A First Course in Geometry and Topology

A Supplementary Exposition

EDWARD CHERNYSH

Contents

Preface

This book was written in preparation for a graduate course to be given at McGill University in the fall semester of the 2018-2019 year. In this spirit, I have tried to both review prerequisite topics and cover material from previous incarnations of the course.

All chapters follow to a tee the material from [MNKS]. This excellent book covers far too much material for a single course, and leaves "holes" in just the right places. Following this book and filling in the details is perhaps the best way to learn the basics of topology. I think of this book as a record of which gaps I had to fill in to understand the "big picture" in each proof.

I would like to warn the reader not to use these notes as a substitute for *any* official course. First, there are likely to be errors in this book (after all, it is only version 1.0.0). But more importantly, I wrote this text *as I was learning*. This means that I have likely glossed over certain important details or, perhaps, neglected to mention crucial points. I assume the reader has attained a certain level of mathematical maturity. Although we introduce topological spaces from first principles, this book will not serve well as a first exposure to topological *notions*. We spend little to no time motivating the concept of a topological space. Moreover, I assume that the reader understands why one should study these spaces. For instance, I assume the reader has a solid understanding of metric spaces, Banach spaces, and so forth. Ideally, an all around decent exposure to abstract mathematics is ideal.

The first part of the text handles general point-set topology. There, we will develop rudimentary notions of continuity and regularity. The first chapter should merely generalize notions familiar to the reader from functional analysis courses. In the second chapter, we break down some important classes of topological spaces (e.g. compact, connected, locally compact/connected). We also introduce quotients and show how certain functions can be "factored through" topological spaces. In the third chapter, we return to a more analytic setting. We begin by discussing the countability axioms, which allows for a nice characterization of analytic structure via sequences. From there, we study regular and normal spaces. At the end of this third chapter, we provide decent introductory exposure to topological vector spaces. Nets will also be covered here.

The second part of these notes focus on algebraic topology and geometry. Here we introduce the basics: fundamental groups, homotopy, covering spaces, and retracts. We will also be proving basic versions of the Brouwer fixed point theorem. Near the end of this part, we introduce geometric notions (e.g. CW-complexes). In this part, we will make use of several internet resources (see the citations). At the end of every chapter, I have listed several exercises from [MNKS] that we feel do a great job of summarizing the "big picture" the chapter conveyed. Full solutions to all of these exercises can be found in Appendix B.

As a final note, I feel the need to remind the reader that this book is not meant as a course substitute. My goal here is to provide a concise treatment of geometry and topology at a graduate level. I provide many details within the proofs, but I have avoid examples as much as possible. This makes the book more terse, but it also allows for quicker reading and places emphasis on full generality. As with anything, this is a work in progress. As I learn more geometry and topology, I will continue to contribute material to this text. During this process, I will do my best to improve this book and correct typographical errors.

Part I

Point-Set Topology

Chapter 1

Topological Spaces and Continuity

This chapter is devoted to the study of *topological spaces*. Generally speaking, a topological space can be viewed as the most basic setting in which one can discuss the continuity of a function. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a function. Of course, one has a natural $\varepsilon - \delta$ notion of continuity based upon the metrics d_X and d_Y. As it turns out, the function f will be continuous if and only if $f^{-1}(U)$ is open in X, for all open $U \subseteq Y$. This offers an alternative characterization of continuous functions (in the setting of a metric space) using only the open sets. Thus, it is natural to make use of these sets when *defining* a minimalist structure over which one can give a reasonable definition of continuity. This begins with a *topology*, which we define below.

1.1 Topologies and Bases

Throughout this chapter, X will denote a non-empty set and $\mathfrak{T}$ will be a non-empty subset of $\mathscr{P}(X)$, the power set of X. Loosely speaking, we give $\mathfrak{T}$ a structure mimicking the behaviour of open sets in $\mathbb{R}^n$.

Definition 1.1. Let X be a non-empty set and $\mathfrak{T}$ a family of subsets of X. We will say that $\mathfrak{T}$ is a *topology* on X provided each of the following properties hold true:

(1) Both $\varnothing$ and X belong to $\mathfrak{T}$. In particular, no topology is empty.

(2) $\mathfrak{T}$ is closed under unions: if $\{U_i\}_{i\in I}$ is a subset of $\mathfrak{T}$ indexed by some index set I (not necessarily countable), then $\bigcup_{i\in I} U_i$ belongs to $\mathfrak{T}$ as well.

(3) $\mathfrak{T}$ is closed under *finitely many* intersections: if $U_1, \ldots, U_n$ are elements of $\mathfrak{T}$, then $\bigcap_1^n U_i \in \mathfrak{T}$.

This topology is the rudimentary structure that gives rise to the fundamental notion of a topological space.

Definition 1.2. Let X be a non-empty set and $\mathfrak{T}$ a topology on X. The pair $(X, \mathfrak{T})$ is then called a *topological space*. The elements of $\mathfrak{T}$ are then deemed to be the *open sets* of the space $(X, \mathfrak{T})$.

A trivial "example" is now in order. For any non-empty set X one can always find a topology on X by taking $\mathfrak{T} := \mathscr{P}(X)$. The topology $\mathscr{P}(X)$ is called the *discrete topology* on X. Similarly, the family $\mathfrak{T} := \{X, \varnothing\}$ is easily seen to be a topology on any non-empty set X. This topology is called the *indiscrete topology*.

