$$|f'(y)| = \lim_{x \to y} \left| \frac{f(x) - f(y)}{x - y} \right| \leq \lim_{x \to y} L = L.$$

Since $y \in I$ was arbitrary, the assertion follows. □

2 Second Tutorial

In relation to the first assignment, we begin by checking that the function

$$f : \mathbb{R} \to \mathbb{R}$$

given by $f(x) = x^{4/3}$ is not uniformly continuous on $\mathbb{R}$. To this end, we construct two sequences (x_n) and (y_n) in $\mathbb{R}$ such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq 1$ for all $n \in \mathbb{N}$. Consider the sequences

$$x_n = n^{3/4} \quad \text{and} \quad y_n := n^{3/4} + \frac{1}{n^{1/4}}$$

Clearly, $|x_n - y_n| = n^{-1/4} \to 0$ as $n \to \infty$. Furthermore, for each $n \in \mathbb{N}$, a straightforward calculation gives

$$\begin{aligned} f(y_n) = \left(\left[n^{3/4} + \frac{1}{n^{1/4}}\right]^4\right)^{1/3} &= \left(n^3 + 4n^2 + 6n + 4 + \frac{1}{n}\right)^{1/3} \\ &> \left(n^3 + 4n^2 + 6n + 4\right)^{1/3} \\ &> \left(n^3 + 3n^2 + 3n + 1\right)^{1/3} \\ &= (n+1)^{3/3} \\ &= n + 1. \end{aligned}$$

In particular, $f(y_n) > n = f(x_n)$. Therefore,

$$|f(y_n) - f(x_n)| = f(y_n) - f(x_n) > (n+1) - n = 1.$$

We infer that f is **not** uniformly continuous on $\mathbb{R}$.

We now return to the notion of differentiability. Let $I \subseteq \mathbb{R}$ be an interval; recall that a function $f : I \to \mathbb{R}$ is said to be differentiable at a point $c \in I$ provided the limit

$$f'(c) := \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. Let us also take a moment to recall the very powerful mean value theorem:

Theorem 2.1 (Mean Value Theorem)**.** *Let $[a, b] \subset \mathbb{R}$ be a bounded interval and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, and differentiable on (a, b). Then, there exists a point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Namely, f' achieves the average change of f on $[a, b]$.

Remark 2.1. Note that the converse does not hold true. More precisely, given a point $c \in (a, b)$, we cannot guarantee that there exist points $x, y \in [a, b]$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}.$$

Indeed, consider the function $f(x) = x^3$ on $[-1, 1]$. Clearly, $f'(0) = 0$ but, since f is strictly monotone increasing and injective,

$$\frac{f(x) - f(y)}{x - y} \neq 0$$

for all $x, y \in [-1, 1]$ with $x \neq y$.

Using Theorem 2.1, we showed that a differentiable function $f : I \to \mathbb{R}$ is increasing (perhaps not strictly) on I if and only if $f'(x) \geq 0$ for all $x \in I$. Of course, such a characterization cannot possibly hold for increasing functions that are not everywhere differentiable.

Corollary 2.2. *Let $f, g : [0, \infty) \to \mathbb{R}$ be differentiable on $[0, \infty)$ and assume that $f'(x) \leq g'(x)$ for all $x \in [0, \infty)$. If $f(0) = g(0)$, then $f \leq g$ on all of $[0, \infty)$.*

This result says that if g increases faster than f and both g and f "start" at the same value, then g is always at least as large as f.

Proof of Corollary. Define $h(x) := g(x) - f(x)$. Clearly, h is differentiable on $[0, \infty)$ and

$$h'(x) = g'(x) - f'(x) \geq 0$$

for all $x \in [0, \infty)$. It follows that h is increasing on $[0, \infty)$. Therefore, $h(x) \geq h(0) = 0$ for all $x \geq 0$. Put otherwise,

$$g(x) - f(x) \geq 0$$

for all $x \geq 0$. This completes the proof. □

Example 2.1. Let $f : I \to \mathbb{R}$ be differentiable on $\mathbb{R}$ and fix a point $c \in \mathbb{R}$ such that $f(c) = 0$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = |f(x)|$. We claim that g is differentiable at c if and only if $f'(c) = 0$. First, if $f'(c) = 0$, then by the reverse triangle inequality,

$$\begin{aligned}\left|\frac{g(x) - g(c)}{x - c} - 0\right| = \left|\frac{|f(x)| - |f(c)|}{x - c}\right| &\leq \frac{|f(x) - f(c)|}{|x - c|} \\ &= \left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| \to 0,\end{aligned}$$

as $x \to c$. Consequently, $g'(c)$ exists and is equal to 0. Conversely, assume that $g'(c)$ exists. Since $g(c) = 0$ and g is non-negative, we see that c is a local minimum of the function g. As g is assumed to be differentiable at c, we must have $g'(c) = 0$. We now deduce that $f'(c) = 0$. Certainly, for all $x \neq c$,

$$\left|\frac{f(x) - f(c)}{x - c}\right| = \frac{|f(x)|}{|x - c|} = \frac{|g(x)|}{|x - c|} = \left|\frac{g(x) - g(c)}{x - c}\right| \to 0$$

as $x \to c$ since $g'(c) = 0$. It follows that $f'(c) = 0$.

Having treated the product and chain rules for derivatives, it is natural to ask what differentiation rule holds for inverse functions. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. By the preservation of intervals theorem, $J := f(I)$ is an interval in $\mathbb{R}$. Moreover, there exists a continuous inverse function $g : J \to I$ to f that is both strictly monotone and continuous on J (see Theorem 5.6.5 in Bartle & Sherbert).

Theorem 2.3 (Inverse Rule for Derivatives)**.** *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be continuous and strictly monotone increasing. Define $J := f(I)$ and let $g : J \to I$ be the continuous and strictly monotone inverse to f. If f is differentiable at $c \in I$ and $f'(c) \neq 0$ then g is also differentiable at $d = f(c)$. Moreover,*

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}.$$

Proof. By Carathéodory's criterion, there exists a function $\varphi : I \to \mathbb{R}$ continuous at c with $\varphi(c) = f'(c)$ such that

$$f(x) - f(c) = \varphi(x)(x - c)$$

for all $x \in I$. By assumption, $\varphi(c) \neq 0$. Since φ is continuous at c and $\varphi(c) \neq 0$, there exists $\delta > 0$ such that $\varphi(x) \neq 0$ on $V_\delta(c) \cap I$. Define now[4]

$$U := f(V_\delta(c) \cap I) \subseteq f(I) = J.$$

By definition of the inverse function g, one has $f(g(y)) = y$ for all $y \in J$. In particular, for all $y \in U$ there holds

$$\begin{aligned} y - d = f(g(y)) - f(c) &= \varphi(g(y))(g(y) - c) \\ &= \varphi(g(y))(g(y) - g(d)). \end{aligned}$$

[4]Since f is strictly increasing and continuous, by the preservation of intervals theorem, U is a non-trivial interval containing d.

Since $\varphi \neq 0$ on $V_\delta(c) \cap I$, it readily follows that

$$g(y) - g(d) = \frac{1}{\varphi(g(y))}\,(y - d)$$

for all $y \in U$. Because $\varphi(g(y))$ is continuous at $y = d$, appealing once more to Carathéodory's criterion shows that $g'(d)$ exists and

$$g'(d) = \frac{1}{\varphi(g(d))} = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}.$$

□

2.1 Further Applications of the Mean Value Theorem

It is natural to ask how the behaviour of f' away from these endpoints affects the existence of the one-sided derivatives $f'(a)$ and $f'(b)$. As the next result shows, the existence of the one-sided limits of f' at a or b is enough to guarantee the existence of $f'(a)$ and $f'(b)$, respectively.

Proposition 2.4. *Let $a < b$ be real numbers and assume that $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) and continuous on $[a, b]$. If*

$$\lim_{x \to a^+} f'(x) = L_1$$

exists, then $f'(a) = L_1$. Similarly, if

$$\lim_{x \to b^-} f'(x) = L_2$$

then $f'(b) = L_2$.

Proof. Assume that $f'(x) \to L_1$ as $x \to a^+$. We want to show that

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = L_1.$$

Let $\varepsilon > 0$ be given; using that $\lim_{x \to a^+} f'(x) = L_1$, we can find $\delta > 0$ such that

$$|f'(x) - L_1| < \varepsilon$$

whenever $0 < x - a < \delta$.[5] For any such x, since f is continuous on $[a, x] \subseteq [a, b]$ and differentiable on (a, x), the mean value theorem guarantees the existence of a point $c_x \in (a, x)$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(c_x).$$

Since $0 < c_x - a < x - a < \delta$, we obtain

$$\left| \frac{f(x) - f(a)}{x - a} - L_1 \right| = |f'(c_x) - L_1| < \varepsilon.$$

We have therefore shown that

$$\left| \frac{f(x) - f(a)}{x - a} - L_1 \right| < \varepsilon$$

whenever $0 < x - a < \delta$. This proves that $f'(a) = L_1$. The second part can be verified by a symmetric argument. □

Consider now a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable everywhere. In particular, f is continuous on all of $\mathbb{R}$. If we impose the additional requirement that

$$\lim_{|x| \to \infty} f(x) = 0,$$

then intuitively the function f should "taper off" at infinity. Consequently, one may wonder whether one can also say that f' vanishes as $|x| \to \infty$. Unfortunately, one can make no such deduction about the derivative of f. Despite this, we have the following:

Proposition 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere and assume that f is bounded on $\mathbb{R}$.[6] Then, there exists a sequence (c_n) in $(0, \infty)$ with $c_n \to \infty$ such that*

$$\lim_{n \to \infty} f'(c_n) = 0.$$

Similarly, there exists a sequence (d_n) in $(-\infty, 0)$ with $d_n \to -\infty$ such that

$$\lim_{n \to \infty} f'(d_n) = 0.$$

[5] As we are taking the limit as $x \to a$ from "above", we are only considering points $x > a$. Therefore, no absolute values are needed here!

[6] This boundedness condition is in particular satisfied when $\lim_{|x| \to \infty} f(x) = 0$.

Proof. We only construct (c_n), leaving the construction of (d_n) as a similar exercise. For each $n \in \mathbb{N}$, the function f is differentiable on $(n, 2n)$ and continuous on $[n, 2n]$. Appealing to the Mean Value Theorem, there exists a point $c_n \in (n, 2n)$ such that
$$f'(c_n) = \frac{f(2n) - f(n)}{2n - n} = \frac{f(2n) - f(n)}{n}.$$
Taking absolute values and using the triangle inequality then gives
$$\begin{aligned} |f'(c_n)| = \frac{|f(2n) - f(n)|}{n} &\leq \frac{|f(2n)| + |f(n)|}{n} \\ &\leq \frac{2M}{n}, \end{aligned}$$
where $M > 0$ is such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. By the Squeeze Theorem, it is readily seen that $f'(c_n) \to 0$ as $n \to \infty$. Finally, $c_n \in (n, 2n)$ implies that $c_n > n$ for all $n \in \mathbb{N}$. Clearly, this means that $c_n \to \infty$ as $n \to \infty$. □

Before proceeding further, let us take a moment to justify an inequality that we have previously made use of during the course (and the previous Math 242).

Example 2.2. We claim that $|\sin(x)| \leq x$ for all $x \in \mathbb{R}$. Certainly, this is almost an immediate consequence of the mean value theorem. Note that this inequality is trivially true at $x = 0$. If $x > 0$, then the function $\sin(x)$ is continuous on $[0, x]$ and differentiable on $(0, x)$. Therefore, there exists by the mean value theorem a point $c_x \in (0, x)$ such that
$$\cos(c_x) = \frac{\sin(x) - \sin(0)}{x - 0} = \frac{\sin(x)}{x}.$$
Therefore,
$$|\sin(x)| = |\cos(c_x)|\,|x| \leq |x|\,.$$
If $x < 0$, then $-x > 0$. Therefore, we may write
$$|\sin(x)| = |-\sin(-x)| = |\sin(-x)| \leq |-x| = |x|\,.$$

2.2 *Interesting Examples and Counter Examples*

In this subsection we consider some classical examples and counter examples. First, we provide an explicit function that is differentiable only at a single point. In fact, this point will be the only continuity point of this function. This means that, although differentiability is a local property, a function f being differentiable at a point c does not guarantee that f is differentiable at even a single point different from c.

Example 2.3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by the rule

$$f(x) := \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We claim that $f'(0)$ exists, but that f is *not* differentiable at any $c \neq 0$. Fix $\varepsilon > 0$ and let $\delta := \varepsilon$. If $0 < |x| < \delta$, then we obtain

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| \frac{f(x)}{x} \right| = \begin{cases} \frac{x^2}{x} & \text{if } x \in \mathbb{Q}, \\ \frac{0}{x} & \text{if } x \notin \mathbb{Q}. \end{cases}$$

This gives

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| \leq |x| < \delta = \varepsilon$$

for all $x \in \mathbb{R}$ with $0 < |x| < \delta$. It follows that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$$

whence $f'(0) = 0$. It remains to show that f' does not exist at any point other than 0. For this, it is enough to show that f is discontinuous at any point $c \neq 0$. If $c \in \mathbb{Q}$, then $f(c) = c^2 \neq 0$. But, because $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, we can find a sequence (ξ_n) of irrational numbers such that $\lim \xi_n = c$. If f were continuous at c, the sequential criterion for continuity would imply that

$$c^2 = f(c) = \lim_{n \to \infty} f(\xi_n) = \lim_{n \to \infty} 0 = 0$$

which is a contradiction. Consequently, f cannot be continuous at any $c \in \mathbb{Q}$ different from 0. Similarly, if $c \in \mathbb{R} \setminus \mathbb{Q}$ and $c \neq 0$, we have $f(c) = 0$. As above, because $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence (r_n) of rational numbers converging to c as $n \to \infty$. If f were continuous at c, the sequential criterion would yield

$$0 = f(c) = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} r_n^2 = c^2 \neq 0.$$

Since this is also a contradiction, we see that f is discontinuous at any $c \in \mathbb{R} \setminus \mathbb{Q}$ with $c \neq 0$. Combining these two cases implies that f is discontinuous at any $c \neq 0$ and hence cannot be differentiable at any of these points.

2.3 On the Continuity of Derivatives (Optional and Time Permitting)

Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be differentiable on I. In particular, f' is a well defined function on I. We say that f is *continuously differentiable* on I provided f' is continuous on I. To say that f is continuously differentiable on I, we sometimes write $f \in C^1(I)$. More generally, $f \in C^k(I)$ if f is k-times differentiable on I and $f', f'', \ldots, f^{(k)}$ are continuous on I.

Example 2.4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that f is differentiable on $\mathbb{R}$, but that f' is not continuous at 0. Hence, not every differentiable function is continuously differentiable. First, note that since $x^2 \sin\left(\frac{1}{x}\right)$ is differentiable away from 0, so must be the function f. In fact, by the product and chain rules, we have

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), \quad \forall x \neq 0.$$

However, it is not obvious that $f'(0)$ exists. To see this, we first fix $\varepsilon > 0$. Take $\delta := \varepsilon$ and let $0 < |x| < \delta$. A straightforward calculation gives

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \frac{x^2 \left|\sin\left(\frac{1}{x}\right)\right|}{|x|} \leq \frac{x^2}{|x|} = |x| < \delta = \varepsilon.$$

Therefore,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$$

so that $f'(0) = 0$. We infer that f is differentiable on $\mathbb{R}$. However, f' is *not* continuous at 0 because

$$\lim_{x \to 0} f'(x)$$

does not exist. To see this, consider the sequence (x_n) given by

$$x_n := \frac{1}{\pi n}.$$

Clearly, $x_n \neq 0$ and $x_n \to 0$ as $n \to \infty$. Furthermore, for every $n \in \mathbb{N}$,

$$f'(x_n) = \frac{2}{\pi n} \sin(\pi n) - \cos(\pi n) = -\cos(\pi n) = (-1)^{n+1}$$

which does not converge. Hence,

$$\lim_{n\to\infty} f'(x_n)$$

does not exist. If f' were continuous at 0, however, we would have

$$\lim_{n\to\infty} f'(x_n) = f'(0).$$

This last example opens up a new question: which conditions *do* guarantee the continuity of the derivative on an interval I? Clearly, mere differentiability is not enough. However, as we will see below, *uniform differentiability* is sufficient.

Definition 2.1. Let $I \subseteq \mathbb{R}$ be an interval. A differentiable function $f : I \to \mathbb{R}$ is said to be uniformly differentiable on I if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon$$

for all $x, y \in I$ with $0 < |x - y| < \delta$.[7]

The definition of uniform differentiability can be compared to that of uniform continuity. Indeed, in the above, both x and y are allowed to "vary" when taking the limit. Namely, we are asking that the $\delta > 0$ obtained above be independent of the point y at which we are taking the derivative.

Proposition 2.6. *Let $f : I \to \mathbb{R}$ be uniformly differentiable on I. Then, f' is uniformly continuous on I. Put otherwise, f is continuously differentiable on I.*

Proof. We must show that f' is uniformly continuous on I. Let $\varepsilon > 0$ be given. By assumption, there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \frac{\varepsilon}{2}. \tag{2.1}$$

[7] Uniform differentiability has some nice applications in numerical analysis.

for all $x, y \in I$ with $0 < |x - y| < \delta$. Now, by the triangle inequality,

$$\begin{aligned}
|f'(x) - f'(y)| &= \left| f'(x) - \frac{f(x) - f(y)}{x - y} + \frac{f(x) - f(y)}{x - y} - f'(y) \right| \\
&\le \left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| + \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| \\
&= \left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| + \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

where we have used (2.1) in this last step. Since $|f'(x) - f'(y)| < \varepsilon$ whenever $x, y \in I$ are such that $|x - y| < \delta$, we infer that f' is uniformly continuous on the interval I. $\square$

2.4 A Word on the Lipschitz Condition

In Proposition 1.8 we proved that if a function f is both Lipschitz continuous and differentiable on an interval I, then f' is bounded by the Lipschitz constant on I. As mentioned there, a converse to this statement holds. Thanks to the mean value theorem, this result is now within reach.

Theorem 2.7. *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Then, f is Lipschitz continuous on I if and only if f' is bounded on I.*

Proof. Note that the forward implication " $\Longrightarrow$ " is precisely what was proven in Proposition 1.8. Therefore, we need only establish the converse. Since f' is bounded on I, there exists $L > 0$ such that $|f'(x)| \le L$ for all $x \in I$. Fix now two points $x, y \in I$; we want to show that

$$|f(x) - f(y)| \le L\,|x - y|\,.$$

Note that this inequality is trivial when $x = y$. Therefore, we may assume without loss of generality that $x < y$. Clearly, we then have $[x, y] \subseteq I$. Thus, f is differentiable on (x, y) and continuous on $[x, y]$. By the mean value theorem, there exists a point $c \in (x, y) \subseteq I$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

whence

$$\left|\frac{f(y)-f(x)}{y-x}\right| = |f'(c)| \le L.$$

This gives $|f(x)-f(y)| \le L\,|x-y|$ which completes the proof. $\square$

Remark 2.2. This theorem offers a nice characterization of Lipschitz continuous functions when the function in question is a priori assumed to be differentiable. One should not make the assumption that every Lipschitz function is differentiable. Indeed, by the inequality

$$||x|-|y|| \le |x-y|, \quad \forall x,y \in \mathbb{R},$$

the function $f(x)=|x|$ is Lipschitz on $\mathbb{R}$. However, f is not differentiable at 0.

Example 2.5. Consider the function $f(x) := \sqrt{x^2+1}$ on $\mathbb{R}$. Clearly, $x \mapsto x^2+1$ is differnetiable at every $x \in \mathbb{R}$ with derivative given by $x \mapsto 2x$. By an example from the previous tutorial, we know that $x \mapsto \sqrt{x}$ is differentiable on $(0,\infty)$ with derivative

$$x \mapsto \frac{1}{2\sqrt{x}}, \quad x > 0.$$

Since $x^2+1>0$ for all $x \in \mathbb{R}$, we infer from the chain rule that f is differentiable on $\mathbb{R}$ with

$$f'(x) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}, \quad \forall x \in \mathbb{R}.$$

Now, $\sqrt{x^2+1} \ge \sqrt{x^2} = |x|$ whence

$$|f'(x)| = \frac{|x|}{\sqrt{x^2+1}} \le 1.$$

By the previous theorem, we see that f is Lipschitz continuous with constant $L=1$.