Definition 1.3. Let $\mathfrak{T}$ and $\mathfrak{W}$ be two topologies on a set X. We say that $\mathfrak{T}$ is *finer* than $\mathfrak{W}$ provided $\mathfrak{T} \supseteq \mathfrak{W}$. In this case, we would say that $\mathfrak{W}$ is *coarser* than $\mathfrak{T}$. If $\mathfrak{T}$ strictly includes $\mathfrak{W}$ then it is called *strictly finer*. In this case, we say that $\mathfrak{W}$ is *strictly coarser* than $\mathfrak{T}$.

The motivation behind this definition is clear: the "larger" a topology, the more specific and selective are the definitions of open sets. This also leads to an important observation. In the setting of a metric space (X, d), open sets are determined by the metric. In a topological space, the open sets are *given* and are unknown in the sense that we do not know how they arise. As far as we are concerned, they may have been arbitrarily prescribed to us!

1.1.1 A Topological Basis

The indeterminate nature of a topology can make it hard to construct meaningful topologies for a given set X. This problem is partially solved by the use of a *basis*. Before we give the definition of a basis, we would like to stress that a basis for a topology in no way resembles that of a vector space. Certainly, there will be no uniqueness property present in our definition.

Definition 1.4. Let X be a non-empty set. A family $\mathcal{B}$ of subsets of X is called a *basis* for a topology on X provided each of the following hold

(1) For every $x \in X$ there exists some $B \in \mathcal{B}$ such that $x \in B$;

(2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ is a basis for X, we define the topology generated by $\mathcal{B}$ on X as the set

$$\mathfrak{T} := \{U \subseteq X : \forall x \in U,\ \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U\}\,. \tag{1.1}$$

That is, $\mathfrak{T}$ consists of all $U \subseteq X$ such that, for every $x \in U$, one can find $B \in \mathcal{B}$ with $x \in B \subseteq U$.

We must now make sure that the above is well defined; this amounts to proving that $\mathfrak{T}$ is indeed a topology on the set X. First, it is obvious that $X, \varnothing \in \mathfrak{T}$. Let I be any index set and suppose $\{U_i\}_{i\in I}$ is an indexed family of sets in $\mathfrak{T}$. Let $U := \bigcup_{i\in I} U_i$, we must show that $U \in \mathfrak{T}$. If $x \in U$ then $x \in U_i$ for some $i \in I$. Since $U_i \in \mathfrak{T}$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq U_i$. But then,

$$x \in B \subseteq U_i \subseteq U\,.$$

To prove that $\mathfrak{T}$ is closed under finitely many intersections, we need only show that if $U_1, U_2 \in \mathfrak{T}$ then $U_1 \cap U_2 \in \mathfrak{T}$ as well. Assume without loss of generality that $U_1 \cap U_2$ is non-empty and choose $x \in U_1 \cap U_2$. We may find sets $B_1, B_2 \in \mathcal{B}$ so that

$$x \in B_1 \subseteq U_1 \quad \text{and} \quad x \in B_2 \subseteq U_2.$$

Clearly, $x \in B_1 \cap B_2$ which means that there exists $B_3 \in \mathcal{B}$ having the property that $x \in B_3 \subseteq B_1 \cap B_2$. Clearly, this means that $x \in B_3 \subseteq U_1 \cap U_2$. As x was arbitrary, we have that $U_1 \cap U_2 \in \mathfrak{T}$. We conclude that $\mathfrak{T}$ is indeed a topology on the set X.

Definition 1.5. Let $(X, \mathfrak{T})$ be a topological space and $\mathcal{B}$ a basis for a topology on X. We say that $\mathcal{B}$ is a basis *for* $\mathfrak{T}$ if the topology on X generated by $\mathcal{B}$ is equal to $\mathfrak{T}$.

Let $(X, \mathfrak{T})$ be a topological space and $\mathcal{B}$ a basis on X that generates $\mathfrak{T}$. It is useful to note that $\mathcal{B} \subseteq \mathfrak{T}$, i.e. the elements of $\mathcal{B}$ are open sets in X. Indeed, since $\mathfrak{T}$ is generated by $\mathcal{B}$, we have

$$\mathfrak{T} := \{U \subseteq X : \forall x \in U,\ \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U\}\,.$$

Clearly, every element of $\mathcal{B}$ satisfies this criterion and so $\mathcal{B} \subseteq \mathfrak{T}$.

Example 1.1. Let (X, d) be a metric space. For $x_0 \in X$ and $\varepsilon > 0$ fixed, we define the open ball about x_0 of radius ε to be

$$B(x_0, \varepsilon) := \{x \in X : d(x, x_0) < \varepsilon\}\,.$$

This set is always non-empty. Let $\mathcal{B}$ be the collection of all such balls and recall that a set $U \subseteq X$ is called open if and only if for every $x_0 \in U$ one can find $\varepsilon > 0$ so that $B(x_0, \varepsilon) \subseteq U$. We then see that the (implicit) topology on (X, d) is generated by $\mathcal{B}$.

Unfortunately, (1.1) only gives us a rather messy way of explicitly describing the elements of the topology generated by $\mathcal{B}$. The following proposition yields a slightly more convenient way of expressing these open sets.

Proposition 1.1. *Let X be a non-empty and set and $\mathcal{B}$ a basis for a topology on X. Let $\mathfrak{T}$ denote the topology on X generated by $\mathcal{B}$. Then $\mathfrak{T}$ is precisely the set of all unions of elements in B.*

Proof. First, it is clear from the definitions that $\mathcal{B} \subseteq \mathfrak{T}$. Since $\mathfrak{T}$ is closed under arbitrary unions, we see that any union of elements belongs to $\mathfrak{T}$. Thus, if we let $\mathfrak{W}$ be the set of all unions of elements in $\mathcal{B}$, then $\mathfrak{W} \subseteq \mathfrak{T}$. For the reverse inclusion, let $U \in \mathfrak{T}$ be given. For each $x \in U$ there exists $B_x \in \mathcal{B}$ containing x such that $B_x \subseteq U$. Clearly, we then have

$$U = \bigcup_{x \in U} B_x$$

whence $U \in \mathfrak{W}$. This establishes the desired equality, $\mathfrak{T} = \mathfrak{W}$. □

As previously mentioned, a topological basis is very much different from an algebraic basis. Certainly, a topological basis $\mathcal{B}$ allows one to express the open sets as unions of elements of $\mathcal{B}$, but it does not guarantee any uniqueness of representation. This is the general flavour of topology: the results are very general but the conclusions often lack power. Despite this, topology is a *very* useful and applicable field that is well worth learning (especially to a mathematician).

Proposition 1.2. *Let $(X, \mathfrak{T})$ be a topological space and assume $\mathcal{B}$ is a collection of open sets in X such that for every $U \in \mathfrak{T}$ and every $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$. Then $\mathcal{B}$ is a basis for the topology $\mathfrak{T}$ on X.*[1]

Proof. We will first check that $\mathcal{B}$ is a basis for a topology on X. If $x \in X$, X is an open set containing X. Therefore, one can find $B \in \mathcal{B}$ such that $x \in B$. Assume now that $B_1, B_2 \in \mathcal{B}$ and that $x \in B_1 \cap B_2$. By definition, both B_1 and B_2 are open

[1]Here we mean that $\mathcal{B}$ is a basis for a topology on X and that the topology it generates is precisely $\mathfrak{T}$.

sets in X. Thus, we can find $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. We see that $\mathcal{B}$ is indeed a basis for a topology on X.

Let now $\mathfrak{W}$ be the topology on X generated by $\mathcal{B}$; we claim that $\mathfrak{T} = \mathfrak{W}$. First, let $W \in \mathfrak{W}$ be given. Invoking Proposition 1.1, we see that W is simply a union of elements in $\mathcal{B}$. But, $\mathcal{B} \subseteq \mathfrak{T}$ by definition and therefore we have $W \in \mathfrak{T}$. As W was arbitrary, $\mathfrak{W} \subseteq \mathfrak{T}$. Fix $U \in \mathfrak{T}$, for each $x \in U$ one can find $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$. Therefore,

$$U = \bigcup_{x \in U} B_x.$$

But, $\mathcal{B} \subseteq \mathfrak{W}$ by definition, and therefore it follows that $U \in \mathfrak{W}$. We conclude that $\mathfrak{W} = \mathfrak{T}$ as was required. ◻

Proposition 1.3. *Let X be a non-empty set and suppose $\mathfrak{T}$ and $\mathfrak{T}'$ are topologies on X with bases $\mathcal{B}$ and $\mathcal{B}'$, respectively. The following statements are equivalent.*

(1) $\mathfrak{T}'$ is finer than $\mathfrak{T}$, i.e. $\mathfrak{T}' \supseteq \mathfrak{T}$;

(2) For every $x \in X$ and each $B \in \mathcal{B}$ containing x, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Proof. Suppose first that $\mathfrak{T}'$ is finer than $\mathfrak{T}$. This means that $\mathfrak{T} \subseteq \mathfrak{T}'$. Let $x \in X$ be given and fix $B \in \mathcal{B}$ such that $x \in B$. Since $\mathcal{B}$ generates $\mathfrak{T}$, we have $B \in \mathfrak{T} \subseteq \mathfrak{T}'$. Therefore, B is open in $(X, \mathfrak{T}')$ and we can therefore find $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. This shows that (2) holds.

Conversely, we fix an open set $U \in \mathfrak{T}$ and prove that $U \in \mathfrak{T}'$. As before, we know that we can write

$$U = \bigcup_{x \in U} B_x,$$

for $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$. By assumption, for each x there exists $B'_x \in \mathcal{B}'$ such that $x \in B'_x \subseteq B_x$. But then, $U = \bigcup_{x \in U} B'_x$ whence $U \in \mathfrak{T}'$. This yields $\mathfrak{T} \subseteq \mathfrak{T}'$ and completes the proof. ◻

Alternatively, one can build up a topology by considering finite intersections of sets. This gives rise to what is known as a *subbasis*.

Definition 1.6. Let X be a non-empty set. A collection $\mathcal{S} \subseteq \mathscr{P}(X)$ is called a subbasis for a topology on X if $\bigcup_{V \in \mathcal{S}} V = X$. In this case, the topology generated by the subbasis $\mathcal{S}$ is defined to be the collection of all unions of finite intersections of elements in $\mathcal{S}$.