3 *Third Tutorial*

We continue to discuss the concept of a derivative on $\mathbb{R}$. Darboux's theorem asserts that derivatives satisfy the intermediate value property. More precisely, let us recall the following class result:

Theorem 3.1 (Darboux). *Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be differentiable. If y is any point between $f'(a)$ and $f'(b)$, there exists a point x between a and b such that $f'(x) = y$.*

Darboux's theorem can sometimes make it easy to show that certain functions cannot be derivatives of any differentiable function. Put otherwise, Darboux's theorem sometimes implies that a given a function has no antiderivative. We provide such an example below.

Example 3.1. Consider the function

$$h(x) := \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

We claim that there does not exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f' = h$ on $\mathbb{R}$. Arguing by contradiction, suppose such a function f does indeed exist. Take $a = -1$ and $b = 1$. Then, $f'(a) = h(a) = 0$ and $f'(b) = h(b) = 1$. By Darboux's theorem, there must exist a point $x \in (-1, 1)$ such that

$$f'(x) = h(x) = \frac{1}{2}.$$

Clearly, no such point exists and we therefore have a contradiction. However, although h is not the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ that is everywhere differentiable, there exist uncountably many functions $f : \mathbb{R} \to \mathbb{R}$, differentiable away from 0, such that $f'(x) = h(x)$ for all $x \neq 0$. Indeed, for each $x_0 \in \mathbb{R}$,

$$f(x) := \begin{cases} x_0 & \text{if } x < 0, \\ x & \text{if } x \geq 0 \end{cases}$$

is differentiable at all points $c \neq 0$ and $f'(c) = h(c)$ at all such c.

Before moving on we provide the following easy preliminary to L'Hôpital's rule.

Proposition 3.2. *Let $f, g : (a, b) \to \mathbb{R}$ be such that $g(x) \neq 0$ if $x \neq c$ and fix a point $c \in (a, b)$. Assume that f, g are differentiable at c and that $g'(c) \neq 0$. If $f(c) = g(c) = 0$, then*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$

Proof. Since $f(c) = g(c) = 0$ we may write

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}}$$

for all $x \neq c$. Now, because $f'(c)$ and $g'(c)$ exist,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad g'(c) = \frac{g(x) - g(c)}{x - c}.$$

Finally, because $g'(c) \neq 0$, it follows from the limit laws that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} = \frac{f'(c)}{g'(c)}.$$

This concludes the proof. □

3.1 *Pathological Functions*

We now discuss functions that are "pathological" in the sense that they defy intuition. We begin with an example of a function that is nowhere continuous (that is, not continuous at any fixed point) despite being given by a very straightforward formula. More precisely, let us consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \tag{$\mathfrak{D}$}$$

The function f defined above is known as the *Dirichlet function*.

Proposition 3.3. *The Dirichlet function f defined in ($\mathfrak{D}$) is nowhere continuous on $\mathbb{R}$. That is, f is discontinuous at every $c \in \mathbb{R}$.*

Proof. We follow closely the argument used in the proof of Example 2.3. Given $c \in \mathbb{R}$, there are two cases to distinguish:

(1) Suppose $c \in \mathbb{Q}$. Since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence (ξ_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $\xi_n \to c$ as $n \to \infty$. Now, by definition of the function f, we have $f(\xi_n) = 0$ for all $n \in \mathbb{N}$. Since constant sequences are always convergent, we find that

$$f(c) = 1 \neq 0 = \lim f(\xi_n)$$

whence f does not satisfy the sequential criterion for continuity at c. We infer that f is not continuous at c.

(2) We now handle the case $c \notin \mathbb{Q}$. As above, we can select a sequence (r_n) in $\mathbb{Q}$ such that $\lim r_n = c$. Because $f(x) = 0$ for all irrationals x, we obtain

$$f(c) = 0 \neq 1 = \lim 1 = \lim f(r_n)$$

which once again shows that the sequential criterion for continuity at c does not hold true. It follows that f is discontinuous at c.

Since f is discontinuous at c in either case, we see that f is not continuous on all of $\mathbb{R}$. □

A more contrived example of a pathological function is the so-called Thomaeś function, defined below.

Definition 3.1. Given $x \in \mathbb{Q} \cap [0, 1]$ we can express it uniquely as a ratio

$$x = \frac{p}{q},$$

where $p \in \mathbb{N}_0$ and $q \in \mathbb{N}$ are co-prime, i.e. $\gcd(p, q) = 1$. We will agree to call this the *standard form of* x. Note that since $x \in [0, 1]$ we must also have $p \leq q$. By convention, we will agree to call $\frac{0}{1}$ the standard form of 0. Define now a function

$$f : [0, 1] \to \mathbb{R}, \quad f(x) := \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

As in the previous example, it is not hard to check that f is discontinuous at every $c \in \mathbb{Q}$. Indeed, although $f(c) \neq 0$, there exists by density a sequence of irrational numbers (ξ_n) in $[0, 1]$ such that $\xi_n \to c$ as $n \to \infty$. Since $f(\xi_n) = 0$ for each $n \in \mathbb{N}$, we cannot have $\lim f(\xi_n) = f(c)$. Therefore, the sequential criterion for continuity cannot hold true at c. What is surprising, however, is that this function is continuous at all irrational numbers.

Proposition 3.4. *Thomae's function described above is continuous at all irrational numbers and discontinuous at all rational numbers.*

Proof. We need only show that f is continuous an arbitrary irrational number $c \in (0, 1)$. Let $\varepsilon > 0$ be given. By the Archimdean property, there exists a natural number $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. We now assert that there exist only finitely many rational numbers $x = \frac{p}{q} \in [0, 1]$ such that $q \leq N$. Clearly, if $x \in [0, 1]$, then

$$0 \leq \frac{p}{q} \leq 1$$

whence $0 \leq p \leq q \leq N$. Hence, $p, q \leq N$ and so there are at most $(N+1)$-possibilities for p and q, respectively. Consequently, there can only be $(N+1)^2$-possible pairs (p, q) as above such that $q \leq N$. Since this is the same as counting fractions $\frac{p}{q} \in [0, 1]$ in standard form, we infer that there are no more than $(N+1)^2$ rational numbers

$$\frac{p}{q} \in [0, 1]$$

in with $q \leq N$. Let now Σ be the set of all these rationals, i.e.

$$\Sigma := \left\{ \frac{p}{q} \in \mathbb{Q} \cap [0, 1] : q \leq N \right\}.$$

By the argument above, Σ is finite. Fix an irrational number $c \in (0, 1)$ and consider the real number δ given by

$$\delta := \min_{x \in \Sigma} |x - c| > 0.$$

Note that δ is indeed positive as $c \notin \mathbb{Q}$ and we are taking the minimum over a finite subset of $\mathbb{Q}$. We now let $x \in [0, 1]$ be such that $|x - c| < \delta$. If $|x - c| = 0$ then $x = c$ so that

$$|f(x) - f(c)| = 0 < \varepsilon.$$

Otherwise, $0 < |x - c| < \delta$. We now distinguish the two possible cases:

- If $x \notin \mathbb{Q}$ then, by definition

  $$|f(x) - f(c)| = |0 - 0| = 0 < \varepsilon.$$

- Assume that x is rational. Since

  $$|x - c| < \delta = \min_{z \in \Sigma} |z - c|,$$

  we cannot have $x \in \Sigma$. Consequently, writing

  $$x = \frac{p}{q}$$

  in standard form, we must have $q > N$. It follows that

  $$|f(x) - f(c)| = |f(x)| = \frac{1}{q} < \frac{1}{N} < \varepsilon.$$

In either case we find that $|f(x) - f(c)| < \varepsilon$ whenever $x \in [0, 1]$ is such that $|x - c| < \delta$. It follows that f is continuous at c. $\square$

3.2 The Riemann Integral

Consider a closed and bounded interval $[a, b] \subseteq \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be an arbitrary function. A *partition* of the interval I is a set of points

$$\mathcal{P} = \{x_0, \dots, x_n\}$$

in $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$. Especially, $I = \bigcup_{j=1}^{n} [x_{j-1}, x_j]$. Informally, $\mathcal{P}$ describes a unique way of breaking the interval $[a, b]$ into non-overlapping (except at the endpoints) compact intervals $[x_{j-1}, x_j] \subseteq I$. A *tagged partition* is a partition $\mathcal{P}$ together with a set of points, called *tags*,

$$\{x_1^*, \dots, x_n^*\}$$

such that $x_j^* \in [x_{j-1}, x_j]$ for each index $j = 1, \dots, n$. To emphasize the fact that a partition $\mathcal{P}$ is equipped with a set of tags, we will write $\dot{\mathcal{P}}$.

Given a (possibly un-tagged) partition $\mathcal{P}$ of an interval $[a, b]$, the *mesh* of $\mathcal{P}$ is defined to be the length of the largest sub-interval defined by $\mathcal{P}$. More precisely, we define

$$\|\mathcal{P}\| := \max_{1 \le j \le n} \left(x_j - x_{j-1} \right) > 0.$$

Definition 3.2. Let $[a, b] \subset \mathbb{R}$ be a compact interval and $f : [a, b] \to \mathbb{R}$ a function. Given a tagged partition $\dot{\mathcal{P}}$ of $[a, b]$ as above, we define the Riemann sum of f over $\dot{\mathcal{P}}$ to be the sum

$$S(f; \dot{\mathcal{P}}) := \sum_{j=0}^{n} f(x_j^*)(x_j - x_{j-1}). \tag{3.1}$$

Often times, we will write $\Delta x_j = (x_j - x_{j-1})$ so that $S(f; \dot{\mathcal{P}}) := \sum_{j=0}^{n} f(x_j^*) \Delta x_j$. The function f is said to be *Riemann integrable* on $[a, b]$ if there exists $\Lambda \in \mathbb{R}$ such that for each $\varepsilon > 0$ there is $\delta > 0$ with the property that

$$\left| S(f; \dot{\mathcal{P}}) - \Lambda \right| < \varepsilon$$

for all tagged partitions $\dot{\mathcal{P}}$ of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \delta$. In this case, we call Λ the Riemann integral of f on $[a, b]$ and denote this quantity by

$$\int_a^b f.$$

Let us take a moment to unpack the definition of Riemann integrability. The Riemann sum $S(f, \dot{\mathcal{P}})$ should be thought of as a rectangular approximation of the area under the "curve" of the function f. Indeed, the step function

$$\varphi(x) := \sum_{j=1}^{n} f(x_j^*)\chi_{[x_{j-1},x_j)}(x)$$

is precisely an approximation of f by a function that takes the constant values $f(x_j^*)$ on each subinterval $[x_{j-1}, x_j)$ of $[a, b]$. Clearly, the classical area under the graph of φ is equal to the Riemann sum

$$S(f, \dot{\mathcal{P}}) = \sum_{j=0}^{n} f(x_j^*)(x_j - x_{j-1}).$$

Note that our choice of tags directly influences the approximations of f we obtain. Luckily, the definition of Riemann integrability guarantees that these Riemann sums $S(f, \dot{\mathcal{P}})$ converge to a meaningful real number, independently of our choice of tags[8], so long as we ensure that $\|\dot{\mathcal{P}}\|$ is sufficiently small, i.e. provided we refine our approximation of f sufficiently.

We now recall some basic properties of the Riemann integral that will/have been seen in the lectures.

Theorem 3.5. *Let $[a, b] \subset \mathbb{R}$ be a compact interval and denote by $\mathcal{R}([a, b])$ the set of all Riemann integrable functions on $[a, b]$. The following properties hold.*

(i) If $f, g \in \mathcal{R}([a, b])$ then $f + g \in \mathcal{R}([a, b])$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

(ii) Given $f \in \mathcal{R}([a, b])$ and $\alpha \in \mathbb{R}$ we have $\alpha f \in \mathcal{R}([a, b])$ with

$$\int_a^b (\alpha f) = \alpha \int_a^b f.$$

Let us now give a detailed example in which we verify the Riemann integrability of an explicit function.

[8] The mesh function $\|\cdot\|$ does not take into account the tag points, and only thinks about the lengths of the subintervals of $[a, b]$ that $\mathcal{P}$ creates.

Example 3.2. Fix a number $c \in (0,1)$ and define $f : [0,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 0 & \text{if } 0 \le x < c, \\ 1 & \text{if } c \le x \le 1. \end{cases}$$

We claim that f is Riemann integrable on $[0,1]$ with integral equal to $(1-c)$. Let $\varepsilon > 0$ and take

$$\delta = \varepsilon.$$

If $\dot{\mathcal{P}}$ is a tagged partition of $[0,1]$ with $\|\dot{\mathcal{P}}\| < \delta$ then every subinterval of $[0,1]$ created by $\dot{\mathcal{P}}$ has length no larger than δ. Let us now enumerate the elements of $\dot{\mathcal{P}}$ as

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

with tags $x_j^* \in [x_{j-1}, x_j]$ for every $j = 1, \ldots, n$. Let $k \ge 1$ be the unique integer such that $x_{k-1} < c \le x_k$. By definition of the function f, we have $f(x_j^*) = 0$ for all $j < k$.[9] There are now two cases that we will distinguish:

(1) Assume that $x_{k-1} \le x_k^* < c \le x_k$. If $k = n$ then $f(x_j^*) = 0$ for all $j = 1, \ldots, n$ so that

$$\left|S(f; \dot{\mathcal{P}}) - (1-c)\right| = |1-c| = |x_k - c| \le \left\|\dot{\mathcal{P}}\right\| < \delta.$$

In this last step we have used that both c and x_k belong to the same subinterval of $\dot{\mathcal{P}}$. If instead $k < n$ then

$$\begin{aligned} \left|S(f; \dot{\mathcal{P}}) - (1-c)\right| &= \left|\sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) - (1-c)\right| \\ &= \left|\sum_{j=k+1}^{n} f(x_j^*)(x_j - x_{j-1}) - (1-c)\right| \\ &= \left|\sum_{j=k+1}^{n} (x_j - x_{j-1}) - (1-c)\right| \\ &= |(x_n - x_k - (1-c)| \\ &= |(1 - x_k) - (1-c)| \\ &= |x_k - c| \\ &< \delta, \end{aligned}$$

[9] Indeed, if $1 \le j < k$ then $x_j^* \le x_{k-1} < c$ so that $f(x_j^*) = 0$.

where we have once again used the fact that x_k and c belong to the same subinterval of $\dot{\mathcal{P}}$

(2) Otherwise, we have $x_{k-1} < c \leq x_k^* \leq x_k$. Then, $f(x_k^*) = 1$ so that, as above,

$$\begin{aligned}
\left|S(f;\dot{\mathcal{P}}) - (1-c)\right| &= \left|\sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) - (1-c)\right| \\
&= \left|\sum_{j=k}^{n} f(x_j^*)(x_j - x_{j-1}) - (1-c)\right| \\
&= \left|\sum_{j=k}^{n}(x_j - x_{j-1}) - (1-c)\right| \\
&= |(x_n - x_{k-1} - (1-c)| \\
&= |x_{k-1} - c| \\
&< \delta.
\end{aligned}$$

In either case we have that

$$\left|S(f;\dot{\mathcal{P}}) - (1-c)\right| < \delta = \varepsilon.$$

Since $\dot{\mathcal{P}}$ was an arbitrary tagged partition of $[0,1]$ with $\|\dot{\mathcal{P}}\| < \delta$, we see that f is Riemann integrable on $[0,1]$ and that

$$\int_0^1 f = (1-c).$$

4 Fourth Tutorial

Let $[a,b] \subseteq \mathbb{R}$ be a compact interval. Recall that a function $f : [a,b] \to \mathbb{R}$ is said to be Riemann integrable on $[a,b]$ with integral $\int_a^b f$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|S(f;\dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon$$

for all tagged partitions $\dot{\mathcal{P}}$ of $[a,b]$ with $\|\dot{\mathcal{P}}\| < \delta$. We have seen that the value $\int_a^b f$ is unique when it exists and that every Riemann integrable function is necessarily bounded. In addition to Example 3.2, let us also show that the "classical" function $f(x) = 1 - x$ is Riemann integrable on $[0,1]$.

Example 4.1. Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = 1 - x$. We claim that $f \in \mathcal{R}([0,1])$ and that

$$\int_0^1 f = \frac{1}{2}.$$

Let $\varepsilon > 0$ be given and define $\delta := \varepsilon$. Let $\dot{\mathcal{P}}$ be a tagged partition of $[0,1]$ with $\|\dot{\mathcal{P}}\| < \delta$. Denote the partition points of $\dot{\mathcal{P}}$ by

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

and let $\{x_1^*, \ldots, x_n^*\}$ be the tags of $\dot{\mathcal{P}}$. Let $\dot{\mathcal{Q}}$ be the tagged partition of $[0,1]$ formed by taking the same partition points as $\dot{\mathcal{P}}$ with tags

$$y_j^* := \frac{x_j + x_{j-1}}{2}, \quad j = 1, \ldots, n.$$

An easy calculation then shows that

$$\begin{aligned}
S(f; \dot{\mathcal{Q}}) = \sum_{j=1}^{n} f(y_j^*)(x_j - x_{j-1}) &= \sum_{j=1}^{n} \left[1 - \frac{x_j + x_{j-1}}{2}\right](x_j - x_{j-1}) \\
&= \sum_{j=1}^{n}(x_j - x_{j-1}) - \frac{1}{2}\sum_{j=1}^{n}\left(x_j^2 - x_{j-1}^2\right) \\
&= 1 - \frac{1}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

On the other hand, using the inequality $|a_1 + \cdots + a_n| \le |a_1| + \cdots + |a_n|$,

$$\begin{aligned}
\left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| &= \left|\sum_{j=1}^{n}\left(f(x_j^*) - f(y_j^*)\right)(x_j - x_{j-1})\right| && (4.1) \\
&\le \sum_{j=1}^{n}\left|f(x_j^*) - f(y_j^*)\right|(x_j - x_{j-1}) && (4.2) \\
&= \sum_{j=1}^{n}\left|x_j^* - \frac{x_j + x_{j-1}}{2}\right|(x_j - x_{j-1}). && (4.3)
\end{aligned}$$

Since $x_j^* \in \left[x_{j-1}, x_j\right]$ and $\frac{x_j + x_{j-1}}{2}$ is the midpoint of this same interval, we find that

$$\left|x_j^* - \frac{x_j + x_{j-1}}{2}\right| \le (x_j - x_{j-1}) \le \left\|\dot{\mathcal{P}}\right\| < \delta,$$

for each index $j = 1, \ldots, n$. Returning to (4.3) gives

$$\left|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})\right| < \delta \sum_{j=1}^{n}(x_j - x_{j-1}) = \delta = \varepsilon.$$

Finally, recalling that $S(f,\dot{\mathcal{Q}}) = \frac{1}{2}$, we see that

$$\left|S(f;\dot{\mathcal{P}}) - \frac{1}{2}\right| = \left|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})\right| < \varepsilon$$

for all tagged partitions $\dot{\mathcal{P}}$ of $[0,1]$ with $\left\|\dot{\mathcal{P}}\right\| < \delta$.

4.1 Approximating by Riemann Sums

Given the striking similarities between the $\varepsilon - \delta$ definition of the limit and our definition of Riemann integrability (see Definition 3.2), it is reasonable to hope that one can approximate $\int_a^b f$ by simply taking a limit of Riemann sums with the mesh of the partitions tending to zero. Informally, by sufficiently refining our tagged partitions, we should be able to approximate the area under the graph of f. This is confirmed by the following:

Lemma 4.1. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. If $\left(\dot{\mathcal{P}}_n\right)$ is a sequence of tagged partitions of $[a,b]$ such that*

$$\lim \left\|\dot{\mathcal{P}}_n\right\| = 0,$$

then

$$\lim S(f;\dot{\mathcal{P}}_n) = \int_a^b f.$$

Proof. Let $\varepsilon > 0$ be given. Since f is Riemann integrable on $[a,b]$, there exists $\delta > 0$ such that

$$\left|S(f;\dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon$$

whenever $\dot{\mathcal{P}}$ is a tagged partition of $[a,b]$ with $\left\|\dot{\mathcal{P}}\right\| < \delta$. Now, because $\left\|\dot{\mathcal{P}}_n\right\| \to 0$ as $n \to \infty$, we can find $N \in \mathbb{N}$ such that

$$\left\|\dot{\mathcal{P}}_n\right\| < \delta$$

for all $n \geq N$. It follows that

$$\left| S(f; \dot{\mathcal{P}}_n) - \int_a^b f \right| < \varepsilon$$

for all $n \geq N$. This shows that $S(f; \dot{\mathcal{P}}_n) \to \int_a^b f$ as $n \to \infty$. $\square$

Remark 4.1. This lemma can sometimes make it relatively easy to extend properties of Riemann sums to the Riemann integral. Informally speaking, if a certain property holds for Riemann sums and this property is preserved by limits, one might use the lemma above to extend this property to the integral. See the next example for an application of such an argument.