Let $\mathcal{S}$ be a subbasis for a topology on X and let $\mathfrak{T}$ be the "topology" it generates. We must verify that $\mathfrak{T}$ is indeed a topology. To this end, let $\mathcal{B}$ be the set of all finite intersections of elements in $\mathcal{S}$. Clearly, $\mathcal{S} \subseteq \mathcal{B}$ and therefore, for each $x \in X$, one can find an element of $\mathcal{B}$ containing x. Let $B_1, B_2 \in \mathcal{B}$ and assume $x \in B_1 \cap B_2$. Write B_i $(i = 1, 2)$ as

$$B_i = S_1^{(i)} \cap \cdots \cap S_{n_i}^{(i)}, \quad S_k^{(i)} \in \mathcal{S}.$$

Then, it is clear that $B_1 \cap B_2 \in \mathcal{B}$ as well. This implies that $\mathcal{B}$ is in fact a basis for a topology on X. However, we see from Proposition 1.1 that the topology generated by $\mathcal{B}$ is precisely $\mathfrak{T}$. We summarize this discussion below.

Lemma 1.4. *Let X be a non-empty set and $\mathcal{S}$ a subbasis for a topology on X. Let $\mathcal{B}$ be the set of all finite intersections of elements in $\mathcal{S}$. Then $\mathcal{B}$ is a basis for the topology generated by $\mathcal{S}$.*

1.1.2 Some Topologies of Interest on $\mathbb{R}$

Typically when speaking of the real line $\mathbb{R}$, it will be endowed with the metric $d(x, y) := |x - y|$. This generates a topology on $\mathbb{R}$ known as the *standard topology*. This is precisely the topology generated by the basis

$$\mathcal{B} := \Big\{(a, b) : a, b \in \mathbb{R},\ a < b\Big\}.$$

When we write $\mathbb{R}$, it will be understood that $\mathbb{R}$ has the standard topology. If instead we endow $\mathbb{R}$ with the topology generated by the basis

$$\Big\{[a, b) : a, b \in \mathbb{R},\ a < b\Big\},$$

we will say that $\mathbb{R}$ has the *lower-limit topology*. We will write $\mathbb{R}_\ell$ to indicate that $\mathbb{R}$ has been given this topology. Finally, define

$$K := \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

and let $\mathcal{K}$ be the set

$$\mathcal{K} := \Big\{(a, b), (a, b) \setminus K\Big\}.$$

It can be shown that $\mathcal{K}$ is a basis for a topology on $\mathbb{R}$. The topology on $\mathbb{R}$ generated by $\mathcal{K}$ is called the K-topology on $\mathbb{R}$. We will write $\mathbb{R}_K$ to indicate that $\mathbb{R}$ comes equipped with the K-topology.

1.2 The Two-Fold Product Topology

For this section, we will work mostly with two fixed topological spaces $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$. Our goal will be to endow the Cartesian product $X \times Y$ with a meaningful topological structure arising from the topologies $\mathfrak{T}$ and $\mathfrak{W}$. To those who have seen product measure spaces, the steps we take should be familiar.

Definition 1.7. Let $\mathcal{B}_{X\times Y}$ be the family of all sets having the form $U \times V$, where $U \in \mathfrak{T}$ and $V \in \mathfrak{W}$. The *product topology*[2] on $X \times Y$ is defined to be the topology generated by the basis $\mathcal{B}_{X\times Y}$.

For the above to make sense, we must first check that $\mathcal{B}_{X\times Y}$ is indeed a basis for a topology on $X \times Y$. This set may be written as

$$\{U \times V : U \in \mathfrak{T},\ V \in \mathfrak{W}\}.$$

Thus, $X \times Y$ belongs to $\mathcal{B}_{X\times Y}$. Hence, for any $(x, y) \in X \times Y$ we can find an element of $\mathcal{B}_{X\times Y}$ containing (x, y). Now, assume that

$$(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$$

for $U_{1,2} \in \mathfrak{T}$ and $V_{1,2} \in \mathfrak{W}$. Clearly,

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

and $U_1 \cap U_2 \in \mathfrak{T}$, $V_1 \cap V_2 \in \mathfrak{W}$, This means that

$$(U_1 \times V_1) \cap (U_2 \times V_2) \in \mathcal{B}_{X\times Y}$$

whence $\mathcal{B}_{X\times Y}$ is certainly a basis for a topology on $X \times Y$. This construction does raise a natural question: can we construct the *same* product topology from only the bases for $\mathfrak{T}$ and $\mathfrak{W}$? The answers is yes, and the explanation is given below.

Theorem 1.5. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces with bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. The collection*

$$C := \{A \times B : A \in \mathcal{B}_1,\ B \in \mathcal{B}_2\}$$

is a basis for the product topology on $X \times Y$.

[2] We would like to point out that the product topology will heavily depend on the topologies given to X and Y.