Example 4.2. Let $f \in \mathcal{R}([a,b])$ be such that $|f|$ is also Riemann integrable on $[a,b]$.[10] Let $(\mathcal{P}_n)$ be a sequence of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ as $n \to \infty$. By the triangle inequality, one has $\left|S(f; \dot{\mathcal{P}}_n)\right| \leq S\left(|f|, \dot{\mathcal{P}}_n\right)$ for each $n \in \mathbb{N}$. By Lemma 4.1 and the continuity of the absolute value function, we have $\left|S(f; \dot{\mathcal{P}}_n)\right| \to \left|\int_a^b f\right|$ and $S\left(|f|, \dot{\mathcal{P}}_n\right) \to \int_a^b |f|$. Since non-strict inequalities are preserved by limits, we infer that

$$\left|\int_a^b f\right| \leq \int_a^b |f| .$$

Note that this argument will fail if we do not know *a priori* that $|f| \in \mathcal{R}([a,b])$.

Example 4.3. Consider the function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) := \begin{cases} \frac{1}{x} & \text{if } x \in \mathbb{Q} \setminus \{0\}, \\ 0 & \text{if } x \notin \mathbb{Q} \text{ or } x = 0. \end{cases}$$

Let (x_n) be the sequence in $[0,1]$ defined by $x_n := \frac{1}{n}$. Clearly, $f(x_n) = n$ for each $n \in \mathbb{N}$ whence f is *unbounded* on the interval $[0,1]$. Consequently, f cannot be Riemann integrable on $[0,1]$. Nonetheless, one can find a sequence $\left(\dot{\mathcal{P}}_n\right)$ of tagged partitions of $[0,1]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ and $\lim S(f; \dot{\mathcal{P}}_n)$ exists. Indeed, given $n \in \mathbb{N}$, divide $[0,1]$ into n-subintervals of equal length and choose from each subinterval an irrational tag (this can be done because $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$). The resulting tagged partition $\dot{\mathcal{P}}_n$ will be such that $\|\dot{\mathcal{P}}_n\| \leq \frac{1}{n}$ and $S(f; \dot{\mathcal{P}}_n) = 0$.

[10] This condition turns out to be redundant. However, we do not yet possess the tools required to show that $f \in \mathcal{R}(a,b])$ implies that $|f| \in \mathcal{R}([a,b])$. We therefore have no qualms about making this assumption in our example.

We also have the following sequential condition for non-integrability:

Corollary 4.2. *Let $f : [a,b] \to \mathbb{R}$ be a function. Let $\left(\dot{\mathcal{P}}_n\right)$ and $\left(\dot{\mathcal{Q}}_n\right)$ two sequences of tagged partitions of $[a,b]$ such that*

$$\lim \left\|\dot{\mathcal{P}}_n\right\| = \lim \left\|\dot{\mathcal{Q}}_n\right\| = 0.$$

*If $\lim S(f;\dot{\mathcal{P}}_n) \neq \lim S(f;\dot{\mathcal{Q}}_n)$, then f is **not** Riemann integrable on $[a,b]$. Moreover, if one of these limits does not exist, then $f \notin \mathcal{R}([a,b])$.*

Proof. By way of contradiction, let us assume that $f \in \mathcal{R}([a,b])$. Citing Lemma 4.1, we must have

$$\lim S(f;\dot{\mathcal{P}}_n) = \int_a^b f = \lim S(f;\dot{\mathcal{Q}}_n)$$

which is a contradiction. □

Consider once more the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ given by ($\mathfrak{D}$). We have already seen that f is discontinuous at every point $c \in \mathbb{R}$. Using Corollary 4.2, we will show that f is not Riemann integrable on any interval $[a,b] \subseteq \mathbb{R}$. Indeed, let $(\mathcal{P}_n)$ be any sequence of partitions of $[a,b]$ with

$$\|\mathcal{P}_n\| \leq \frac{b-a}{n}$$

for each n.[11] From each subinterval of $\mathcal{P}_n$, we choose (by density) a rational tag and denote the resulting tagged partition by $\dot{\mathcal{P}}_n$. Similarly, from every subinterval of $\mathcal{P}_n$ we choose an irrational tag (again by density) and let $\dot{\mathcal{Q}}_n$ be the corresponding tagged partition of $[a,b]$. Since $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ have the same partition points as $\mathcal{P}_n$, it is easy to see that

$$\lim \left\|\dot{\mathcal{P}}_n\right\| = \lim \left\|\dot{\mathcal{Q}}_n\right\| = 0.$$

On the other hand, because every tag of $\dot{\mathcal{P}}_n$ is rational, we have $S(f;\dot{\mathcal{P}}_n) = 1$ for each $n \in \mathbb{N}$. But, as every tag of $\dot{\mathcal{Q}}_n$ is irrational, we instead have $S(f;\dot{\mathcal{Q}}_n) = 0$ for each $n \in \mathbb{N}$. Citing Corollary 4.2, we infer that $f \notin \mathcal{R}([a,b])$.

[11] Such a sequence $(\mathcal{P}_n)$ can be obtained by dividing $[a,b]$ into n-subintervals of equal length.

4.2 Cauchy's Criterion for Riemann Integrability

We begin by recalling the following Theorem from class:

Theorem 4.3 (Cauchy's Criterion). *Let $f : [a, b] \to \mathbb{R}$ be a function. Then, f is Riemann integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| < \varepsilon$$

for all tagged partitions $\dot{\mathcal{P}}$, $\dot{\mathcal{Q}}$ of $[a, b]$ such that $\|\dot{\mathcal{P}}\| < \delta$ and $\|\dot{\mathcal{Q}}\| < \delta$.

Negating the condition given above, we see that a function $f : [a, b] \to \mathbb{R}$ is **not** Riemann integrable on $[a, b]$ if and only if there exists $\varepsilon_0 > 0$ such that, for every $\delta > 0$, one can find tagged partitions $\dot{\mathcal{P}}_\delta$ and $\dot{\mathcal{Q}}_\delta$ of $[a, b]$, each having mesh strictly less than δ, such that

$$\left|S(f; \dot{\mathcal{P}}_\delta) - S(f; \dot{\mathcal{Q}}_\delta)\right| \geq \varepsilon_0.$$

In this case, taking $\delta := \frac{1}{n}$ gives two tagged partitions $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ such that

$$\left\|\dot{\mathcal{P}}_n\right\| < \frac{1}{n} \quad \text{and} \quad \left\|\dot{\mathcal{Q}}_n\right\| < \frac{1}{n}$$

with $\left|S(f; \dot{\mathcal{P}}_n) - S(f; \dot{\mathcal{Q}}_n)\right| \geq \varepsilon_0$. Conversely, assume that we are given two sequences of tagged partitions $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ such that

$$\lim \left\|\dot{\mathcal{P}}_n\right\| = \lim \left\|\dot{\mathcal{Q}}_n\right\| = 0$$

but $\left|S(f; \dot{\mathcal{P}}_n) - S(f; \dot{\mathcal{Q}}_n)\right| \geq \varepsilon_0 > 0$ for each $n \in \mathbb{N}$. Can we conclude from this that $f \notin \mathcal{R}([a, b])$? Indeed, for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$\left\|\dot{\mathcal{P}}_n\right\| < \delta \quad \text{and} \quad \left\|\dot{\mathcal{Q}}_n\right\| < \delta$$

for all $n \geq N$. In particular, for $n = N$ we have

$$\left\|\dot{\mathcal{P}}_N\right\| < \delta \quad \text{and} \quad \left\|\dot{\mathcal{Q}}_N\right\| < \delta.$$

On the other hand,

$$\left|S(f; \dot{\mathcal{P}}_N) - S(f; \dot{\mathcal{Q}}_N)\right| \geq \varepsilon_0.$$

Taking $\dot{\mathcal{P}}_\delta := \dot{\mathcal{P}}_N$ and $\dot{\mathcal{Q}}_\delta := \dot{\mathcal{Q}}_n$, we see that the negation of Cauchy's criterion holds true and f is *not* Riemann integrable on $[a, b]$. To summarize, we have proven the following:

Corollary 4.4. *Let $f : [a, b] \to \mathbb{R}$ be a function. Then, f is not Riemann integrable on $[a, b]$ if and only if there exists $\varepsilon_0 > 0$ and two sequences $(\dot{\mathcal{P}}_n)$, $(\dot{\mathcal{Q}}_n)$ of tagged partitions of $[a, b]$ such that*

(i) $\|\dot{\mathcal{P}}_n\| \to 0$ *and* $\|\dot{\mathcal{Q}}_n\| \to 0$ *as* $n \to \infty$*;*

(ii) $\left|S(f; \dot{\mathcal{P}}_n) - S(f; \dot{\mathcal{Q}}_n)\right| \geq \varepsilon_0$ *for all* $n \in \mathbb{N}$*.*

Example 4.4. This criterion makes it possible to show that a large class of functions are not Riemann integrable. Let $g : \mathbb{Q} \to \mathbb{R}$ be a function such that $g(x) \geq \varepsilon_0 > 0$ for all $x \in \mathbb{Q}$ and some $\varepsilon_0 > 0$. Define $f : [a, b] \to \mathbb{R}$ by

$$f(x) := \begin{cases} g(x) & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Note that by taking $g = 1$ we recover the Dirichlet function in ($\mathfrak{D}$). We now claim that f is *not* Riemann integrable on $[a, b]$. To prove this, we will make use of the criterion proven in Corollary 4.4. For each $n \in \mathbb{N}$ we can create an *untagged* partition $\mathcal{P}_n$ of $[a, b]$ by dividing $[a, b]$ into n-subintervals of equal length. Clearly, this gives us a sequence of partitions of $[a, b]$ such that

$$\|\mathcal{P}_n\| = \frac{b - a}{n} \to 0$$

as $n \to \infty$. For fixed $n \in \mathbb{N}$ we can choose by density a rational tag from each subinterval of $\mathcal{P}_n$; doing so gives us a tagged partition $\dot{\mathcal{P}}_n$ of $[a, b]$ having only rational tags and the same partition points as $\mathcal{P}_n$. Similarly, we build from $\mathcal{P}_n$ a tagged partition $\dot{\mathcal{Q}}_n$ of $[a, b]$ having only irrational tags and the same partition points as $\mathcal{P}_n$. Since $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ have the same partition points as $\mathcal{P}_n$, it is obvious that

$$\lim \left\|\dot{\mathcal{P}}_n\right\| = \lim \left\|\dot{\mathcal{Q}}_n\right\| = 0.$$

On the other hand, because f vanishes at every irrational number, $S(f; \dot{\mathcal{Q}}_n) = 0$ for all $n \in \mathbb{N}$. Consequently, letting $x_0, \ldots, x_k$ and $\left\{x_j^*\right\}_{j=1}^{k}$ be the partition points and tags (respectively) of $\dot{\mathcal{P}}_n$ for fixed n, we infer that

$$\begin{aligned}
\left|S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{Q}}_n)\right| = \left|S(f;\dot{\mathcal{P}}_n)\right| &= \left|\sum_{j=1}^{k} f(x_j^*)(x_j - x_{j-1})\right| \\
&= \left|\sum_{j=1}^{k} g(x_j^*)(x_j - x_{j-1})\right| \\
&= \sum_{j=1}^{k} g(x_j^*)(x_j - x_{j-1}) \\
&\geq \varepsilon_0 \sum_{j=1}^{k} (x_j - x_{j-1}) \\
&= \varepsilon_0 (b - a).
\end{aligned}$$

Since $n \in \mathbb{N}$ was arbitrary, we see from Corollary 4.4 that f cannot be Riemann integrable on $[a, b]$.

4.3 A Warning about Compositions

Composition has thus far interacted nicely with the analytic concepts we have explored. For instance, composition preserves continuity, uniform continuity, and even differentiability. Unfortunately, Riemann integrability is not a member of this club, as the next example dictates.

Example 4.5. Consider the function

$$f : [0,1] \to \mathbb{R}, \quad f(x) := \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly, f is Riemann integrable on $[0, 1]$. Next, let g be Thomae's function described in Definition 3.1. It is also known (see Bartle & Sherbert §7.1) that g is Riemann integrable on $[0, 1]$. Note that $g([0, 1]) \subseteq [0, 1]$ and so $f \circ g$ is well defined on $[0, 1]$. However,

$$f(g(x)) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

which is precisely the non-integrable Dirichlet function from ($\mathfrak{D}$). Hence, $f \circ g$ is not an element of $\mathcal{R}([0,1])$ despite being the composition of two Riemann integrable function.[12]

5 Fifth Tutorial

Let us first recall some of what we have discussed in the lectures. We know that if $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$, then modifying the function f at *finitely* many points does not affect the integrability of f or the value $\int_a^b f$.[13] Furthermore, you have proven in your fourth assignment that the Riemann integral is monotone in the following sense:

Proposition 5.1 (Monotoncity of the Integral)**.** *Let $f, g \in \mathcal{R}([a,b])$ and assume that $f(x) \leq g(x)$ for all $x \in [a,b]$. Then,*

$$\int_a^b f \leq \int_a^b g.$$

Let us introduce one more piece of notation. Given an interval $[a,b] \subset \mathbb{R}$ and a subset $E \subseteq [a,b]$, the *indicator* function (or *characteristic* function) of E is a function $\mathbf{1}_E(x)$ (sometimes denoted $\chi_E(x)$) defined on $[a,b]$ that is equal to 1 on E and 0 outside of E. Symbolically,

$$\mathbf{1}_E : [a,b] \to \mathbb{R}, \quad \mathbf{1}_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

For any subinterval $[c,d]$ of $[a,b]$, you have proven (also in your fourth assignment) that the function

$$\mathbf{1}_{[c,d]}(x) := \begin{cases} 0 & \text{if } a \leq x < c, \\ 1 & \text{if } c \leq x \leq d, \\ 0 & \text{if } d < x \leq b \end{cases}$$

[12] Broadly speaking, the problem is that the function f is not everywhere continuous. In the future, we will see that the integrability of g is preserved provided f is continuous.

[13] One should note that modifying f at *countably* many points need not preserve the integrability of f. Indeed, the Dirichlet function given in ($\mathfrak{D}$) is obtained by modifying the constant function $f \equiv 0$ on the countable set $\mathbb{Q}$.

is Riemann integrable with $\int_a^b \mathbf{1}_{[c,d]} = d - c$. Since modifying the values of Riemann integrable functions at *finitely* many points does not affect integrability or alter the value of the integral, we see that the functions $\mathbf{1}_{(c,d)}, \mathbf{1}_{[c,d)}$ and $\mathbf{1}_{(c,d]}$ are all integrable on $[a, b]$ with integral equal to $d - c$. Consequently, we obtain the following:

Corollary 5.2. *If φ is a step function on $[a, b]$, φ is Riemann integrable on $[a, b]$.*

Proof. A step function φ on $[a, b]$ is by definition a finite linear combination of functions having the for $\mathbf{1}_{(c,d)}, \mathbf{1}_{[c,d)}$ or $\mathbf{1}_{(c,d]}$. Since finite linear combinations of Riemann integrable functions are integrable, the assertion follows. □

Remark 5.1. Step functions are often used to approximate a Riemann integrable function (and therefore to approximate the value of its integral). Such approximations are particularly useful since the integral of a step function is very easy to compute. For instance, the step function

$$\varphi : [0, 2] \to \mathbb{R}, \quad \varphi(x) := \begin{cases} 0 & \text{if } x \in [0, 1], \\ 1 & \text{if } x \in (0, 1), \\ 2 & \text{if } x \in [1, 2] \end{cases}$$

can be written as $\varphi(x) = \mathbf{1}_{(0,1)} + 2\mathbf{1}_{[1,2]}$. Then,

$$\int_0^1 \varphi = \int_0^2 \mathbf{1}_{(0,1)} + 2 \int_0^2 \mathbf{1}_{[1,2]} = 1 + 2 = 3.$$

However, we should note that there exists functions φ, taking only finitely many values, that are **not** step functions. Indeed, the Dirichlet function in ($\mathfrak{D}$) takes only the values $0, 1$ but is not a step function.

Proposition 5.3. *Let $f : [a, b] \to \mathbb{R}$ be a function and let $S(f; \dot{\mathcal{P}})$ be a Riemann sum of f. There exists a step function $\varphi : [a, b] \to \mathbb{R}$ such that $\int_a^b \varphi = S(f; \dot{\mathcal{P}})$.*

Proof. Let

$$a = x_0 < \cdots < x_n = b$$

denote the partition points of $\dot{\mathcal{P}}$ and let $x_1^*, \ldots, x_n^*$ be the tags of $\dot{\mathcal{P}}$. By definition,

$$S(f; \dot{\mathcal{P}}) = \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}).$$

Consider the step function

$$\varphi(x) := \sum_{j=1}^{n} f(x_j^*)\mathbf{1}_{(x_{j-1},x_j]}.$$

Then, $\varphi \in \mathcal{R}([a,b])$ and

$$\int_a^b \varphi = \sum_{j=1}^{n} f(x_j^*) \int_a^b \mathbf{1}_{(x_{j-1},x_j]} = \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) = S(f;\dot{\mathcal{P}}).$$

□

5.1 The Squeeze Theorem

In addition to the Cauchy Criterion in Theorem 4.3, we possess the following criterion for Riemann integrability:

Theorem 5.4 (Squeeze Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a function. Then, f is Riemann integrable on $[a,b]$ if and only if, for each $\varepsilon > 0$, there exists two Riemann integrable functions $\alpha_\varepsilon, \omega_\varepsilon : [a,b] \to \mathbb{R}$ such that*

$$\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x), \quad \forall x \in [a,b],$$

and

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon.$$

To see how Theorem 5.4 can be used in practice, let us consider the following example.

Example 5.1. Consider the function $f : [0,1] \to \mathbb{R}$ given by $x \mapsto x$. Using the Squeeze Theorem, we will prove that f is Riemann integrable on $[0,1]$ and that $\int_0^1 x = \frac{1}{2}$. Given a $n \in \mathbb{N}$, let $\mathcal{P}_n$ be the partition of $[0,1]$ obtained by dividing $[0,1]$ into n-subintervals of equal length. Clearly,

$$\lim \|\mathcal{P}_n\| = \lim \frac{1}{n} = 0.$$

Fix *any* $n \in \mathbb{N}$. The partition points of $\mathcal{P}_n$ are given by

$$x_j = \frac{j}{n}, \quad j = 0, \ldots, n$$

Define two functions α_n, ω_n on $[0,1]$ by

$$\alpha_n(x) := \begin{cases} \frac{j-1}{n} & \text{if } \frac{j-1}{n} \le x < \frac{j}{n}, \\ 1 & \text{if } x = 1, \end{cases} \quad \text{and} \quad \omega_n(x) := \begin{cases} \frac{j}{n} & \text{if } \frac{j-1}{n} \le x < \frac{j}{n}, \\ 1 & \text{if } x = 1. \end{cases}$$

and note that $\alpha_n \le f \le \omega_n$ on $[0,1]$. Since step functions are always integrable, α_n and ω_n are both Riemann integrable on $[0,1]$. Now, a direct calculation shows that

$$(\omega_n - \alpha_n)(x) = \begin{cases} \frac{1}{n} & \text{if } \frac{j-1}{n} \le x < \frac{j}{n}, \\ 0 & \text{if } x = 1 \end{cases}$$

whence

$$\int_0^1 (\omega_n - \alpha_n) = \sum_{j=1}^{n} \frac{1}{n}\left(\frac{j}{n} - \frac{j-1}{n}\right) = \frac{1}{n^2}\sum_{j=1}^{n} 1 = \frac{1}{n}.$$

In particular, for any n such that $\frac{1}{n} < \varepsilon$, one has

$$\alpha_n \le f \le \omega_n \quad \text{and} \quad \int_0^1 (\omega_n - \alpha_n) < \varepsilon.$$

It follows from the Squeeze theorem that $f \in \mathcal{R}([0,1])$ and that $\int_0^1 f$ exists. Now, note that

$$\begin{aligned} \int_0^1 \alpha_n &= \sum_{j=1}^{n} \frac{j-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}\sum_{j=1}^{n} (j-1) = \frac{1}{n^2}\left[\frac{n(n+1)}{2} - n\right] \\ &= \frac{1}{2} - \frac{1}{n}. \end{aligned}$$

Similarly,

$$\int_0^1 \omega_n = \sum_{j=1}^{n} \frac{j}{n} \cdot \frac{1}{n} = \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{1}{2} + \frac{1}{2n}.$$

By monotonicity of the integral,

$$\frac{1}{2} - \frac{1}{n} = \int_0^1 \alpha_n \le \int_0^1 f \le \int_0^1 \omega_n = \frac{1}{2} + \frac{1}{2n}.$$

Since $n \in \mathbb{N}$ was arbitrary, we can take the limit as $n \to \infty$ and deduce from the Squeeze Theorem (for sequences) that $\int_0^1 f = \frac{1}{2}$.