Proof. This proof will rely on Proposition 1.2. Let $W \subseteq X \times Y$ be an open set and choose $(x, y) \in W$. By definition of the product topology, there exists a set $A_1 \times A_2 \in \mathcal{B}_{X\times Y}$ such that

$$(x, y) \in A_1 \times A_2 \subseteq W.$$

Now, A_1 is open in $(X, \mathfrak{T})$ and we can thus find $B_1 \in \mathcal{B}_1$ such that

$$x \in B_1 \subseteq A_1.$$

Likewise, we can choose $B_2 \in \mathcal{B}_2$ with $y \in B_2 \subseteq A_2$. Thus,

$$(x, y) \in B_1 \times B_2 \subseteq A_1 \times A_2 \subseteq W.$$

Now, all elements in C are open in $X \times Y$ by definition. Thus, by Proposition 1.2, we see that C is a basis for the product topology on $X \times Y$. □

We now turn towards the relationship between the product topology and the "original" topologies. This involves the so-called *standard projections.*

Definition 1.8. Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces and assume $X \times Y$ is given the product topology. There are two canonical *projection mappings* π_X and π_Y defined in the obvious way:

$$\pi_X : X \times Y \twoheadrightarrow X, \quad (x, y) \mapsto x,$$

and

$$\pi_Y : X \times Y \twoheadrightarrow Y, \quad (x, y) \mapsto y.$$

Notice that both π_X and π_Y are surjections.

These projection maps have nice preservation properties that we will soon come to know as *continuity*. Let $U \subseteq X$ be an open set and notice that $\pi_X^{-1}(U)$ will be of the form

$$\pi_X^{-1}(U) = U \times Y,$$

which is open in $X \times Y$. The second projection π_Y also enjoys this property. This gives way to the following result.

Proposition 1.6. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces and let $X \times Y$ have the product topology. Define now*

$$\mathcal{S} := \left\{\pi_X^{-1}(U) : U \in \mathfrak{T}\right\} \cup \left\{\pi_Y^{-1}(V) : V \in \mathfrak{W}\right\}.$$

Then, $\mathcal{S}$ is a subbasis for the product topology on $X \times Y$.

Proof. First, is is clear that $\mathcal{S}$ is a subbasis as

$$\pi_X^{-1}(X), \pi_Y^{-1}(Y) = X \times Y.$$

In any case, let P be the product topology on $X \times Y$ and P' that generated by $\mathcal{S}$. Now, elements in $\mathcal{S}$ are always open in $X \times Y$ and hence belong to P. The topology P' consists of all unions of finite intersections of items in $\mathcal{S}$, and therefore $P' \subseteq P$.

Conversely, the elements $\{U \times V : U \in \mathfrak{T},\ V \in \mathfrak{W}\}$ form a basis for P. But, it is easy to see that

$$U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V).$$

By Proposition 1.1, all the elements of the topology P can be represented as the union of sets having the form $U \times V$, with $U \in \mathfrak{T}$ and $V \in \mathfrak{W}$. Hence, every element of P lives in P' by definition. □

1.3 *The Subspace Topology*

In this section, we discuss how to give a subset of a topological space its own topological structure. Let $(X, \mathfrak{T})$ be a topological space and let $Y \subseteq X$ be a non-empty subset. We define the subspace topology of Y as

$$\mathfrak{T}_Y := \{Y \cap U : U \in \mathfrak{T}\}. \tag{1.2}$$

That is, $\mathfrak{T}_Y$ consists of all open sets in X intersected with Y. We then call the pair $(Y, \mathfrak{T}_Y)$ a subspace of $(X, \mathfrak{T})$. It remains to justify the fact that $\mathfrak{T}_Y$ is a topology on Y. Clearly, taking $U = \varnothing$ or $U = X$, we see that $\varnothing, Y \in \mathfrak{T}_Y$. If $\{V_i\}_{i \in I}$ is an indexed family in $\mathfrak{T}_Y$, then every V_i has the form

$$V_i = Y \cap U_i, \quad U_i \in \mathfrak{T}.$$

Therefore,

$$\bigcup_{i \in I} V_i = \bigcup_{i \in I} Y \cap U_i = Y \cap \bigcup_{i \in I} U_i =: Y \cap \tilde{U}$$

for $\tilde{U} \in \mathfrak{T}$. Also, if $V_1, \ldots, V_n \in \mathfrak{T}_Y$ we write each V_j as

$$V_j = Y \cap U_j, \quad U_j \in \mathfrak{T}.$$

But then, it is easy to verify that for some $U' \in \mathfrak{T}$:

$$\bigcap_{j=1}^{n} V_j = \bigcap_{j=1}^{n} Y \cap U_j = Y \cap \bigcap_{j=1}^{n} U_j =: Y \cap U'.$$

Hence, $\mathfrak{T}_Y$ is indeed a topology on Y.

Lemma 1.7. *Let $\mathcal{B}$ be a basis for a topology $\mathfrak{T}$ on X. If $Y \subseteq X$, then*

$$\mathcal{B}_Y := \{Y \cap B : B \in \mathcal{B}\}$$

is a basis for $\mathfrak{T}_Y$ on Y.

Proof. Let us fix an open set V of Y. Then, V is of the form $U \cap Y$ for U open in X. Let $y \in V = U \cap Y$ be given, since $\mathcal{B}$ generates the topology $\mathfrak{T}$, we can find $B \in \mathcal{B}$ such that $y \in B \subseteq U$. From this, we also get that $y \in B \cap Y \subseteq U \cap Y = V$. Since $\mathcal{B}$ is a basis for $\mathfrak{T}$, it is clear that $\mathcal{B} \subseteq \mathfrak{T}$. But then, $\mathcal{B}_Y \subseteq \mathfrak{T}_Y$. The statement then follows from Proposition 1.2. □

When dealing with subspaces and subsets of topological spaces, it is important to clarify some terminology. Let $(X, \mathfrak{T})$ be a topological space and $Y \subseteq X$. If we say that Y is a *subset* of X, we are choosing to view Y relative to X and Y is not given a topology. If we say that Y is a *subspace* of X, we are viewing Y as a subset of itself (relative to the subspace topology $\mathfrak{T}_Y$). Thus, if we have a subset $A \subseteq Y \subseteq X$, we will say that A is open in Y if $A \in \mathfrak{T}_Y$.

Proposition 1.8. *Let $(X, \mathfrak{T})$ be a topological space and suppose that Y is a subspace of X with Y open in X. If U is open in Y then U is open in X.*

Proof. Since U is open in Y, it must be of the form $U = V \cap Y$ for some V that is open in X. However, Y is open in X and topologies are closed under finite intersections. It follows that U is open in X. □

If A and B are subspaces of two topological spaces X and Y, which topology should we give to $A \times B$? Naturally, $A \times B$ is a subspace of $X \times Y$ and therefore can inherit a subspace topology. Similarly, A is a subspace of X and B a subspace of Y, and thus both A and B inherit subspace topologies. This means that $A \times B$ could also be given a product topology! Thankfully, the following result shows that both of these topologies are the same.

Theorem 1.9. *Let* $(X, \mathfrak{T})$ *and* $(Y, \mathfrak{W})$ *be topological spaces and give* $X \times Y$ *the product topology. Let* A *and* B *be subspaces of* X *and* Y*, respectively. The topology that* $A \times B$ *inherits as a subspace of* $X \times Y$ *is precisely the product topology on* $A \times B$*.*

Proof. Let U be open in X and V open in Y. We know that products of the form $U \times V$ are basis elements for the topology on $X \times Y$. Therefore, sets of the form

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$$

will be basis elements for the topology $A \times B$ inherits as a subspace of $X \times Y$. On the other hand, the $A \cap U$ form a basis for the subspace topology of A in X. Similarly, the $B \cap V$ will make up a basis for B as a subspace of Y. By definition of the product topology, this means that

$$\mathcal{B} := \left\{(A \cap U) \times (B \cap V) : U \in \mathfrak{T},\ V \in \mathfrak{W}\right\}$$

will form a basis for the topology on $A \times B$ (when viewed as a product). To summarize, we have shown that $\mathcal{B}$ is a basis for both the product topology on $A \times B$, and that which it inherits from $X \times Y$. This means that both these topologies must be identical. This completes the proof. □

1.4 Closed Sets and Limit Points

In the previous sections, we gave rigorous and abstract formulations for what it means to be an open set relative to a space. Let us now do the same for closed sets.

Definition 1.9. Let $(X, \mathfrak{T})$ be a topological space and suppose that F is a subset of X. We will say that F is *closed* in X if $F^c = X \setminus F$ is open in the space X, i.e. if $F^c \in \mathfrak{T}$.

This gives us an intimate relationship between the open and closed subsets of a topological space. This weak-symmetry makes it possible for one to develop all of topology starting instead from the closed sets. It has, however, become convention to axiomatically formulate the open sets and to study the resulting closed sets.

The algebraic structure of a topology allows us to easily deduce facts regarding closed sets. We give some of these below.

Proposition 1.10. *Let* $(X, \mathfrak{T})$ *be a topological space. There holds*

(1) *Both* X *and* $\varnothing$ *are closed in* X.

(2) *If* $\{F_i\}_{i\in I}$ *is any indexed collection of closed sets in* X, *then* $\bigcap_{i\in I} F_i$ *is closed in* X.

(3) *If* $F_1, \ldots, F_n$ *is a finite collection of closed sets in* X, *then* $\bigcup_1^n F_i$ *is closed in* X.

Proof. These properties are very easy to verify, but we give the proof for the sake of completeness. The first property is immediate since $X, \varnothing$ are open in X. For (2), simply note that

$$\left(\bigcap_{i\in I} F_i\right)^{\mathsf{c}} = \bigcup_{i\in I} F_i^{\mathsf{c}}.$$

Now, every F_i^{c} is open in X since F_i is closed in X. Since unions of open sets are open, we conclude from the above that $(\bigcap_{i\in I} F_i)^{\mathsf{c}}$ is closed in X, as was required. The final point is handled similarly. Write

$$(F_1 \cup \cdots \cup F_n)^{\mathsf{c}} = F_1^{\mathsf{c}} \cap \cdots \cap F_n^{\mathsf{c}}.$$

Just as before, $F_j^{\mathsf{c}} \in \mathfrak{T}$ for every index j. Since finite intersections of open sets are open, we get that finite unions of closed sets are closed. □

When discussing subspaces, a familiar issue arises with closed sets. If $(X, \mathfrak{T})$ is a topological space and $Y \subseteq X$, we can give Y the subspace topology. We then say that $A \subseteq Y$ is closed in Y provided A is closed *relative* to the topology $\mathfrak{T}_Y$.

Theorem 1.11. *Let* $(X, \mathfrak{T})$ *be a topological space and let* Y *be a subspace of* X. *Assume that* $A \subseteq Y$. *Then* A *is closed in* Y *if and only if it is the intersection of a closed set in* X *with* Y.