5.2 Integrating Continuous Functions

We know that all monotone increasing functions $[a, b] \to \mathbb{R}$ are Riemann integrable. Perhaps even more importantly, it has been proven that continuous functions are always Riemann integrable. The Riemann integral enjoys much nicer properties when restricted to continuous functions. Namely, much more can be said about $\int_a^b f$ when f is assumed to be continuous on $[a, b]$. Since compositions of continuous functions are continuous, $|f|$ is Riemann integrable on $[a, b]$ whenever f is a continuous function on $[a, b]$. Recalling Example 4.2, we obtain the following:

Proposition 5.5. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f, |f|$ are Riemann integrable on $[a, b]$ and*

$$\left| \int_a^b f \right| \leq \int_a^b |f| .$$

Theorem 5.6 (Mean Value Theorem for Integrals)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, there exists a point $c \in [a, b]$ such that*

$$\frac{1}{b-a} \int_a^b f = f(c).$$

Proof. Since $[a, b]$ is compact and f is continuous, there f achieves an absolute minimum m and an absolute maximum M on $[a, b]$. By your fourth assignment (see Problem 4 there), we must have

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Consequently,

$$m \leq \frac{1}{b-a} \int_a^b f \leq M.$$

By the intermediate value theorem, f must achieve every value in $[m, M]$. In particular, there exists a point $c \in [a, b]$ such that $f(c) = \frac{1}{b-a} \int_a^b f$. This completes the proof. □

Corollary 5.7. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous. If $\int_a^b f = \int_a^b g$, there exists a point $c \in [a, b]$ such that $f(c) = g(c)$.*

Proof. Applying the Mean Value Theorem for Integrals to $f-g$ implies the existence of a point $c \in [a,b]$ such that

$$(f-g)(c) = f(c) - g(c) = \frac{1}{b-a}\int_a^b (f-g) = \frac{\int_a^b f - \int_a^b g}{b-a} = 0.$$

That is, $f(c) = g(c)$. $\square$

5.3 Riemann Sums and the Darboux Integral

Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and let $\left(\dot{\mathcal{P}}_n\right)$ be a sequence of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$, as $n \to \infty$. Lemma 4.1 states that the Riemann sums $S(f;\dot{\mathcal{P}}_n)$ converge to $\int_a^b f$, i.e.

$$\lim S(f;\dot{\mathcal{P}}_n) = \int_a^b f.$$

As demonstrated in Example 4.2, this can sometimes help us extend properties of Riemann *sums* to the Riemann *integral*. This type of argument is very common in analysis and is extremely useful. We provide another application of this argument below:

Proposition 5.8. *Let $a > 0$ and $f : [-a,a] \to \mathbb{R}$ be continuous. In particular, f is Riemann integrable on every closed subinterval of $[-a,a]$. If f is even, i.e. if $f(x) = f(-x)$ for all $x \in [-a,a]$, then*

$$\int_{-a}^a f = 2\int_0^a f.$$

Proof. For each $n \in \mathbb{N}$ we construct a partition $\mathcal{P}_n$ of $[0,a]$ by dividing this interval in to n-subintervals of equal length. Clearly, $\|\mathcal{P}_n\| = \frac{a}{n}$ for each $n \in \mathbb{N}$. Let $x_1^*, \ldots, x_k^*$ be any set of tags for $\mathcal{P}_n$ and let $\dot{\mathcal{P}}_n$ denote the resulting tagged partition of $[a,b]$. Let

$$0 = x_0 < \cdots < x_n = a$$

be the partition points of $\dot{\mathcal{P}}_n$. Let $\mathcal{Q}_n$ be the tagged partition of $[-a,a]$ with partition points

$$-a = -x_n < -x_{n-1} < \cdots < -x_1 < x_0 < \cdots < x_n = a.$$

Finally, we give $\mathcal{Q}_n$ the tags

$$\begin{cases} x_j^* \in \left[x_{j-1}, x_j\right] & 1 \leq j \leq n, \\ -x_j^* \in \left[-x_j, -x_{j-1}\right] & 1 \leq j \leq n. \end{cases}$$

Then, $\dot{\mathcal{Q}}_n$ is tagged partition of $[-a, a]$ with $\|\dot{\mathcal{Q}}_n\| = \|\dot{\mathcal{P}}_n\| = \frac{a}{n}$. Since f is an even function,

$$\begin{aligned} S(f; \dot{\mathcal{Q}}_n) &= \sum_{j=1}^{n} f(-x_j^*) \left(-x_{j-1} - (-x_j)\right) + \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) + \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) \\ &= 2S(f; \dot{\mathcal{P}}_n). \end{aligned}$$

As $n \in \mathbb{N}$ was arbitrary, this gives us a sequence of tagged partitions $\dot{\mathcal{P}}_n$ of $[0, a]$, and a related sequence $\dot{\mathcal{Q}}_n$ of tagged partitions of $[-a, a]$, such that

$$\lim \left\|\dot{\mathcal{Q}}_n\right\| = \lim \left\|\dot{\mathcal{P}}_n\right\| = \lim \frac{a}{n} = 0.$$

Consequently, two applications of Lemma 4.1 implies

$$\int_{-a}^{a} f = \lim S(f; \dot{\mathcal{Q}}_n) = 2 \lim S(f; \dot{\mathcal{P}}_n) = 2 \int_{0}^{a} f.$$

□

A similar argument will allow us to obtain the following:

Proposition 5.9. *Let $a > 0$ and $f : [-a, a] \to \mathbb{R}$ be continuous. In particular, f is Riemann integrable on every closed subinterval of $[-a, a]$. If f is odd, i.e. if $-f(x) = f(-x)$ for all $x \in [-a, a]$, then*

$$\int_{-a}^{a} f = 0.$$

Proof. For each $n \in \mathbb{N}$ we defined tagged partitions $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ as in the proof of the Proposition 5.8. However, in this case, we obtain

$$\begin{aligned}
S(f;\dot{\mathcal{Q}}_n) &= \sum_{j=1}^{n} f(-x_j^*)\left(-x_{j-1}-(-x_j)\right) + \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) \\
&= -\sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) + \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) \\
&= 0.
\end{aligned}$$

By the same argument as before, Lemma 4.1 yields

$$\int_{-a}^{a} f = \lim S(f;\dot{\mathcal{Q}}_n) = 0.$$

□

In Example 5.1, we proved that the function $f(x) = x$ is Riemann integrable on $[0,1]$ using the Squeeze Theorem. There, we approximated f above and below by conveniently chosen step functions. More precisely, we divided $[0,1]$ into n-subintervals $\{I_j\}_{j=1}^n$ of equal length and defined two step functions α_n and ω_n by taking α_n to be the minimum of f over I_j and ω_n to be the maximum. In this sense, we approximating f from above and below by step functions taking on the extrema of f (on these I_j). This is precisely the idea behind the *Darboux integral*, which we briefly discuss.

Let $I = [a,b]$ be a compact interval and let $\mathcal{P}$ be a partition of $[a,b]$. Let

$$a = x_0 < \cdots < x_n = b$$

be the partition points of $\mathcal{P}$. Given a bounded function $f : [a,b] \to \mathbb{R}$, we can define the *upper* and *lower* Darboux sums of f on $\mathcal{P}$, respectively, by

$$L(f;\mathcal{P}) := \sum_{j=1}^{n} m_j(x_j - x_{j-1}), \quad m_j := \inf_{x\in[x_{j-1},x_j]} f(x), \tag{5.1}$$

$$U(f;\mathcal{P}) := \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \quad M_j := \sup_{x\in[x_{j-1},x_j]} f(x). \tag{5.2}$$

Since $m_j \le M_j$ for all $j = 1,\dots,n$, we see that $L(f;\mathcal{P}) \le U(f;\mathcal{P})$. Furthermore, let $M > 0$ be such that $-M \le f(x) \le M$ on $[a,b]$. Then, $L(f;\mathcal{P})$ is bounded from above independently of $\mathcal{P}$. Indeed,

$$L(f;\mathcal{P}) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}) \leq \sum_{j=1}^{n} M(x_j - x_{j-1}) = M(b-a).$$

Similarly,

$$U(f;\mathcal{P}) \geq \sum_{j=1}^{n} -M(x_j - x_{j-1}) = -M(b-a).$$

Hence, $U(f;\mathcal{P})$ is bounded from below independently of $\mathcal{P}$.

Definition 5.1. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function. Denote by $\mathscr{P}(I)$ the collection of all partitions $\mathcal{P}$ of $[a, b]$. The lower and upper Darboux integrals of f are defined, respectively, by

$$L(f) := \sup_{\mathcal{P}\in\mathscr{P}(I)} L(f;\mathcal{P}) \quad \text{and} \quad U(f) := \inf_{\mathcal{P}\in\mathscr{P}(I)} U(f;\mathcal{P}).$$

Note that these quantities exist by our previous argument. We say that f is Darboux integrable on $[a, b]$ if $U(f) = L(f)$. In this case the Darboux integral of f on $[a, b]$ is defined as:

$$\int_a^b f := U(f) = L(f).$$

One can show (see Bartle §7.4) that a function is Darboux integrable on $[a, b]$ if and only if it is Riemann integrable.

Unlike the Riemann integral, the Darboux integral declares a function f to be integrable if it can be approximated from above and below by step functions. Put otherwise, a bounded function f is considered integrable if and only if if can be "squeezed" between two step functions. This bears a notable resemblance to the statement of the squeeze theorem 5.4.

Although the Riemann and Darboux integrals are equivalent, the Riemann one is "better" in the sense that it more easily generalizes. By making a minor modification to the definition of the Riemann integral (replacing δ with something called a *gauge*), one obtains the so-called **gauge integral**. As with the Riemann integral, the gauge integral is defined on intervals. In fact, the gauge integral can be defined on *unbounded* intervals. On an interval I, it turns out that the gauge integral is *more* general than the Lebesgue integral. However, the Lebesgue integral can be defined in *much* more general settings than the gauge integral. For instance, the Lebesgue integral can be defined on subsets of $\mathbb{R}$ that are not intervals.

5.4 A Remark About Additivity

You will shortly see in the lectures that the Riemann integral is additive in the following sense:

Theorem 5.10. *Let $f : [a, b] \to \mathbb{R}$ be a function. Then, f is Riemann integrable on $[a, b]$ if and only if, for each $a < c < b$, f is Riemann integrable on $[a, c]$ and on $[c, b]$. In this case, one has*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Let $f : [a, b] \to \mathbb{R}$ be a function and assume that $f \in \mathcal{R}([c, b])$ for all $c \in (a, b)$. Does it follow that $f \in \mathcal{R}([a, b])$? Unfortunately, one cannot conclude that f is Riemann integrable on $[a, b]$. Consider the function

$$f(x) := \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Since f is continuous on $[c, 1]$ for every $0 < c < 1$, we see that f is Riemann integrable on $[c, 1]$ for each such c. On the other hand, because f is unbounded on $[0, 1]$, it cannot be Riemann integrable there.

6 Sixth Tutorial

We begin by recalling two major results that are very familiar to us from single variable calculus. Paired together, these two theorems are often referred to as the *fundamental theorem of calculus.*

Theorem 6.1 (Fundamental Theorem of Calculus Form 1)**.** *Let $f, F : [a, b] \to \mathbb{R}$ be functions and assume there exists a finite set $E \subset [a, b]$ such that*

(1) F is continuous on $[a, b]$;

(2) $F'(x)$ exists for all $x \notin E$;

(3) $F'(x) = f(x)$ for each $x \notin E$.

Then, if f is Riemann integrable on $[a, b]$,

$$\int_a^b f = F(b) - F(a).$$

Example 6.1. Consider the Lipschitz continuous function $f(x) := |x|$ on $\mathbb{R}$. Given $a < b$, we have

$$\int_a^b f = \begin{cases} \frac{b^2-a^2}{2} & \text{if } 0 \le a < b, \\ \frac{b^2+a^2}{2} & \text{if } a < 0 < b, \\ \frac{a^2-b^2}{2} & \text{if } a < b \le 0. \end{cases}$$

Indeed, when $0 \le a < b$ we have $f(x) := x$ on $[a, b]$ whence the Fundamental Theorem of Calculus with $F(x) = \frac{1}{2}x^2$ gives

$$\int_a^b f = F(b) - F(a) = \frac{b^2 - a^2}{2}.$$

When $a < b \le 0$ we instead find that $f(x) := -x$ on $[a, b]$. Therefore, the Fundamental Theorem with $F(x) := -\frac{1}{2}x^2$ implies that

$$\int_a^b f = F(b) - F(a) = \frac{a^2 - b^2}{2}.$$

Finally, if $a < 0 < b$, the additivity theorem together with the previous two cases shows that

$$\int_a^b f = \int_a^0 f + \int_0^b = \frac{a^2 - 0^2}{2} + \frac{b^2 - 0^2}{2} = \frac{a^2 + b^2}{2}.$$

Example 6.2. We claim that there does not exist a continuously differentiable function

$$f : [0, 2] \to \mathbb{R}$$

such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \le 2$ on $[0, 2]$. To justify this, we proceed by contradiction. If such a function f exists, then f' must be Riemann integrable on $[0, 2]$ by continuity. The Fundamental Theorem of Calculus would then imply

$$5 = f(2) - f(0) = \int_0^2 f' \le \int_0^2 2 = 4$$

which is absurd.

We also have the following counterpart to Theorem 6.1.

Theorem 6.2 (Fundamental Theorem of Calculus Form 2)**.** *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and set*

$$F : [a, b] \to \mathbb{R}, \quad F(x) := \int_a^x f.^{14}$$

The function F is Lipschitz continuous on $[a, b]$. Furthermore, if f is continuous at a point $c \in [a, b]$, then $F'(c)$ exists and is equal to $f(c)$.

We should point out that, without the continuity assumption of f at c, one need not have $F'(c) = f(c)$. Indeed, consider the function

$$f(x) := \begin{cases} 1 & \text{if } x \in \left\{\frac{1}{n} : n \in \mathbb{N}\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

You have proven in your fifth assignment that f is Riemann integrable on $[0, 1]$ and that $\int_0^1 f = 0$. Fix a point $x \in [0, 1]$ and note that, because $f \geq 0$,

$$0 \leq F(x) = \int_0^x f \leq \int_0^x f + \int_x^1 f = \int_0^1 f = 0.$$

Put otherwise, F is the constant function $F \equiv 0$. In particular, F is differentiable on $[0, 1]$ with $F' \equiv 0$.

- Note that f is discontinuous at 0. To see this, observe that

  $$f(0) = 0 \neq 1 = \lim f\left(\frac{1}{n}\right).$$

  That is, f does not obey the sequential criterion for continuity at 0 and therefore cannot be continuous at 0. Despite this, $F'(0) = f(0)$.

- On the other hand, by a similar argument, f is discontinuous at $\frac{1}{n}$ for each $n \in \mathbb{N}$. Moreover,

  $$F'\left(\frac{1}{n}\right) = 0 \neq 1 = f\left(\frac{1}{n}\right).$$

Especially, this example shows that the continuity assumption in of Theorem 6.2 cannot be dropped. Before proceeding, there is a monotonicity argument in the above example that deserves to be acknowledged:

[14] By convention, we define $\int_a^a f = 0$.

Lemma 6.3. *Let $f \in \mathcal{R}([a,b])$ be such that $f \geq 0$ on all of $[a,b]$. Then, for any $x \in [a,b]$, one has*

$$\int_a^x f \leq \int_a^b f.$$

Proof. Since f is Riemann integrable on $[a,b]$, the Additivity Theorem ensures that f is Riemann integrable on $[a,x]$ and $[x,b]$. Then, because the integral is monotone (see your assignments),

$$\int_x^b f \geq \int_x^b 0 = 0.$$

Consequently, by the Additivity Theorem,

$$\int_a^b f = \int_a^x f + \int_x^b f \geq \int_a^x f.$$

This completes the proof. □

6.1 Null Sets

We now introduce the notion of a null set. This concept is intimately related to the Lebsegue measure (and therefore measure theory) and is of fundamental importance to the theory of integration.

Definition 6.1 (Null Sets)**.** Let $Z \subseteq \mathbb{R}$. We say that Z is a *null set* (or of Lebesgue measure zero) if, for each $\varepsilon > 0$, one can find *countably*[15] many open intervals $\{I_k\}_k$ such that

$$Z \subseteq \bigcup_k I_k$$

and

$$\sum_k |I_k| \leq \varepsilon,$$

where $I_k = (a_k, b_k)$ and $|I_k| = b_k - a_k$ is the length of the interval I_k.

One can think of null sets as being "small" in the sense of "volume". This intuitive description is partially motivated by the following example:

[15]We allow for the possibility of a countably infinite family.

Example 6.3. We claim that every subset of a null set is also a null set. Let Z be a null set and fix $Y \subseteq Z$. Given $\varepsilon > 0$, there exists a countable family $\{I_k\}_k$ of open intervals such that $Z \subseteq \bigcup_k I_k$ and $\sum_k |I_k| \leq \varepsilon$. Since $Y \subseteq Z$, we also have $Y \subseteq \bigcup_k I_k$. As $\varepsilon > 0$ was arbitrary, we see that Y is a null set.

Perhaps the most intuitive examples of null sets are the finite subsets of $\mathbb{R}$. This precisely what we discuss in the next example.

Example 6.4. We claim that every finite set is a null set. Indeed, let $\{x_1, \ldots, x_n\}$ be a finite set. Given $\varepsilon > 0$, define for each $k = 1, \ldots, n$ the open interval

$$I_k := \left(x_k - \frac{\varepsilon}{2n}, x_k + \frac{\varepsilon}{2n}\right).$$

Since $x_k \in I_k$ for every index k, we have $\{x_1, \ldots, x_n\} \subseteq \bigcup_{k=1}^n I_k$. Furthermore,

$$\sum_{k=1}^n |I_k| = \sum_{k=1}^n \frac{\varepsilon}{n} = \varepsilon.$$

We infer that $\{x_1, \ldots, x_n\}$ is a null set.