Proof. Assume first that A is closed in Y. This is equivalent to saying that $Y \setminus A$ is open in Y. We can then find $U \subseteq X$ that is open in X having the property that $Y \setminus A = U \cap Y$. It is easily seen that

$$A = Y \setminus (U \cap Y) = Y \cap (X \setminus U)$$

since $U \cap Y$ is the portion of U that is contained in $Y \subseteq X$. As U is open in X, we see that $X \setminus U$ is closed in X which means that $A = Y \cap C$ for a closed set C

in X. Conversely, assume that $A = Y \cap C$ where C is closed in X. We will show that A is open in Y. It is easy to see that

$$Y \setminus A = Y \setminus (Y \cap C) = Y \cap (X \setminus C)$$

where $X \setminus C$ is open in X. This means that $A \in \mathfrak{T}_Y$ and that A is open *in* Y. This completes the proof. ◻

Just as with open subspaces, the topologies can satisfy a nice "transitive" property.

Proposition 1.12. *Let $(X, \mathfrak{T})$ be a topological space and let Y be a subspace of X. Assume in addition that Y is closed in X. If A is closed in Y, then A is closed in X.*

Proof. Since A is closed in Y, we have by definition that $A \subseteq Y$. From the previous theorem, there exists a closed set C in X having the property that $A = Y \cap C$. Since Y is closed in X and intersections of closed sets are again closed, we see that A is closed in X. ◻

Example 1.2. Let X be a non-empty set, and endow X with the following:

$$\mathfrak{T} := \{U \subseteq X : U^{\mathsf{c}} \text{ is finite or } U = \varnothing\}\,.$$

Then, we say that X has the *finite complement topology*. It is easy to check that $(X, \mathfrak{T})$ is indeed a topological space. Since $\{x\}$ is finite, for each $x \in X$, its complement $X \setminus \{x\}$ is *open* in X. Put otherwise, $\{x\}$ is closed for every point $x \in X$. This makes $(X, \mathfrak{T})$ into what we will soon come to know as a "T_1-space".

1.4.1 Closures and Interiors

We continue to consider an arbitrary topological space $(X, \mathfrak{T})$. Let $A \subseteq X$ be any subset. The *interior* of A, denoted $\operatorname{Int}(A)$ is defined to be the largest subset of A that is open in X. Since arbitrary unions of open sets are again open, the interior of A is given by

$$\operatorname{Int}(A) = \bigcup_{\substack{U \subseteq A \\ U \text{ open in } X}} U. \tag{Int}$$

Analogously, the closure of A, written $\operatorname{Cl}(A)$ or $\overline{A}$, is defined to be the smallest closed subset of X that contains A. Of course, since intersections of closed sets return closed sets, this is precisely

$$\mathrm{Cl}\,(A) = \bigcap_{\substack{A \subseteq F \subseteq X \\ F \text{ closed in } X}} F. \tag{Cl}$$

The following proposition is immediate from the definitions.

Proposition 1.13. *Let* $(X, \mathfrak{T})$ *be a topological space and* A *a subset of* X.

(1) $\mathrm{Int}\,(A)$ *is the largest open subset of* X *contained in* A.

(2) $\mathrm{Cl}\,(A)$ *is the smallest closed subset of* X *containing* A.

(3) $\mathrm{Int}\,(A) \subseteq A \subseteq \mathrm{Cl}\,(A)$.

Naturally, we should ask ourselves how closures behave when discussing subspaces.

Proposition 1.14. *Let* $(X, \mathfrak{T})$ *be a topological space and* Y *a subspace of* X. *Let* $A \subseteq Y$ *and denote by* $\mathrm{Cl}\,(A)$ *the closure of* A *in* X. *The closure of* A *in* Y *is equal to* $Y \cap \mathrm{Cl}\,(A)$.

Proof. $\mathrm{Cl}\,(A)$ is closed in X and so Theorem 1.11 guarantees that $Y \cap \mathrm{Cl}\,(A)$ is closed in Y. Let $\bar{A}$ be the closure of A in Y. Since $\mathrm{Cl}\,(A) \cap Y$ is closed in Y and contains A, we must have that $\bar{A} \subseteq Y \cap \mathrm{Cl}\,(A)$. Likewise, $\bar{A}$ being closed in Y allows us to choose a closed set C in X having the property that $\bar{A} = Y \cap C$. This implies that $A \subseteq C$ whence $A \subseteq \mathrm{Cl}\,(A) \subseteq C$. Taking intersections, we have $\mathrm{Cl}\,(A) \cap Y \subseteq C \cap Y = \bar{A}$. This completes the proof. □

This only partially solves the issue of classification. In general, it is incredibly difficult to describe the closure of a set in detail. In $\mathbb{R}$ there are some easy examples. For one, the closure of $(0, 1)$ in $\mathbb{R}$ is equal to $[0, 1]$. The closure of $\mathbb{Q}$ in $\mathbb{R}$ is the whole of $\mathbb{R}$ (this is known as density, and we will return to it shortly). However, there are also some very nasty instances. Let $V \subset \mathbb{R}$ be a non-measurable (with respect to the Lebesgue measure) set. Since closed sets are Lebesgue measurable, $\mathrm{Cl}\,(V)$ is measurable and hence $V \neq \mathrm{Cl}\,(A)$. However, explicitly describing $\mathrm{Cl}\,(V)$ is no easy task (and I do not know of a way to do so).