One should be aware that null sets can be infinite, and even unbounded. As it turns out, there are many examples of such sets (e.g. $\mathbb{Z}$ and $\mathbb{Q}$). In fact every countable set is necessarily a null set. To establish this, we recall an elementary identity. Fix $r \in (0, 1)$ and note that

$$(1 - r)(1 + r + r^2 + \cdots + r^n) = 1 - r^{n+1}.$$

Especially,

$$\sum_{k=0}^n r^k = \frac{1 - r^{n+1}}{1 - r}. \tag{6.1}$$

Lemma 6.4. *Every countable subset of* $\mathbb{R}$ *is a null set.*

Proof. In our previous example we showed that finite sets are null. Therefore, we must only show that every countably infinite set is null. Let $E \subset \mathbb{R}$ be countably infinite and enumerate its elements as

$$x_1, \ldots, x_n, \ldots.$$

Let $\varepsilon > 0$ be given. For each natural number n let us define

$$I_n := \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right).$$

Clearly, $x_n \in I_n$ and so $E = \{x_n : n \in \mathbb{N}\} \subseteq \bigcup_{n\in\mathbb{N}} I_n$. Moreover, by (6.1),

$$\begin{aligned}
\sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\varepsilon}{2^n} &= \varepsilon \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{2}\right)^n \\
&= \varepsilon \lim_{N\to\infty} \left[\sum_{n=0}^{N} \left(\frac{1}{2}\right)^n - 1\right] \\
&= \varepsilon \lim_{N\to\infty} \left[\frac{1 - \left(\frac{1}{2}\right)^{N+1}}{1 - \frac{1}{2}} - 1\right] \\
&= \varepsilon \left[\frac{1}{\frac{1}{2} - 1} - 1\right] \\
&= \varepsilon.
\end{aligned}$$

It follows that Z is null. ☐

Corollary 6.5. *The set of rational numbers $\mathbb{Q}$ is a null set (i.e. $\mathbb{Q}$ has Lebesgue measure zero).*

Remark 6.1. We have shown that every countable set is a null set. It is, however, *not true* that every null set is countable. An example of a set which is not null is the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$.

Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. We say that f is **continuous almost everywhere** on A if the set of points of discontinuities of f is a null set. That is, if

$$D := \{x \in A : f \text{ is discontinuous at } x\}$$

is a null set. As a sanity check we note that because the empty set $\varnothing$ is null, a continuous function $f : A \to \mathbb{R}$ is, in particular, continuous almost everywhere. What is perhaps more surprising, however, is that every monotone function also satisfies this property:

Theorem 6.6. *Let $f : [a, b] \to \mathbb{R}$ be monotone. Then, f is continuous almost everywhere on $[a, b]$. More precisely, the set of discontinuity points*

$$D := \{d \in A : f \text{ is discontinuous at } d\}$$

is countable.

Proof. By replacing f with $-f$, we may assume without loss of generality that f is monotone increasing on $[a, b]$. Namely, we may assume that $x \leq y$ implies $f(x) \leq f(y)$. If D is empty then there is nothing to prove. Otherwise, choose a point $d \in D$. Since f is continuous at d if and only if

$$\lim_{x \to d^-} f(x) = \lim_{x \to d^+} f(x),$$

we must have

$$\lim_{x \to d^-} f(x) < \lim_{x \to d^+} f(x).$$

Using that $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a rational number r_d such that

$$\lim_{x \to d^-} f(x) < r_d < \lim_{x \to d^+} f(x).$$

Doing allows us to associate a rational number r_d to each $d \in D$ such that the above holds. Next, we consider the mapping

$$\Gamma : D \to \mathbb{Q}, \quad d \mapsto r_d.$$

We claim that Γ is injective. To this end, let $d, e \in D$ be such that $d \neq e$. Without loss of generality, we can assume that $d < e$. Since f is monotone increasing on $[a, b]$, we have

$$r_d < \lim_{x \to d^+} f(x) \leq \lim_{x \to e^-} f(x) < r_e.$$

In particular, $r_d \neq r_e$. That is, $\Gamma(d) \neq \Gamma(e)$ whenever $d \neq e$. It follows that Γ is indeed and injection from D into $\mathbb{Q}$. Since there exists an injection from D into a countably infinite set, we see that D is countable. This completes the proof. □

Consider once more the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ defined in ($\mathfrak{D}$). This function was shown to be discontinuous at every point in $\mathbb{R}$. Therefore, the set of discontinuities of the Dirichlet function f is exactly equal to $\mathbb{R}$, which can be shown to not be null. Consequently, f is an example of a function that is **not** continuous almost everywhere.

6.2 Lebesgue's Integrability Criterion

Thus far we have proven several conditions guaranteeing the Riemann integrability of certain functions on compact intervals $[a, b]$. Some of this are easy to apply (e.g. checking whether the function is monotone or continuous) but are unfortunately merely sufficient conditions for integrability. Furthermore, the "if and only if" conditions we possess for Riemann integrability are typically very technical and tedious to apply (e.g. Cauchy's Criterion and the Squeeze Theorem). Ideally, one would like a necessary and sufficient condition for Riemann integrability that can be verified by simply inspecting the function is question. This is precisely what Lebesgue's criterion achieves:

Theorem 6.7 (Lebesgue)**.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable on $[a, b]$ if and only if f is continuous almost everywhere on $[a, b]$. That is, $f \in \mathcal{R}([a, b])$ if and only if*

$$\{x \in [a, b] : f \text{ is discontinuous at } x\}$$

is a null set.

Corollary 6.8. *Every monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.*

Proof. A monotone function f on $[a, b]$ is clearly bounded. Furthermore, it is a consequence of Theorem 6.6 that f is continuous almost everywhere on $[a, b]$. It then follows from Lebesgue's criterion that f is Riemann integrable on $[a, b]$. □

Example 6.5. Lebesgue's Criterion implies at once the Riemann integrability of Thomae's function from Definition 3.1. Indeed, it was proven in Proposition 3.4 that Thomae's function was continuous at every irrational number in $[0, 1]$ and discontinuous at each rational number of $[0, 1]$. Therefore, the set of discontinuities of Thomae's function is precisely equal to $\mathbb{Q} \cap [0, 1] \subseteq \mathbb{Q}$. Since $\mathbb{Q} \cap [0, 1]$ is countable and hence a null set, it follows that Thomae's function is continuous almost everywhere on $[0, 1]$.

Lebesgue's criterion is followed by a myriad of applications. In fact, Lebesgue's Theorem allows for a short and elegant proof of the following major property of integration:

Theorem 6.9 (Composition Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function on $[a, b]$ and let $J \supseteq f([a, b])$. If $\varphi : J \to \mathbb{R}$ is continuous on J, the composition $\varphi \circ f$ is Riemann integrable on $[a, b]$.*

Corollary 6.10. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then, $|f| \in \mathcal{R}([a, b])$ and*

$$\left| \int_a^b f \right| \leq \int_a^b |f| \,. \tag{6.2}$$

Proof. Taking $\varphi(x) := |x|$ in Theorem 6.9 implies the Riemann integrability of $|f| = \varphi \circ f$. The identity (6.2) follows at once from Example 4.2. $\square$

Corollary 6.11. *Let $f, g \in \mathcal{R}([a, b])$.*

(i) The product fg is Riemann integrable on $[a, b]$.

(ii) $\max(f, g)$ and $\min(f, g)$ are both Riemann integrable on $[a, b]$.

Proof. First, taking $\varphi(x) := x^2$ in Theorem 6.9 shows that $f^2 = \varphi \circ f$ and $g^2 = \varphi \circ g$ are Riemann integrable on $[a, b]$. Since $\mathcal{R}([a, b])$ is a closed under finite linear combinations it follows that

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2}$$

is Riemann integrable on $[a, b]$. This establishes (i). Now, since $f + g \in \mathcal{R}([a, b])$, we see from Corollary 6.10 that $|f + g| \in \mathcal{R}([a, b])$. Writing

$$\max(f, g) = \frac{f + g + |f + g|}{2},$$

we get that $\max(f, g)$ is Riemann integrable as well. Similarly, because

$$\min(f, g) = \frac{f + g - |f - g|}{2},$$

we infer that $\min(f, g) \in \mathcal{R}([a, b])$. $\square$

7 Seventh Tutorial/Review

In preparation for the midterm exam, we will give a brief summary of what we have covered thus far and solve various problems related to this material. After having reviewed the notions of uniform continuity, differentiability and the Riemann integral, we will quickly cover the concept of uniform convergence and provide a worked out example. Note that the bare bones of uniform convergence *is* examinable.

7.1 Uniform Continuity

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Recall that f is called *uniformly continuous* on A if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ satisfy $|x - y| < \delta$. Clearly, if f is uniformly continuous then it is also continuous. The main difference between uniform continuity and continuity is that the $\delta > 0$ stated above can only depend on ε, i.e. it cannot depend on the point y. This makes uniform continuity a global property rather than a local property. Below we list some important properties that uniformly continuous functions satisfy:

- Uniformly continuous functions always map Cauchy sequences to Cauchy sequences. This need not be the case for continuous functions.
- If $A \subseteq \mathbb{R}$ is bounded and $f : A \to \mathbb{R}$ is uniformly continuous on A, then f is bounded. This fails for functions that are merely continuous (e.g. $1/x$ on $(0, 1)$).
- Sums and compositions of uniformly continuous functions remain uniformly continuous. However, products of uniformly continuous functions are not necessarily uniformly continuous. Indeed, $f(x) = x$ is uniformly continuous (even Lipschitz continuous) on $\mathbb{R}$ but $x \mapsto x^2$ is not.
- If $K \subset \mathbb{R}$ is compact and f is continuous on K, then f must be *uniformly* continuous on K. This of course need not hold if K is not closed and bounded.

Example 7.1. We claim that $f(x) := \sqrt{x^2 + 1}$ is uniformly continuous on $\mathbb{R}$. For this, it is enough to show that f is Lipschitz continuous. To see this, let $x, y \in \mathbb{R}$

be arbitrary and observe that

$$
\begin{aligned}
\left|\sqrt{x^2+1}-\sqrt{y^2+1}\right| &= \left|\left(\sqrt{x^2+1}-\sqrt{y^2+1}\right)\cdot\frac{\sqrt{x^2+1}+\sqrt{y^2+1}}{\sqrt{x^2+1}+\sqrt{y^2+1}}\right| \\
&= \frac{\left|x^2-y^2\right|}{\sqrt{x^2+1}+\sqrt{y^2+1}} \\
&= \frac{|x+y|\,|x-y|}{\sqrt{x^2+1}+\sqrt{y^2+1}} \\
&\le \left(\frac{|x|+|y|}{\sqrt{x^2+1}+\sqrt{y^2+1}}\right)|x-y| \\
&\le \left(\frac{|x|}{\sqrt{x^2+1}}+\frac{|y|}{\sqrt{y^2+1}}\right)|x-y| \\
&\le 2\,|x-y|\,.
\end{aligned}
$$

We infer that f is Lipschitz with constant $L = 2$.

Since working directly from the definition of uniform continuity is difficult when establishing *non-uniform* continuity, the following two-sequence criterion should be well understood:

Proposition 7.1 (Two Sequence Criterion)**.** *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. The following are equivalent:*

*(i) f is **not** uniformly continuous on A;*

(ii) there exists $\varepsilon_0 > 0$ and two sequences $(x_n), (y_n)$ in A such that

$$\lim |x_n - y_n| = 0$$

but $|f(x_n) - f(y_n)| \ge \varepsilon_0$ for all $n \in \mathbb{N}$.

Remark 7.1 (One Sequence Test). Because uniformly continuous functions necessarily take Cauchy sequences to Cauchy sequences, it is also enough to exhibit a Cauchy sequence (x_n) in A such that $(f(x_n))$ is not Cauchy. However, this is not an "if and only if" test as above.

Example 7.2. We prove that the function

$$f(x) := \sin\left(\frac{1}{x}\right)$$

is not uniformly continuous on $(0,\infty)$ using both the Two Sequence Criterion and the One Sequence Test. For the Two Sequence Criterion, consider the sequences

$$x_n := \frac{1}{2\pi n + \frac{\pi}{2}}, \quad y_n := \frac{1}{2\pi n}.$$

Clearly, $x_n, y_n \to 0$ and so $\lim(x_n - y_n) = 0$. In particular, $\lim |x_n - y_n| = 0$. However,

$$|f(x_n) - f(y_n)| = \left|\sin\left(2\pi n + \frac{\pi}{2}\right) + \sin(2\pi n)\right| = |1| = 1.$$

Hence, we can apply the criterion with $\varepsilon_0 := 1$. If we instead want to use the One Sequence Test, we define

$$x_n := \frac{1}{\pi n}.$$

Since $x_n \to 0$, it is convergence and therefore Cauchy in $(0,\infty)$. However, $f(x_n) = \sin(\pi n) = (-1)^n$ which diverges and therefore cannot be Cauchy.

Example 7.3. Similarly, we can see that the function

$$f(x) := \left|\sin\left(\frac{1}{x}\right)\right|$$

is not uniformly continuous on $(0,\infty)$. Indeed, define

$$x_n := \frac{1}{2\pi n + \frac{\pi}{2}}, \quad y_n := \frac{1}{2\pi n}$$

so that $|x_n - y_n| \to 0$ as $n \to \infty$. Then, as in the previous example, one has $|f(x_n) - f(y_n)| = 1$ for every $n \in \mathbb{N}$. We infer from the two sequence criterion that f is not uniformly continuous on $(0,\infty)$.

7.2 Differentiation

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ a function, and fix a point $c \in I$. We say that f is differentiable at c with derivative $f'(c)$ if the limit

$$f'(c) := \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

exists. If $f'(c)$ exists at all points $c \in I$ then we say that f is differentiable on I and the function $x \mapsto f'(x)$ is called the derivative of f on I. As above, we list some important properties below.

- If f is differentiable at c, then it must also be continuous there. Certainly, we have

$$\begin{aligned}\lim_{x\to c}(f(x)-f(c)) &= \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\cdot(x-c)\right]\\ &= \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c)\\ &= f'(c)\lim_{x\to c}(x-c)\\ &= 0.\end{aligned}$$

  Thus, $\lim_{x\to c} f(x) = f(c)$ whence f is continuous at c.

- Linear combinations of differentiable functions are differentiable. Indeed, if $f, g : I \to \mathbb{R}$ are differentiable at $c \in I$ then, for each $\alpha, \beta \in \mathbb{R}$, the function $(\alpha f + \beta g)$ is differentiable at c and moreover

$$(\alpha f + \beta g)'(c) = \alpha f'(c) + \beta g'(c).$$

- Similarly, if f, g are differentiable at $c \in I$, then $(fg)'(c)$ exists and is given by

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

- Let $f : I \to \mathbb{R}$ be differentiable at c and let $J \supseteq f(I)$ be an interval. If g is differentiable at $f(c)$, then the composition $g \circ f$ is also differentiable at c with

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Perhaps the most important result of this section is the *Mean Value Theorem.* This theorem lives at the very heart of many important results including Darboux's theorem and the Fundamental Theorem of Calculus. We restate and prove the Mean Value Theorem below.

Theorem 7.2 (Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then, there exists a point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Consider the function

$$g(x) := \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) - (f(x) - f(a))\,.$$

Clearly, g is continuous on $[a, b]$ and is differentiable on the open interval (a, b). Since $g(b) = g(a) = 0$, an application of Rolle's theorem implies the existence of a point $c \in (a, b)$ such that $g'(c) = 0$. However,

$$0 = g'(c) = \frac{f(b) - f(a)}{b - a} - f'(c)$$

yields

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

as was required. □

We now list several important consequences of the Mean Value Theorem. In each of these, unless stated otherwise, $I \subseteq \mathbb{R}$ is an arbitrary interval and $f : I \to \mathbb{R}$ is a differentiable function.

- The function f is increasing on I if and only if $f' \geq 0$ on I. Similarly, f is decreasing if and only if $f' \leq 0$ on I.

- Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). If $f'(c) = 0$ for all $c \in (a, b)$, then f is constant on I. Indeed, given any point $x \in [a, b]$ with $x > a$ we can apply the Mean Value Theorem on the interval $[a, x]$ to obtain a point $c \in (a, x) \subseteq (a, b)$ with the property that

  $$f'(c) = \frac{f(x) - f(a)}{x - a}.$$

  Since $f'(c) = 0$, this forces $f(x) = f(a)$. This proves that f is constant.

- Darboux's Theorem states that derivatives, despite not necessarily being continuous, obey the intermediate value property. This is sometimes useful when proving that certain functions cannot be derivatives.

Example 7.4. Let I be an interval and $f : I \to \mathbb{R}$ a differentiable function. If $f'(x) > 0$ for all $x \in I$, then f is *strictly increasing* on I. Indeed, fix $x < y$ in

I. Since f is continuous on $[x, y]$ and differentiable on (x, y), the Mean Value Theorem guarantees the existence of a point $c \in (x, y) \subseteq I$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c).$$

Or, rather, $f(y) - f(x) = f'(c)(y - x) > 0$. This proves that f is strictly increasing on I. However, the converse does not hold. Certainly, the function $f(x) := x^3$ is strictly increasing on $\mathbb{R}$ but $f'(0) = 0$.

Example 7.5. Let I be an interval and $f : I \to \mathbb{R}$ a differentiable function on I. We show that if f' is bounded on I then f is Lipschitz on I. To this end, let $L > 0$ be such that

$$|f'(x)| \le L$$

for all $x \in I$. Fix $x, y \in I$ with $x \neq y$. We may assume without loss of generality that $x < y$. Since f is continuous on $[x, y]$ and differentiable on (x, y), we may apply the Mean Value Theorem to obtain a point $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Therefore,

$$\left|\frac{f(y) - f(x)}{y - x}\right| = |f'(c)| \le L.$$

It follows that $|f(y) - f(x)| \le L|y - x|$ for all $x \neq y$ in I. Since this holds trivially for $x = y$, we see that f is Lipschitz continuous on I.

7.3 Riemann Integration

A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable on an interval $[a, b]$ if there exists a real number $\int_a^b f$ having the property that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|S(f; \dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon$$

whenever $\dot{\mathcal{P}}$ is a tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \delta$.

Example 7.6. Fix $c \in (0, 1)$ and let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} 1 & \text{if } 0 \le x < c, \\ 0 & \text{if } c \le x \le 1. \end{cases}$$

We claim that f is Riemann integrable on $[0,1]$ with $\int_0^1 f = c$. To see this, let $\varepsilon > 0$ be given and let $\delta := \varepsilon$. Let $\dot{\mathcal{P}}$ be a tagged partition of $[0,1]$ with $\|\dot{\mathcal{P}}\| < \delta$. Denote the partition points of $\dot{\mathcal{P}}$ by

$$0 = x_0 < \cdots < x_n = 1$$

and let $x_1^*, \ldots, x_n^*$ be the tag points. Let $k := \min\left\{j \geq 1 : x_j \geq c\right\}$. Note that $c \in (x_{k-1}, x_k]$. If $j > k$ then we must have $x_j^* \geq x_{j-1} \geq x_k$ whence $f(x_j^*) = 0$. It follows that

$$S(f;\dot{\mathcal{P}}) = \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) = \sum_{j=1}^{k} f(x_j^*)(x_j - x_{j-1}).$$

If $k = 1$ then the above is simply

$$S(f;\dot{\mathcal{P}}) = f(x_1^*)(x_1 - x_0) = f(x_1^*)x_1 = \begin{cases} x_1 & \text{if } f(x_1^*) = 1, \\ x_0 & \text{if } f(x_1^*) = 0. \end{cases}$$

We now assume that $k > 1$. Then, $f(x_j^*) = 1$ for all $1 \leq j < k$. Indeed, by the minimality of k we must have $x_j < c$. Consequently, $x_j^* \leq x_j < c$. This gives

$$\begin{aligned} S(f;\dot{\mathcal{P}}) = \sum_{j=1}^{k} f(x_j^*)(x_j - x_{j-1}) &= \sum_{j=1}^{k-1} f(x_j^*)(x_j - x_{j-1}) + f(x_k^*)(x_k - x_{k-1}) \\ &= \sum_{j=1}^{k-1} (x_j - x_{j-1}) + f(x_k^*)(x_k - x_{k-1}) \\ &= (x_{k-1} - x_0) + f(x_k^*)(x_k - x_{k-1}) \\ &= \begin{cases} x_k & \text{if } f(x_k^*) = 1, \\ x_{k-1} & \text{if } f(x_k^*) = 0. \end{cases} \end{aligned}$$

Thus, no matter the value of k, we obtain

$$S(f;\dot{\mathcal{P}}) = \begin{cases} x_k & \text{if } f(x_k^*) = 1, \\ x_{k-1} & \text{if } f(x_k^*) = 0. \end{cases}$$

Finally, we obtain

$$\left|S(f;\dot{\mathcal{P}}) - c\right| = \begin{cases} |x_k - c| & \text{if } f(x_k^*) = 1, \\ |x_{k-1} - c| & \text{if } f(x_k^*) = 0 \end{cases}$$

so that $\left|S(f;\dot{\mathcal{P}}) - c\right| \leq \|\dot{\mathcal{P}}\| < \delta = \varepsilon$. This completes the proof.

Having given an example using only the definition, we should reiterate all the important results we have gathered for the Riemann integrable. Note that many of these properties were proven by you in your assignments!

- Every Riemann integrable function is bounded. Furthermore, every monotone function on a compact interval is Riemann integrable there. The same can be said for continuous functions.
- We know that all step functions are Riemann integrable.
- If f, g are Riemann integrable on $[a, b]$, then so is $\alpha f + \beta g$ (where $\alpha, \beta \in \mathbb{R}$). Moreover,
$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g.$$
- If $f, g \in \mathcal{R}([a, b])$ with $f \leq g$ on $[a, b]$ then $\int_a^b f \leq \int_a^b g$.
- Modifying a Riemann integrable function at **finitely** many points does not affect the integrability of the function or the value of its integral.

By way of a practice example, we re-prove the following lemma about approximating by Riemann sums.

Lemma 7.3. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and let $\left(\dot{\mathcal{P}}_n\right)$ be a sequence of tagged partitions of $[a, b]$ such that*
$$\lim \left\|\dot{\mathcal{P}}_n\right\| = 0.$$
Then,
$$\lim S\left(f; \dot{\mathcal{P}}_n\right) = \int_a^b f.$$

Proof. Let $\varepsilon > 0$. Because f is Riemann integrable on $[a, b]$, there exists by definition some $\delta > 0$ such that
$$\left|S(f; \dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon \tag{7.1}$$
for all tagged partitions $\dot{\mathcal{P}}$ of $[a, b]$ with mesh strictly less than δ. On the other hand, because $\|\dot{\mathcal{P}}_n\| \to 0$ as $n \to \infty$, there exists a natural number $N \in \mathbb{N}$ such that
$$\left\|\dot{\mathcal{P}}_n\right\| = \left|\left\|\dot{\mathcal{P}}_n\right\| - 0\right| < \delta$$

for all $n \geq N$. Using $\dot{\mathcal{P}}_n$ in (7.1) we see that

$$\left| S(f; \dot{\mathcal{P}}_n) - \int_a^b f \right| < \varepsilon$$

for all $n \geq N$. By definition, this means that $S(f; \dot{\mathcal{P}}_n) \to \int_a^b f$ as $n \to \infty$. □

Next we provide an example in which the squeeze theorem can be used to establish the integrability of a function. This next example is very similar to two problems you've already handed in, but the idea behind the proof is extremely versatile.

Example 7.7. We prove (using the Squeeze Theroem) that

$$f(x) := \begin{cases} \left|\sin\left(\frac{1}{x}\right)\right| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is Riemann integrable on $[0, 1]$. Fix an arbitrary $\varepsilon > 0$ and let $N \geq 2$ be such that $\frac{1}{N} < \frac{\varepsilon}{2}$.[16] Since f is continuous on $\left[\frac{1}{N}, 1\right]$, it must be integrable there. Now, it is clear that $0 \leq f(x) \leq 1$ on $\left[0, \frac{1}{N}\right]$. Furthermore,

$$\int_0^{\frac{1}{N}} (1 - 0) = \frac{1}{N} < \varepsilon$$

Consider the functions

$$\alpha_\varepsilon(x) := \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{N}, \\ f(x) & \text{if } \frac{1}{N} \leq x \leq 1 \end{cases} \quad \text{and} \quad \omega_\varepsilon(x) := \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{N}, \\ f(x) & \text{if } \frac{1}{N} \leq x \leq 1. \end{cases}$$

Clearly, $\alpha_\varepsilon \leq f \leq \omega_\varepsilon$ on all of $[0, 1]$. Since f is Riemann integrable on $\left[\frac{1}{N}, 1\right]$ and $\alpha_\varepsilon = f$ there, α_ε is also Riemann integrable on $\left[\frac{1}{N}, 1\right]$. On the other hand, α_ε is equal to a constant function on $\left[0, \frac{1}{N}\right]$ except at the one point $x = \frac{1}{N}$. Thus, α_ε is integrable on the interval $\left[0, \frac{1}{N}\right]$. By the Additivity Theorem, we infer that

[16] I only want $N \geq 2$ to guarantee that $\frac{1}{N} < 1$. This way, both $[0, 1/N]$ and $[1/N, 1]$ are non-trivial intervals.

$\alpha_\varepsilon \in \mathcal{R}([0,1])$. Similarly, $\omega_\varepsilon \in \mathcal{R}([0,1])$. Finally, another application of the Additivity Theorem gives

$$\begin{aligned}
\int_0^1 (\omega_\varepsilon - \alpha_\varepsilon) &= \int_0^{1/N} (\omega_\varepsilon - \alpha_\varepsilon) + \int_{1/N}^1 (\omega_\varepsilon - \alpha_\varepsilon) \\
&= \int_0^{1/N} (1-0) + \int_{1/N}^1 0 \\
&= \frac{1}{N} \\
&< \varepsilon.
\end{aligned}$$

By the Squeeze Theorem, we infer that $f \in \mathcal{R}([0,1])$.

7.4 The Bare-Bones of Uniform Convergence

We now come to the last examinable topic of the midterm exam. In future tutorials we will cover the concept of uniform convergence in greater depth and detail. For now, we will focus on explicit examples to help with the intuition and general idea.

Definition 7.1 (Pointwise Convergence)**.** Let $A \subseteq \mathbb{R}$ and (f_n) be a sequence of functions defined on A. That is, for each $n \in \mathbb{N}$ we have an associated function $f_n : A \to \mathbb{R}$. We say that (f_n) converges *pointwise* to a function $f : A \to \mathbb{R}$ (as $n \to \infty$) if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for every *fixed* $x \in A$. Here, as x is fixed, the limit above is understood as the limit of a sequence of real numbers.

For example, let us consider the sequence of functions on $[0,1]$ defined by

$$f_n(x) := x^n.$$

At $x = 0$, one has $f_n(x) = f_n(0) = 0$ for each $n \in \mathbb{N}$. Therefore, $f_n(0)$ is the constant sequence $f_n(0) = 0$ and therefore $\lim_{n\to\infty} f_n(0) = 0$. Similarly, $\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1 = 1$. Now, if $0 < x < 1$, we have seen in analysis 1 that $x^n \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0.$$

Combining all possible cases for x, we see that

$$\lim_{n\to\infty} f_n(x) = f(x) := \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

This means that $f_n \to f$ *pointwise* on $[0, 1]$.

Definition 7.2. Let $A \subseteq \mathbb{R}$ and (f_n) a sequence of functions defined on A. We say that f_n converges *uniformly* to a function $f : A \to \mathbb{R}$ (as $n \to \infty$) if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in A$ and all $n \ge N$ there holds

$$|f_n(x) - f(x)| < \varepsilon.$$

Note that here the natural number N is *not* allowed to depend on the point x. Equivalently, we say that $f_n \to f$ uniformly on A provided, for each $\varepsilon > 0$, one can find $N \in \mathbb{N}$ such that

$$\sup_{x\in A} |f_n(x) - f(x)| \le \varepsilon$$

for all $n \ge N$.

Example 7.8. Define on $\mathbb{R}$ the sequence of functions

$$f_n(x) := \sqrt{x^2 + \frac{1}{n^2}}.$$

Clearly, each f_n is continuous (even differentiable) on all of $\mathbb{R}$. Now, for each fixed $x \in \mathbb{R}$, the standard limit laws show that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n^2}} = \sqrt{x^2} = |x| .$$

Letting $f(x) := |x|$, we see that $f_n \to f$ pointwise on $\mathbb{R}$. We now inspect the possible uniform convergence of this sequence. First, we observe that

$$\begin{aligned} |x| = \sqrt{x^2} \le f_n(x) = \sqrt{x^2 + \frac{1}{n^2}} &\le \sqrt{|x|^2 + \frac{2\,|x|}{n} + \frac{1}{n^2}} \\ &= \sqrt{\left(|x| + \frac{1}{n}\right)^2} \\ &= |x| + \frac{1}{n}. \end{aligned}$$

In particular, for each $x \in \mathbb{R}$ and all $n \in \mathbb{N}$,

$$|x| - \frac{1}{n} \leq f_n(x) \leq |x| + \frac{1}{n}$$

whence

$$|f_n(x) - f(x)| \leq \frac{1}{n}, \quad \forall x \in \mathbb{R} \text{ and } \forall n \in \mathbb{N}.$$

Given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then, for all $n \geq N$ and all $x \in \mathbb{R}$,

$$|f_n(x) - f(x)| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

This shows that $f_n \to f$ uniformly on $\mathbb{R}$.

We also recall the following theorem proven yesterday in class. In general, this result makes it much easier to show that certain pointwise convergences cannot be uniform.

Theorem 7.4. *Let $A \subseteq \mathbb{R}$ and let (f_n) be a sequence of functions defined on A converging uniformly to a function $f : A \to \mathbb{R}$ on the set A. If each f_n is continuous at a point $c \in A$, then so is f.*

Proof. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly on A, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$$

for all $x \in A$ and every $n \geq N$. Now, f_N is continuous at the point $c \in A$. Thus, there exists $\delta > 0$ such that

$$|f_N(x) - f_N(c)| < \frac{\varepsilon}{3}$$

for all $x \in A$ with $|x - c| < \delta$. Finally, note that the triangle inequality implies

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \frac{2\varepsilon}{3} + |f_N(x) - f_N(c)|. \end{aligned}$$

If in addition $|x - c| < \delta$, then

$$|f(x) - f(c)| < \frac{2\varepsilon}{3} + |f_N(x) - f_N(c)| < \varepsilon.$$

This completes the proof. □

Corollary 7.5. *Let $A \subseteq \mathbb{R}$ and let (f_n) be a sequence of functions defined on A converging uniformly to a function $f : A \to \mathbb{R}$ on the set A. If each f_n is continuous on A, then so is f.*

Equipped with this result, let us once again consider the sequence of functions defined in Example 7.8.

Example 7.9. By the chain rule, every function f_n defined in the previous example is differentiable on $\mathbb{R}$ with derivative given by

$$f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}}.$$

However, the uniform limit of these f_n's is not differentiable at 0. Indeed, the function $f(x) = |x|$ does not have a derivative at $x = 0$. Despite this, the functions f_n' still converge pointwise on $\mathbb{R}$. More precisely,

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}} = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

But, is it true that f_n' converges uniformly to this function described above? To answer this, denote by g the pointwise limit of (f_n'), i.e.

$$g(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

We claim that f_n' does **not** converge uniformly to g on any compact interval $[-a, a]$. To see this, we argue by way of contradiction. Assume that $f_n' \to g$ *uniformly* on $[-a, a]$. Since every f_n' is continuous on $[-a, a]$, the previous Corollary implies that the uniform limit g must be continuous on $[-a, a]$ as well. However, g is clearly discontinuous at 0. Hence, the convergence cannot be uniform.

To further illustrate why certain sequences may converge pointwise but not uniformly, we offer one last example. Here, however, the pointwise limit will indeed be continuous. Therefore, we will *not* be able to apply Corollary 7.5 to directly obtain a contradiction (as in the previous example).

Example 7.10. Consider the sequence of functions f_n defined by

$$f_n(x) := \frac{nx}{1 + x^2 n^2}$$

on $\mathbb{R}$. Clearly, for every fixed $x \in \mathbb{R} \setminus \{0\}$, we have

$$0 \le f_n(x) = \frac{nx}{1 + x^2 n^2} \le \frac{nx}{x^2 n^2} = \frac{1}{x} \cdot \frac{1}{n}$$

where $\frac{1}{n} \to 0$ as $n \to \infty$. By the Squeeze Theorem for sequences, we see that $f_n(x) \to 0$ as $n \to \infty$. This is simply the statement that $f_n \to 0$ pointwise on $\mathbb{R} \setminus \{0\}$. Since $f_n(0) = 0 \to 0$ as $n \to \infty$, we see that $f_n \to 0$ pointwise on all of $\mathbb{R}$.

Let $a > 0$ be given; we claim that $f_n \to 0$ uniformly on $[a, \infty)$. Indeed, for all $x \in [a, \infty)$ and every $n \ge 1$ one has

$$|f_n(x) - 0| = f_n(x) = \frac{nx}{1 + x^2 n^2} \le \frac{nx}{n^2 x^2} = \frac{1}{nx} \le \frac{1}{na}.$$

By the Archimedean property, there exists $N \in \mathbb{N}$ such that $\frac{1}{Na} < \varepsilon$, where $\varepsilon > 0$ is arbitrary but fixed. Then, for all $n \ge N$ and all $x \in [a, \infty)$,

$$|f_n(x) - 0| \le \frac{1}{na} \le \frac{1}{Na} < \varepsilon.$$

It follows that $f_n \to 0$ uniformly on $[a, \infty)$ for all $a > 0$. Nonetheless, f_n does not converge uniformly to 0 on $[0, \infty)$.[17] To see this, assume by contradiction that $f_n \to 0$ uniformly on $[0, \infty)$. By definition, one could find $N \in \mathbb{N}$ such that

$$|f_n(x) - 0| = \frac{nx}{1 + x^2 n^2} < \frac{1}{2}$$

for all $n \ge N$ and every $x \in [0, \infty)$. In particular, for each $n \ge N$,

$$\frac{1}{2} = \left| f_n\left(\frac{1}{n}\right) - 0 \right| < \frac{1}{2}$$

which is a contradiction. This shows that f_n does not converge to 0 uniformly on $[0, \infty)$.

[17] Note that because the pointwise limit $g = 0$ is continuous, we cannot apply Corollary 7.5. Instead, we must resort to a more technical argument.

Remark 7.2. This last example shows that one must sometimes resort to direct methods when showing that a sequence of functions does not converge uniformly. Although this was not necessary for Example 7.9, it is still helpful to see how one could solve this example using only the definition of uniform convergence. More precisely, how can one show that $f_n' \not\to g$ uniformly on $[-a, a]$ using only the definition of uniform convergence (and not Corollary 7.5)?

By way of contradiction, let us assume that $f_n' \to g$ uniformly on $[-a, a]$. Then, taking $\varepsilon := \frac{1}{2}$ we can find $N \in \mathbb{N}$ such that

$$|f_n'(x) - g(x)| < \frac{1}{2}, \quad \forall n \geq N,$$

and all $x \in [-a, a]$. In particular, for all $n \geq N$ and every $x \in (0, a]$,

$$\left|f_N'(x) - g(x)\right| < \frac{1}{2}.$$

However, this implies that

$$\left|\frac{x}{\sqrt{x^2 + \frac{1}{N^2}}} - 1\right| < \frac{1}{2}, \quad \forall x \in (0, a].$$

Consider now the sequence (x_k) given by $x_k := \frac{1}{k}$. Since $x_k \to 0$ as $k \to \infty$, there exists $K \in \mathbb{N}$ such that $x_k \in (0, a] \subseteq [-a, a]$ for all $k \geq K$.[18] Therefore,

$$\left|\frac{x_k}{\sqrt{x_k^2 + \frac{1}{N^2}}} - 1\right| = \left|f_N'(x_k) - g(x_k)\right| < \frac{1}{2}$$

for all $k \geq K$. Taking the limit as $k \to \infty$, we find that

$$|0 - 1| = \left|\lim_{k\to\infty} \frac{x_k}{\sqrt{x_k^2 + \frac{1}{N^2}}} - 1\right| = \lim_{k\to\infty}\left|\frac{x_k}{\sqrt{x_k^2 + \frac{1}{N^2}}} - 1\right| \leq \frac{1}{2}$$

which is a contradiction. Therefore, f_n' does not converge to g uniformly.

[18] By the $\varepsilon - N$ definition of convergence for sequences, there exists $K \in \mathbb{N}$ such that $|x_k| < a$ for all $k \geq K$. Thus, $x_k \in (0, a)$ for all $k \geq K$.

8 Eighth Tutorial

In this tutorial, we continue to cover examples and applications of uniform convergence. We recall that a given sequence of functions $f_n : A \to \mathbb{R}$ converges uniformly to a function $f : A \to \mathbb{R}$, as $n \to \infty$, if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in A$ and all $n \geq N$. In practice, to show that (f_n) converges uniformly to f on A, we try to find a sequence (a_n) *converging to* 0 such that

$$|f_n(x) - f(x)| \leq a_n$$

for all $x \in A$ and all $n \in \mathbb{N}$. This is sufficient because, for each $\varepsilon > 0$, one can find $N \in \mathbb{N}$ such that $a_n < \varepsilon$ for all $n \geq N$. Consequently, for all $n \geq N$ and all $x \in A$ there holds

$$|f_n(x) - f(x)| \leq a_n < \varepsilon$$

whence $f_n \to f$ uniformly on A. We restate this below in the form of a lemma:

Lemma 8.1. *Let $A \subseteq \mathbb{R}$ and (f_n) be a sequence of functions on A and fix a function $f : A \to \mathbb{R}$. Assume that there exists a sequence (a_n) such that*

$$|f_n(x) - f(x)| \leq a_n$$

for all $n \in \mathbb{N}$ and every $x \in A$. If $a_n \to 0$ as $n \to \infty$, then $f_n \to f$ uniformly as $n \to \infty$.

Example 8.1. Consider the sequence of functions (f_n) on $\mathbb{R}$ defined by

$$f_n(x) := \frac{x}{1 + nx^2}.$$

To identify the uniform limit of these f_n's (if it exists), we should first try and determine their pointwise limit. Clearly, for every fixed $x \neq 0$ we have

$$|f_n(x)| \leq \frac{|x|}{1 + nx^2} \leq \frac{|x|}{nx^2} = \frac{1}{n\,|x|} \xrightarrow{n\to\infty} 0.$$

Thus, if $x \neq 0$ is fixed, then the Squeeze Theorem for sequences implies that

$$\lim_{n\to\infty} f_n(x) = 0.$$

Moreover,

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0.$$

We infer that $f_n(x) \to 0$, as $n \to \infty$, for each $x \in \mathbb{R}$. This is precisely the statement that $f_n \to 0$ pointwise on $\mathbb{R}$. We now assert that $f_n \to 0$ uniformly on $\mathbb{R}$. For this, we will require the following identity:

$$\frac{|t|}{1+t^2} \leq \frac{1}{2}, \quad \forall t \in \mathbb{R}. \tag{8.1}$$

To prove (8.1), it suffices to check that

$$t^2 - 2|t| + 1 \geq 0, \quad \forall t \in \mathbb{R}.$$

However, $t^2 - 2|t| + 1 = (|t| - 1)^2 \geq 0$ for all $t \in \mathbb{R}$. This verifies (8.1). Returning to the example, we can apply (8.1) with $t = \sqrt{n}x$ to obtain the following uniform estimate:

$$\begin{aligned} |f_n(x) - 0| = \frac{|x|}{1+nx^2} &= \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{n}\,|x|}{1+nx^2} \\ &= \frac{1}{\sqrt{n}} \cdot \frac{\left|\sqrt{n}x\right|}{1+\left(\sqrt{n}x\right)^2} \\ &\overset{(8.1)}{\leq} \frac{1}{2\sqrt{n}} \end{aligned}$$

for all $x \in \mathbb{R}$ and every $n \in \mathbb{N}$. Applying Lemma 8.1 with $a_n := \frac{1}{2\sqrt{n}}$ then shows that $f_n \to 0$ uniformly.

Remark 8.1. We have seen in the lectures and tutorials that uniform convergence does *not* imply the uniform convergence of the derivatives. As is turns out, counter examples to such a statement are typically neither rare nor contrived. In fact, the example above is yet another instance of such a sequence. Indeed, by the chain rule and quotient rule, each f_n is continuously differentiable with derivative given by

$$f_n'(x) = \frac{(1+nx^2) - 2nx^2}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}.$$

Clearly,

$$\lim_{n\to\infty} f_n'(0) = \lim_{n\to\infty} 1 = 1.$$

On the other hand, if $x \neq 0$ is fixed,

$$\left|\frac{1-nx^2}{(1+nx^2)^2}\right| \leq \frac{1+nx^2}{(1+nx^2)^2} = \frac{1}{1+nx^2} \leq \frac{1}{nx^2}$$

which tends to 0 as $n \to \infty$. Hence, $f_n'(x) \to 0$, as $n \to \infty$, for all $x \neq 0$. Put otherwise,

$$f_n'(x) \to g(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

However, defining $f(x) := 0$ on $\mathbb{R}$, we have shown above that $f_n \to f$ uniformly. Despite this, $f_n'(0) \not\to f'(0)$. Thus, $f_n' \not\to f'$ pointwise on $\mathbb{R}$ even though f_n' *does* converge pointwise. Furthermore, (f_n') does not converge uniformly to g on $\mathbb{R}$ since g is discontinuous at 0.

8.1 Uniform Convergence of the Derivatives

It should now be clear that uniform convergence does not imply the uniform convergence of the derivatives. In fact, if (f_n) is a sequence of differentiable functions converging uniformly to a differentiable function f, the sequence (f_n') may not even converge pointwise to f'. Therefore, deducing properties about the convergence of (f_n') using the uniform convergence of (f_n) can in general be quite tricky. However, the reverse does sometimes hold. Namely, under very reasonable assumptions, one can show that $f_n \to f$ uniformly if $f_n' \to f'$ uniformly. To establish this, we first recall the following class result:

Theorem 8.2. *Let (f_n) be a sequence of Riemann integrable functions on $[a, b]$ converging uniformly to a function $f : [a, b] \to \mathbb{R}$ on $[a, b]$. Then, f is Riemann integrable on $[a, b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

In fact, one can say slightly more:

Corollary 8.3. *If (f_n) is a sequence of Riemann integrable functions on $[a, b]$ converging uniformly to a function $f : [a, b] \to \mathbb{R}$ on $[a, b]$, then f is Riemann integrable on $[a, b]$ and*

$$\lim_{n\to\infty} \int_a^b |f_n - f| = 0.$$

Proof. Since f_n is Riemann integrable for every $n \in \mathbb{N}$ and $f \in \mathcal{R}([a, b])$ by the previous theorem, $(f_n - f)$ is a sequence of Riemann integrable functions on $[a, b]$. By Corollary 6.10, $|f_n - f|$ is also Riemann integrable on $[a, b]$. Because $f_n \to f$ uniformly if and only if $|f_n - f| \to 0$ uniformly, another application of the previous theorem ensures that

$$\lim_{n\to\infty} \int_a^b |f_n - f| = \int_a^b 0 = 0.$$

□

Theorem 8.4. *Let (f_n) be a sequence of continuously differentiable functions on $[a, b]$ and let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function such that $f_n \to f$ pointwise on $[a, b]$. Let $g : [a, b] \to \mathbb{R}$ be a function such that $f_n' \to g$ uniformly on $[a, b]$. Then, f is continuously differentiable with $f' = g$ on $[a, b]$. Moreover, $f_n \to f$ uniformly on $[a, b]$ as $n \to \infty$. In fact,*

$$f(x) = f(a) + \int_a^x g \tag{8.2}$$

for all $x \in [a, b]$.

Proof. We begin by establishing (8.2). Fix a point $x \in [a, b]$; by the Fundamental Theorem of Calculus we may write

$$f_n(x) = f_n(a) + \int_a^x f_n'. \tag{8.3}$$

Since $f_n \to f$ pointwise on $[a, b]$, we have

$$f_n(x) \to f(x) \quad \text{and} \quad f_n(a) \to f(a)$$

as $n \to \infty$. Note that by assumption every f_n' is continuous on $[a, b]$. Then, as the uniform limit of continuous functions, g is continuous on $[a, b]$. Furthermore, $f_n' \to g$ uniformly on $[a, x]$ for each $x \in [a, b]$. Invoking Theorem 8.2, we obtain

$$\lim_{n\to\infty} \int_a^x f_n' = \int_a^x g.$$

Using all of this in (8.3),

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(f_n(a) + \int_a^x f_n' \right) = f(a) + \int_a^x g.$$

This verifies (8.2). We now prove the main result. Since g is continuous on $[a, b]$, the Fundamental Theorem of Calculus implies that $x \mapsto \int_a^x g$ is differentiable on $[a, b]$ with derivative given by $g(x)$, for each $x \in [a, b]$. Because $f(a)$ is just a constant, it follows from (8.2) that

$$f'(x) = g(x)$$

on all of $[a, b]$. As g is continuous, this means that f is continuously differentiable. It only remains to check that $f_n \to f$ uniformly on $[a, b]$. To this end, we fix $\varepsilon > 0$. Since $f_n' \to g$ uniformly on $[a, b]$, and application of Corollary 8.3 shows that

$$\lim_{n\to\infty} \int_a^b |f_n' - g| = 0.$$

Thus, there exists $N_1 \in \mathbb{N}$ such that

$$\int_a^b |f_n' - g| < \frac{\varepsilon}{2}$$

for all $n \geq N_1$. Since $f_n \to f$ pointwise, there exists (as above) $N_2 \in \mathbb{N}$ such that

$$|f_n(a) - f(a)| < \frac{\varepsilon}{2}$$

for all $n \geq N_2$. Let now $N := \max(N_1, N_2)$ and fix $n \geq N$. For any such n, (8.2) and (8.3) together with Corollary 6.10 assert that

$$\begin{aligned}
|f_n(x) - f(x)| &= \left|\left(f_n(a) - \int_a^x f_n'\right) - \left(f(a) - \int_a^x g\right)\right| \\
&\leq |f_n(a) - f(a)| + \left|\int_a^x (f_n' - g)\right| \\
&\leq |f_n(a) - f(a)| + \int_a^x |f_n' - g| \\
&\leq |f_n(a) - f(a)| + \int_a^b |f_n' - g| \,.
\end{aligned}$$

In this last step, we have made use of Lemma 6.3. It follows that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for any such n and all $x \in [a, b]$. We infer that $f_n \to f$ uniformly on $[a, b]$. $\square$

8.2 More On of Uniform Limits

At this point it is quite convincing that uniform limits enjoy nicer properties than simple pointwise limits. This section explores more of these nice properties, the first of which is an elegant boundedness result.

Proposition 8.5. *Let $A \subseteq \mathbb{R}$ and (f_n) be a sequence of functions converging uniformly to a function $f : A \to \mathbb{R}$ on the set A. If each f_n is bounded, then so is f.*[19] *That is, uniform limits of bounded functions are bounded.*

Proof. Let $\varepsilon = 1$. Since $f_n \to f$ uniformly on A, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1$$

for all $n \geq N$ and every $x \in A$. Since f_N is bounded, one has

$$\begin{aligned} |f(x)| = |f(x) - f_N(x) + f_N(x)| &\leq |f(x) - f_N(x)| + |f_N(x)| \\ &< 1 + M_N \end{aligned}$$

for any $x \in A$. This is precisely the statement that f is bounded on A. □

We also offer a slight improvement of our previous statement:

Corollary 8.6. *Let (f_n) be a sequence of functions, each defined on a set $A \subseteq \mathbb{R}$, converging uniformly to a function $f : A \to \mathbb{R}$ on A. If each f_n is bounded, then the sequence (f_n) is uniformly bounded. More precisely, if there exists for each $n \in \mathbb{N}$ some $M_n > 0$ such that*

$$|f_n(x)| \leq M_n, \quad \forall x \in A,$$

then there exists $M > 0$ such that

$$|f_n(x)| \leq M$$

for all $x \in A$ and all $n \in \mathbb{N}$. Moreover,

$$|f(x)| \leq M$$

for every $x \in A$.

[19] Here we assume that, for each $n \in \mathbb{N}$, there exists $M_n > 0$ such that $|f_n(x)| \leq M_n$ for all $x \in A$. It is allowed that these M_n depend on the index n of the sequence!

Proof. Citing the previous proposition, the uniform limit f is bounded on A. Thus, there exists a constant $C > 0$ such that $|f(x)| \leq C$ for all $x \in A$. Now, as $f_n \to f$ uniformly on A, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1$$

for all $n \geq N$ and each $x \in A$. In particular, we have

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| < 1 + C$$

for every $n \geq N$ and any $x \in A$. Since $|f_n| \leq M_n$ on A for every $n \in \mathbb{N}$, it follows from the above that, given any $n \in \mathbb{N}$,

$$|f_n(x)| \leq \max\{M_1, M_2, \ldots, M_{N-1}, 1 + C\} =: M.$$

for every point $x \in A$. It follows that $|f_n(x)| \leq M$ for every $x \in A$ and all $n \in \mathbb{N}$. Since $|f(x)| \leq C < 1 + C \leq M$ on A, we are done. □

By a similar argument to the proof of Proposition 8.5, we also obtain the following:

Proposition 8.7. *Let $A \subseteq \mathbb{R}$ and (f_n) be a sequence of functions converging uniformly to a function $f : A \to \mathbb{R}$ on the set A. If every f_n is unbounded, then f is also unbounded.*

Proof. We argue by contradiction. Assuming that f is bounded on A, there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$. For $\varepsilon := 1$, we can also find $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1$$

for all $n \geq N$ and every $x \in A$. In particular,

$$|f_N(x)| = |f_N(x) - f(x) + f(x)| \leq |f_N(x) - f(x)| + |f(x)| < 1 + M$$

for every $x \in A$. This implies that f_N is bounded on A, which is a contradiction. □

We have proven that uniform limits of continuous functions are continuous. More precisely, we proved in Theorem 7.4 that if $f_n \to f$ uniformly and every f_n is continuous at a point c, then so is the limit f. It is therefore reasonable to ask whether uniform limits also preserve points of *discontinuitiy*. This answer is negative and this is justified by the following example:

Example 8.2. For each $n \in \mathbb{N}$ consider the function

$$f_n : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} \frac{1}{n} & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We claim that every f_n is discontinuous on all of $\mathbb{R}$. Indeed, if f_n were continuous at a point $c \in \mathbb{R}$, then so would be the function nf_n. However, nf_n is precisely the Dirichlet function from ($\mathfrak{D}$), which was shown to be everywhere discontinuous in Proposition 3.3. Hence, f_n is discontinuous at every point $c \in \mathbb{R}$.

Despite this, $f_n \to 0$ uniformly on $\mathbb{R}$, where the constant function $x \mapsto 0$ is everywhere continuous (even infinitely differentiable). To see this, note that for each $n \in \mathbb{N}$ one has

$$|f_n(x)| = f_n(x) \leq \frac{1}{n}$$

for all $x \in \mathbb{R}$. By Lemma 8.1, it follows that $f_n \to 0$ uniformly on $\mathbb{R}$.

Proposition 8.8 (Composition Theorem for Uniform Convergence)**.** *Let $A \subseteq \mathbb{R}$ and let (f_n) be a sequence of functions converging uniformly to a function f on A. Assume that every f_n is bounded on A. If $g : \mathbb{R} \to \mathbb{R}$ is continuous, then*

$$g \circ f_n \to g \circ f$$

uniformly on A.

Proof. By virtue of Corollary 8.6, there exists $M > 0$ such that $|f_n(x)| \leq M$ and $|f(x)| \leq M$ for every $n \in \mathbb{N}$ and each $x \in A$. Let $\varepsilon > 0$ be given. Since $[-M, M]$ is compact, g is uniformly continuous on $[-M, M]$. Thus, there exists $\delta > 0$ such that

$$|g(y) - g(v)| < \varepsilon \tag{8.4}$$

for all $y, v \in [-M, M]$ with $|y - v| < \delta$. On the other hand, because $f_n \to f$ uniformly on A, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \delta$$

for all $n \geq N$. For any such n and all $x \in A$, we see from (8.4) (with $y := f_n(x)$ and $v := f(x)$) that

$$|(g \circ f_n)(x) - (g \circ f)(x)| = |g(f_n(x)) - g(f(x))| < \varepsilon.$$

Since this holds for all $n \geq N$ and every $x \in A$, it follows that $g \circ f_n \to g \circ f$ uniformly on A. □

Remark 8.2. Note that the result above continues to hold if g is merely assumed to be continuous on the interval $[-M, M]$.

8.3 Uniform Limits of Uniformly Continuous Functions

Having shown that uniform limits preserve certain local properties (e.g. continuity), we now ask which global properties survive limit. Having already seen that boundedness is preserved by uniform limits, we now check that uniform limits of *uniformly* continuous functions are still uniformly continuous.

Proposition 8.9. *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Let (f_n) be a sequence of uniformly continuous functions defined on A and assume that $f_n \to f$ uniformly on A. Then, f is uniformly continuous on A.*

Proof. We can use the same proof as in Theorem 7.4. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ and each $t \in A$,

$$|f_n(t) - f(t)| < \frac{\varepsilon}{3}.$$

For any $x, u \in A$ there holds

$$\begin{aligned} |f(x) - f(u)| &= |f(x) - f_N(x) + f_N(x) - f_N(u) + f_N(u) - f(u)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(u)| + |f_N(u) - f(u)| \\ &\leq \frac{2\varepsilon}{3} + |f_N(x) - f_N(u)|\,. \end{aligned}$$

Now, f_N is uniformly continuous on A. Thus, there exists $\delta > 0$ such that

$$|f(x) - f(u)| < \frac{\varepsilon}{3}$$

whenever $|x - u| < \delta$. So, if $|x - u| < \delta$, our calculations above indicate that

$$|f(x) - f(u)| \leq \frac{2\varepsilon}{3} + |f_N(x) - f_N(u)| < \varepsilon$$

whence f is uniformly continuous on A. □

9 Ninth Tutorial

Let (a_n) be a sequence of real numbers. We say that the series $\sum_{n=1}^{\infty} a_n$ converges to a number $S \in \mathbb{R}$, provided the sequence of partial sums (S_N) defined by

$$S_N := \sum_{n=1}^{N} a_n$$

converges to S as $N \to \infty$. That is, $\sum_{n=1}^{\infty} a_n$ converges to $S \in \mathbb{R}$ if and only if

$$\lim_{N\to\infty} S_N = S.$$

In this case, we will write $\sum_{n=1}^{\infty} a_n$ to denote the limit of the partial sums, i.e.

$$\sum_{n=1}^{\infty} a_n := \lim_{N\to\infty} S_N.$$

Additionally, we say that the series $\sum_{n=1}^{\infty} a_n$ converges *absolutely* whenever $\sum_{n=1}^{\infty} |a_n|$ converges in the traditional sense. Before stating some of the various convergence tests we have seen in the lectures, we recall an example of a series that we encountered previously when discussing null sets in §6.1.

Example 9.1. Let $|r| < 1$. Then, the series $\sum_{n=0}^{\infty} r^n$ is convergent, with limit

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Now, if a series $\sum_{n=1}^{\infty} a_n$ is convergent, then the sequence of terms (a_n) must *also* converge to 0 as $n \to \infty$. This can be reformulated as follows:

Theorem 9.1 (Divergence Test)**.** *Let (a_n) be a sequence of real numbers. If $\sum_{n=1}^{\infty} a_n$ is convergent, then* $\lim a_n = 0$. *Therefore, if $a_n \not\to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n$ does not converge.*

The so-called Divergence Test is easy to use when it applies. As seen in the lectures, this provides us with an easy proof that the alternating series $\sum_{n=1}^{\infty} (-1)^n$ diverges. We provide another straightforward example using the Divergence Test below.

Example 9.2. The series

$$\sum_{n=1}^{\infty} \frac{n^3+3}{2n^3+1}$$

does not converge. Indeed, it is not hard to see that

$$\lim \frac{n^3+3}{2n^3+1} = \frac{1}{2} \neq 0.$$

Thus, $\sum_{n=1}^{\infty} \frac{n^3+3}{2n^3+1}$ diverges by the Divergence Test.

We should carefully note that the Divergence Test is not an “if and only if” condition. More precisely, Theorem 9.1 asserts that $\sum_{n=1}^{\infty} a_n$ diverges whenever $a_n \not\to 0$ as $n \to \infty$. This theorem does *not* claim that the series converges when $a_n \to 0$. In fact, this implication cannot possibly be true. Certainly, the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent despite the fact that $\lim \frac{1}{n} = 0$.

Proposition 9.2. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with $a_n \geq 0$ for each $n \in \mathbb{N}$. Let (b_n) be a bounded sequence of non-negative real numbers. Then,*

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Proof. Let $L > 0$ be such that $0 \leq b_n \leq L$ for all $n \in \mathbb{N}$. Given $N \in \mathbb{N}$, we define S_N to be the partial sum

$$S_N := \sum_{n=1}^{N} a_n b_n.$$

The statement amounts to proving that the sequence (S_N) is convergent. Now, since $a_n b_n \geq 0$ for every natural number n, the sequence of partial sums (S_N) is increasing in N. Consequently, by the Monotone Convergence Theorem, (S_N) converges if and only if it is bounded. Using that $\sum_{n=1}^{\infty} a_n$ is convergent, it follows that the partial sums

$$\sum_{n=1}^{N} a_n$$

converge and are hence bounded by some constant $M > 0$.[20] Therefore, given $N \in \mathbb{N}$ there holds

$$0 \leq S_N = \sum_{n=1}^{N} a_n b_n \leq L \sum_{n=1}^{N} a_n \leq LM < \infty.$$

It follows that (S_N) is a bounded increasing sequence, whence the Monotone Convergence Theorem implies that (S_N) converges as $N \to \infty$. $\square$

[20] In fact, since the partial sums $\sum_{n=1}^{N} a_n$ are increasing, we can take M to be the value of the series $\sum_{n=1}^{\infty} a_n$. However, this is not necessary for our argument.

Let us now take a moment to point out several important consequences of this result.

Corollary 9.3. *Let $\sum_{n=1}^{\infty} a_n$ converge absolutely and let (b_n) be a bounded sequence of real numbers. Then, $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.*

Proof. By definition of absolute convergence, $\sum_{n=1}^{\infty} |a_n|$ is convergent. Since (b_n) is bounded, the sequence $(|b_n|)$ is also bounded. Therefore, the series

$$\sum_{n=1}^{\infty} |a_n b_n| = \sum_{n=1}^{\infty} |a_n|\,|b_n|$$

converges by the previous proposition. By definition, this means that $\sum_{n=1}^{\infty} a_n b_n$ converges *absolutely.* □

Corollary 9.4. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series with $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series*

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Proof. By Theorem 9.1, we have $\lim b_n = 0$. In particular, (b_n) is bounded. Hence, Proposition 9.2 applies. □

Corollary 9.5. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with $a_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series*

$$\sum_{n=1}^{\infty} a_n^2$$

is convergent.

Proof. This follows from Corollary 9.4 with $b_n := a_n$. □

9.1 Linearity of Series

In this subsection we ask whether the convergence of series is a linear property. Perhaps the most glaring question here is whether multiplication by a constant alters the convergence of a series. Namely, given $\lambda \in \mathbb{R}$ and a convergent series $\sum_{n=1}^{\infty} a_n$, what can be said about $\sum_{n=1}^{\infty} \lambda a_n$? Clearly, if $\lambda \geq 0$ and each $a_n \geq 0$, then the series $\sum_{n=1}^{\infty} \lambda a_n$ converges as a consequence of Proposition 9.2. However, this does not give us any information about the value of $\sum_{n=1}^{\infty} \lambda a_n$. Furthermore, it is reasonable to expect this type of property to hold without making any sign assumptions on (a_n) and λ.

Theorem 9.6. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and fix $\lambda \in \mathbb{R}$. Then, $\sum_{n=1}^{\infty} \lambda a_n$ converges. Moreover,*

$$\sum_{n=1}^{\infty} \lambda a_n = \lambda \sum_{n=1}^{\infty} a_n.$$

Proof. Given $N \in \mathbb{N}$, we denote by S_N the partial sum

$$S_N := \sum_{n=1} \lambda a_n.$$

Since this is a *finite* sum, we see that

$$S_N = \lambda \sum_{n=1}^{N} a_n,$$

where $\sum_{n=1}^{N} a_n \to \sum_{n=1}^{\infty} a_n$ as $N \to \infty$ because the series $\sum_{n=1}^{\infty} a_n$ is assumed to be convergent. Consequently, by the limit laws for sequences, (S_N) converges and

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\lambda \sum_{n=1}^{N} a_n \right) = \lambda \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lambda \sum_{n=1}^{\infty} a_n.$$

This is precisely the statement that $\sum_{n=1}^{\infty} \lambda a_n = \lambda \sum_{n=1}^{\infty} a_n$. □

Remark 9.1. In the previous result, we employed an argument that is very common throughout analysis. In fact, we have seen similar arguments when treating the Riemann integral (see Example 4.2 and Corollary 6.10). Essentially, we observed that a constant can be "pulled out" of a *finite* sum, and lifted this property to series by taking a limit. In general, if a property that is preserved by limits holds for finite sums, then it holds for series by taking a limit.

Similarly, we obtain the following:

Theorem 9.7. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then, $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and*

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

Proof. We exploit the proof-strategy described in Remark 9.1. Given any $N \in \mathbb{N}$, we have

$$\sum_{n=1}^{N} (a_n + b_n) = \sum_{n=1}^{N} a_n + \sum_{n=1}^{N} b_n,$$

where $\sum_{n=1}^{N} a_n \to \sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{N} b_n \to \sum_{n=1}^{\infty} b_n$ as $N \to \infty$. Therefore, by the limit laws for sequences,

$$\lim_{N\to\infty} \sum_{n=1}^{N} (a_n + b_n) = \lim_{N\to\infty} \sum_{n=1}^{N} a_n + \lim_{N\to\infty} \sum_{n=1}^{N} b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

This completes the proof. □

If $\sum_{n=1}^{\infty} a_n$ is convergent, then we know from Theorem 9.1 $a_n \to 0$, as $n \to \infty$. We now show that, if (a_n) is decreasing and non-negative, then more can be said about the limiting behaviour of the terms a_n.

Proposition 9.8. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series such that*

(i) $a_n \geq 0$ *for all* $n \in \mathbb{N}$*;*

(ii) (a_n) *is decreasing.*

Then, $\lim na_n = 0$.

Proof. Define $b_n := na_n$; we must show that $b_n \to 0$. For this, it is enough to show that both the subsequences b_{2n} and b_{2n+1} converge to 0 as $n \to \infty$. To this end, we consider the partial sums

$$S_N := \sum_{n=1}^{N} a_n$$

which form, by assumption, a convergent sequence. Especially, (S_N) is Cauchy. Therefore, given $\varepsilon > 0$, we can find $K \in \mathbb{N}$ such that

$$|S_M - S_N| < \frac{\varepsilon}{2} \tag{9.1}$$

whenever $N, M \geq K$. In particular, if $2N > N \geq K$,

$$|S_{2N} - S_N| = S_{2N} - S_N = \sum_{n=N+1}^{2N} a_n < \frac{\varepsilon}{2}.$$

Since the sequence (a_n) is decreasing, it follows that

$$2Na_{2N} = 2 \sum_{n=N+1}^{2N} a_{2N} \leq 2 \sum_{n=N+1}^{2N} a_n < \varepsilon.$$

Since this holds for all $N \geq K$ and $\varepsilon > 0$ was arbitrary, we infer that

$$b_{2N} = 2Na_{2N} \to 0$$

as $N \to \infty$. On the other hand, if $2N + 1 > N \geq K$, then

$$\begin{aligned}(2N+1)a_{2N+1} < 2(N+1)a_{2N+1} &= 2\sum_{n=N+1}^{2N+1} a_{2N+1} \\ &\leq 2\sum_{n=N+1}^{2N+1} a_n < \varepsilon,\end{aligned}$$

where in this last step we have applied (9.1) with $M := 2N + 1$. As before, since this holds for all $N > K$, it follows that $b_{2N+1} \to 0$ as $N \to \infty$. Because b_{2N} and b_{2N+1} both converge to 0 as $N \to \infty$, we must have $\lim b_N = 0$. That is,

$$\lim na_n = 0.$$

□

9.2 Convergence Tests and Examples

We begin by recalling the most important convergence tests we have seen thus far. Perhaps the most intuitive test to which we have access, is the comparison test. For the sake of completeness, we restate this result below.

Theorem 9.9 (Comparison Test)**.** *Let (a_n) and (b_n) be two sequences of non-negative real numbers and assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$.*

(1) If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

(2) If $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$

We should also take a moment to recall the limit comparison test:

Theorem 9.10 (Limit Comparison Test)**.** *Let (a_n) and (b_n) be two sequences of non-negative real numbers. Assume further that*

$$L := \lim \left(\frac{a_n}{b_n}\right)$$

exists.

(1) If $L > 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

(2) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

As previously mentioned, the Divergence Test an only test for non-convergence. This next test, due to Cauchy, offers a *characterization* of convergence for a large class of series.

Theorem 9.11 (Cauchy's Condensation Test)**.** *Let (a_n) be a decreasing sequence of non-negative real numbers. Then, the series*

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if the series

$$\sum_{n=1}^{\infty} 2^n a_{2^n}$$

is convergent.

We now provide several examples in which we analyze the convergence of series. However, we first recall that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is known to be convergent if and only if $p > 1$. In fact, when p is an even integer, it is possible to explicitly compute the value of $\sum_{n=1}^{\infty} \frac{1}{n^p}$. On the other hand, the precise value of this series is not known for any odd integer p.

Example 9.3. We prove by comparison that the series

$$\sum_{n=1}^{\infty} \frac{1}{2(n+1)}$$

diverges. Indeed, note that

$$\lim \frac{\frac{1}{2(n+1))}}{\frac{1}{n}} = \lim \frac{n}{2n+2} = \frac{1}{2} > 0.$$

Therefore, by the Limit Comparison Test, the series

$$\sum_{n=1}^{\infty} \frac{1}{2(n+1)}$$

is divergent.

Example 9.4. Consider the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}.$$

Multiplying by the conjugate, we see that

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)}$$

where every term is a non-negative real number. Clearly,

$$\frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)} < \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}, \quad \forall n \in \mathbb{N}.$$

Therefore, the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$$

converges by comparison with the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

A slight modification to this series can, unfortunately, spoil its convergence.

Example 9.5. We claim that the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}}$$

is divergent. To see this, first note that

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\left(\sqrt{n+1}+\sqrt{n}\right)}.$$

Now, for every $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{n}\left(\sqrt{n+1}+\sqrt{n}\right)} \geq \frac{1}{\sqrt{n+1}\left(\sqrt{n+1}+\sqrt{n+1}\right)} = \frac{1}{2(n+1)}.$$

By comparison with the series

$$\sum_{n=1}^{\infty} \frac{1}{2(n+1)}$$

treated in Example 9.3, we infer that

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}}$$

is divergent.

Example 9.6. Using the Cauchy condensation test, we will prove that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$. First, note that the terms of our series are decreasing in n for any $p \geq 1$. Therefore, Cauchy's Condensation Test does indeed apply. Now, the p-series above converges if and only if

$$\sum_{n=1}^{\infty} \frac{2^n}{(2^n)^p}$$

converges. For $p = 1$, this reduces to the series

$$\sum_{n=1}^{\infty} \frac{2^n}{(2^n)^p} = \sum_{n=1}^{\infty} 1$$

which diverges by Theorem 9.1. If $p > 1$, we instead obtain

$$\sum_{n=1}^{\infty} \frac{2^n}{(2^n)^p} = \sum_{n=1}^{\infty} \frac{1}{(2^{p-1})^n}$$

which is a convergent geometric series since

$$0 < \frac{1}{2^{p-1}} < 1.$$

Remark 9.2. If we omit the assumption that the terms of the series $\sum_{n=1}^{\infty} a_n$ are decreasing, then the Cauchy Condensation Test may fail. To see this, consider the series

$$\sum_{n=1}^{\infty} a_n$$

where each a_n is given by

$$a_n := \begin{cases} 0 & \text{if } n = 2^k \text{ for some } k \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly,

$$\sum_{n=1}^{\infty} 2^n a_{2n} = \sum_{n=1}^{\infty} 0 = 0$$

is convergent. However, the series $\sum_{n=1}^{\infty} a_n$ cannot converge because it fails the Divergence Test (Theorem 9.1). Indeed, the sequence (a_n) has infinitely many terms equal to 1, and hence contains a subsequence that does not converge to 0.

10 Tenth/Final Tutorial

As this will be the final tutorial, we will first review some of the important convergence tests we covered in the previous section. Additionally, we will use this as an opportunity to provide additional examples.

Theorem 10.1 (Comparison Test)**.** *Let (a_n) and (b_n) be two sequences of non-negative real numbers and assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$.*

(1) If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

(2) If $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$

We should also take a moment to recall the limit comparison test. Unlike its predecessor, it does not assume the inequality $a_n \leq b_n$ for all $n \in \mathbb{N}$.

Theorem 10.2 (Limit Comparison Test)**.** *Let (a_n) and (b_n) be two sequences of non-negative real numbers. Assume further that*

$$L := \lim \left(\frac{a_n}{b_n} \right)$$

exists.

(1) If $L > 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

(2) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

There is also the Cauchy Condensation test, which is often applicable when the series involves a logarithm.

Theorem 10.3 (Cauchy's Condensation Test)**.** *Let (a_n) be a decreasing sequence of non-negative real numbers. Then, the series*

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if the series

$$\sum_{n=1}^{\infty} 2^n a_{2^n}$$

is convergent.

Thankfully, we now have access to several additional tests, especially some that merely test for convergence (and not necessarily absolute convergence). The first of these new tests is the infamous alternating series test below:

Theorem 10.4 (Alternating Series Test)**.** *A series of the form*

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

is said to be alternating provided every a_n has the same sign. If $(|a_n|)$ is monotone decreasing and $\lim a_n = 0$, then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

Remark 10.1. Note that this theorem does not assert that the series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges absolutely. Indeed, by the Alternating Series Test, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges. However, the series cannot be absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series.

A first example is now in order:

Example 10.1. We claim that the series

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{1/3}}$$

is convergent. Indeed, $\cos(\pi n) = (-1)^n$ for each $n \in \mathbb{N}$ whence the series above is equivalently given by

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}.$$

Since the sequence

$$|a_n| = a_n = \frac{1}{n^{1/3}}$$

is monotone decreasing and $a_n \to 0$ as $n \to \infty$, it follows from the Alternating Series Test that

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{1/3}}$$

converges.

10.1 The Ratio and Root Tests

Here we recall and discuss the root and ratio tests for series. Unfortunately, it is only in rare cases that the root test proves to be useful. However, the opposite can be said about the ratio test.

Theorem 10.5 (Root Test). *Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers and assume that the limit*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = C.$$

If $C < 1$ then the series converges absolutely. If $C > 1$, then the series must diverge. Nothing can be said if $C = 1$.

Example 10.2. Consider the series

$$\sum_{n=1}^{\infty} \left(\frac{\sin(n^2)}{\pi^3 n!}\right)^n$$

In situations like this (when there is an exponent n attached to every term in the series), the root test is worth trying out. Clearly,

$$\lim \sqrt[n]{\left|\left(\frac{\sin(n^2)}{\pi^3 n!}\right)^n\right|} = \lim \left|\frac{\sin(n^2)}{\pi^3 n!}\right| = \lim \frac{\left|\sin(n^2)\right|}{\pi^3 n!} \to 0$$

by the Squeeze Theorem for sequences. Hence, the root test implies that our series converges.

As is the case above, the examples that use the root test can feel somewhat artificial. We now recall the statement of the ratio test, which will be much more useful in practice than the root test.

Theorem 10.6 (Ratio Test)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a series and assume that*

$$L := \lim \left|\frac{a_{n+1}}{a_n}\right|$$

exists.

(1) If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(2) If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

As with the root test, the test is inconclusive when $L = 1$.

10.2 Abel's Test

We now treat examples relating to Abel's test, which we recall below from the class notes.

Theorem 10.7 (Abel's Test)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and let (b_n) be a bounded monotone sequence. Then, the series*

$$\sum_{n=1}^{\infty} a_n b_n$$

is convergent.

Example 10.3. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot \left(1 + \frac{1}{n}\right)^n .$$

This can be written as

$$\sum_{n=1}^{\infty} a_n b_n$$

where

$$a_n := \frac{(-1)^n}{n} \quad \text{and} \quad b_n := \left(1 + \frac{1}{n}\right)^n .$$

Now, the sequence

$$|a_n| = \frac{1}{n}$$

is monotone decreasing and converges to 0. Therefore, the Alternating Series Test ensures that $\sum_{n=1}^{\infty} a_n$ is convergent. If we can show that (b_n) is bounded and monotone, it will follow from Abel's test that the original series $\sum_{n=1}^{\infty} a_n b_n$ converges. Luckily for us, this is already a known fact! Certainly, it was proven Analysis 1 that

$$b_n = \left(1 + \frac{1}{n}\right)^n$$

is monotone increasing with $\lim b_n = e$.

Example 10.4. We prove that the series

$$\sum_{n=1}^{\infty} \left(\frac{2n^2 - 3n + 1}{4n^5 - 3} \sum_{k=1}^{n} \frac{1}{k^2} \right)$$

is convergent. With the hope of applying Abel's test, we will write this series as $\sum_{n=1}^{\infty} a_n b_n$ where

$$a_n := \frac{2n^2 - 3n + 1}{4n^5 - 3} \quad \text{and} \quad b_n := \sum_{k=1}^{n} \frac{1}{k^2}.$$

To successfully apply Abel's test, the following must be verified:

(i) $\sum_{n=1}^{\infty} a_n$ converges;

(ii) (b_n) is monotone;

(iii) (b_n) is bounded.

Let us first verify (ii)-(iii). Clearly, for each $n \in \mathbb{N}$,

$$b_n = \sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{n+1} \frac{1}{k^2} = b_{n+1}$$

whence (b_n) is monotone increasing. Now, every b_n is a partial sum for the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ which was shown to be convergent in the previous section (or tutorial). Consequently, (b_n) is convergent and hence bounded. Therefore, it only remains to establish (i). Here we will make use of the Limit Comparison Test. By the p-test, we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is convergent. Since

$$\begin{aligned} \lim \frac{\frac{2n^2-3n+1}{4n^5-3}}{1/n^3} = \lim \frac{n^3\left(2n^2 - 3n + 1\right)}{4n^5 - 3} &= \lim \frac{2n^5 - 3n^4 + 1}{4n^5 - 3} \\ &= \frac{1}{2} > 0 \end{aligned}$$

it follows from the Limit Comparison Test that $\sum_{n=1}^{\infty} a_n$ converges. By our previous remarks and Abel's test, we infer that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

10.3 Dirichlet's Test

Theorem 10.8 (Dirichlet's Test)**.** *Let (a_n) be a decreasing sequence of real numbers such that* $\lim a_n = 0$*. Let (b_n) be a sequence such that there exists $M > 0$ with the property that*

$$\left|\sum_{n=1}^{N} b_n\right| \leq M$$

for all $N \geq 1$. Then,

$$\sum_{n=1}^{\infty} a_n b_n$$

is convergent.

Example 10.5. We will prove that the series

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\ln n}$$

is convergent using Dirichlet's test. For this, we define

$$a_n := \frac{1}{\ln n} \quad \text{and} \quad b_n := \cos(\pi n).$$

Clearly, (a_n) is monotone decreasing and converges to 0 as $n \to \infty$. Therefore, to apply Dirichlet's test, we need only check that there exists $M > 0$ such that

$$\left|\sum_{n=1}^{N} b_n\right| = \left|\sum_{n=1}^{N} \cos(\pi n)\right| \leq M$$

for all $N \in \mathbb{N}$. To see this, we note that

- $\sum_{n=1}^{1} \cos(\pi n) = \cos(\pi) = -1$;
- $\sum_{n=1}^{2} \cos(\pi n) = \cos(\pi) + \cos(2\pi) = -1 + 1 = 0$;
- $\sum_{n=1}^{3} \cos(\pi n) = \cos(\pi) + \cos(2\pi) + \cos(3\pi) = -1 + 1 - 1 = -1$;
- $\sum_{n=1}^{4} \cos(\pi n) = \cos(\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) = -1 + 1 - 1 + 1 = 0.$

By induction, it follows that

$$\left|\sum_{n=1}^{N} b_n\right| = \left|\sum_{n=1}^{N} \cos(\pi n)\right| \leq 1$$

for all $N \in \mathbb{N}$. Hence, Dirichlet's test applies.

Example 10.6. Consider the series

$$\sum_{n=1}^{\infty} \frac{2^{2n} n^2}{e^n n!} \frac{1}{\ln \sqrt{n+1}}.$$

Using Dirichlet's test, we will show that the series above converges. To this end, we define

$$a_n := \frac{1}{\ln \sqrt{n+1}} \quad \text{and} \quad b_n := \frac{2^{2n} n^2}{e^n n!}.$$

Clearly, since $\sqrt{x}$ and $\ln x$ are both increasing on $(0, \infty)$, we have

$$a_{n+1} = \frac{1}{\ln \sqrt{n+2}} \leq \frac{1}{\ln \sqrt{n+1}} = a_n.$$

Hence, (a_n) is decreasing. Furthermore, it is clear that $a_n \to 0$ as $n \to \infty$. For Dirichlet's test to apply, we need only show that the partial sums $\left(\sum_{n=1}^{N} b_n\right)$ are bounded independently of N. For this, it is enough to show that the series

$$\sum_{n=1}^{\infty} b_n$$

is convergent. Indeed, observe that

$$\begin{aligned}
\lim \left| \frac{b_{n+1}}{b_n} \right| = \lim \frac{\frac{2^{2(n+1)}(n+1)^2}{e^{n+1}(n+1)!}}{\frac{2^{2n}n^2}{e^n n!}} &= \lim \left(\frac{2^{2(n+1)}(n+1)^2}{e^{n+1}(n+1)!} \cdot \frac{e^n n!}{2^{2n}n^2} \right) \\
&= \lim \frac{2^2(n+1)^2}{e(n+1)n^2} \\
&= \frac{4}{e} \lim \frac{n+1}{n^2} \\
&= \frac{4}{e} \lim \left(\frac{1}{n} + \frac{1}{n^2} \right) \\
&= 0.
\end{aligned}$$

Therefore the series $\sum_{n=1}^{\infty} b_n$ converges by the ratio test. In particular, the partial sums $\sum_{n=1}^{N} b_n$ are bounded.

Example 10.7. Consider the series

$$\sum_{n=1}^{\infty} \frac{c_n}{\sqrt{n}}$$

where $(c_n) = (1, -4, 1, 2, 1, -4, 1, 2, 1, \dots)$. Using Dirichlet's test, we will prove that this series is convergent. Clearly, $\frac{1}{\sqrt{n}}$ is a decreasing sequence of real numbers converging to 0. It remains to show that the partial sums $\sum_{n=1}^{N} c_n$ are bounded in N. As before, we try to notice a pattern in these partial sums:

- For $N = 1$ we have $\sum_{n=1}^{N} c_n = 1$;

- For $N = 2$ we have $\sum_{n=1}^{N} c_n = 1 - 4 = -3$;
- For $N = 3$ we have $\sum_{n=1}^{N} c_n = 1 - 4 + 1 = -2$;
- For $N = 4$ we have $\sum_{n=1}^{N} c_n = 1 - 4 + 1 + 2 = 0$;
- For $N = 4$ we have $\sum_{n=1}^{N} c_n = 1 - 4 + 1 + 2 + 1 = 1$;

and so forth. Therefore,

$$\left|\sum_{n=1}^{N} c_n\right| \leq 3$$

for all $N \in \mathbb{N}$. Dirichlet's test thus yields the convergence of our series.

Example 10.8. Consider the series

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$$

Namely, we consider the harmonic series where each 3^{rd} term is negative. By way of contradiction, let us assume that this series converges. Let (S_N) denote the sequence of partial sums. By assumption, (S_N), and hence (S_{3N}), is convergent. Now, each S_{3N} is given by

$$S_{3N} = \sum_{n=1}^{N} \left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n}\right).$$

Since $\lim S_{3N}$ exists by assumption, we see that the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n}\right) = \sum_{n=1}^{\infty} \frac{9n^2-2}{3n(3n-2)(3n-1)}.$$

must be convergent. Note that every term of the series

$$\sum_{n=1}^{\infty} \frac{9n^2-2}{3n(3n-2)(3n-1)}$$

is non-negative. On the other hand,

$$\lim \frac{\frac{9n^2-2}{3n(3n-2)(3n-1)}}{\frac{1}{n}} = \lim \frac{9n^3-2n}{3n(3n-2)(3n-1)} = \frac{1}{3} > 0.$$

Using the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, we infer that the series

$$\sum_{n=1}^{\infty} \frac{9n^2 - 2}{3n(3n-2)(3n-1)}$$

is divergent. This contradiction shows that (S_N) cannot be convergent.