Despite this great difficulty, there are useful characterizations of closures and interiors. These will not have accurate pointwise descriptions, but will be nicely categorized according to their topological properties.

Theorem 1.15. *Let $(X, \mathfrak{T})$ be a topological space and A a subset of X. Fix a point $x \in X$.*

(1) $x \in \mathrm{Cl}(A)$ if and only if every open set U containing x intersects[3] A.

(2) If $\mathcal{B}$ is a basis for the topology $\mathfrak{T}$ on X, then $x \in \mathrm{Cl}(A)$ if and only if every $B \in \mathcal{B}$ containing x intersects A.

Proof. We begin with the proof of the first point. Equivalently, we will prove that $x \notin \mathrm{Cl}(A)$ if and only if there exists an open set $U \ni x$ in X with $U \cap A = \emptyset$. First, let $x \notin \mathrm{Cl}(A)$ so that $x \in X \setminus \mathrm{Cl}(A)$. Since $\mathrm{Cl}(A)$ is closed in X, we see that $X \setminus \mathrm{Cl}(A)$ is in X and certainly does not intersect A. Conversely, assume one can find an open set U containing X such that $U \cap A = \emptyset$. Then U^{c} is a closed set containing A but not x. By definition of the closure, we see that $x \notin \mathrm{Cl}(A)$.

The second statement follows immediately from Proposition 1.1 which states that every open set in X is a union of elements from $\mathcal{B}$. □

We now introduce terminology used by many mathematicians in both topology and in analysis. Fix a topological space $(X, \mathfrak{T})$ and a point $x_0 \in X$. A set $U \subseteq X$ is called a *neighbourhood* of x_0 if

(1) U is open in X;

(2) U contains the point x_0.

If $A \subseteq X$ then a neighbourhood of A is simply an open set containing the entire set A. This terminology will become increasingly useful as the complexity of our statements increase.

1.4.2 Limit Points of Sets

We again consider a general topological space $(X, \mathfrak{T})$. The concept of a limit point is familiar from real analysis, and also carries over nicely to the abstract setting in which we work. If $x \in X$ is *any* point and $A \subseteq X$ is a set, we will say that x is a limit point of A if any neighbourhood of x intersects A at a point other than x. Formulated rigorously, we obtain the following definition.

Definition 1.10. Let $x \in X$ and $A \subseteq X$. We say that x is a *limit point* of A if x belongs to the closure of $A \setminus \{x\}$.

[3]Let E and F be sets. We say that E intersects F if $E \cap F$ is non-empty.

Notice that we do not require that x belong to the set A. For example, consider the interval $(0,1) \subset [0,1] \subset \mathbb{R}$. Clearly, $[0,1]$ is the closure of $(0,1)$ and so 0 is a limit point of $(0,1)$. Actually, $[0,1]$ is the set of all limit points of $(0,1) \subset \mathbb{R}$. This nicely illustrates the following general relationship between the limit points of a set and its closure.

Theorem 1.16. *Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. Let A' denote the set of all limits points of A. Then,* $\mathrm{Cl}(A) = A \cup A'$.

Proof. We first show that $\mathrm{Cl}(A) \subseteq A \cup A'$. Suppose x belongs to the closure of A. If $x \in A$, then $x \in A \cup A'$. If instead $x \notin A$, the previous theorem guarantees that every neighbourhood U of x intersects A. Since $x \in U$ and $x \notin A$, we actually get that U intersects $A \setminus \{x\}$. Since this holds for all neighbourhoods U of x, it follows from Theorem 1.15 that $x \in A'$. We conclude that $\mathrm{Cl}(A) \subseteq A \cup A'$.

Conversely, notice that $A \subseteq \mathrm{Cl}(A)$. If $x \in A'$ then $x \in \mathrm{Cl}(A \setminus \{x\})$. Since $A \setminus \{x\} \subseteq A \subseteq \mathrm{Cl}(A)$, we have by definition that

$$x \in \mathrm{Cl}(A \setminus \{x\}) \subseteq \mathrm{Cl}(A).$$

This means that $A' \subseteq \mathrm{Cl}(A)$. It follows that $A \cup A' \subseteq \mathrm{Cl}(A)$ and this completes the proof. □

As a corollary, we obtain a significant generalization of a statement familiar to anyone with a background in real analysis.

Corollary 1.17. *Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. Then A is closed in X if and only if A contains all of its limit points.*

Proof. Let A' again denote the family of limit points of A. If A is closed, then $A = \mathrm{Cl}(A) = A' \cup A \supseteq A'$. Conversely, if $A' \subseteq A$ then

$$A \subseteq \mathrm{Cl}(A) = A' \cup A \subseteq A$$

whence $A = \mathrm{Cl}(A)$. □

1.5 Hausdorff Spaces and Sequences

Much of our geometric and analytic intuition from $\mathbb{R}^m$ and $\mathbb{C}^n$ can fail when working over general topological spaces. However, this should not be a deterrent, as one can easily impose *very weak* restrictions upon the topology to rectify this issue. Even without these restrictions, meaningful work can be carried out.

Sequences and convergence are topics familiar from calculus and real analysis. A *sequence* is merely a function from a countably infinite set N into some set X. Of course, we might as well assume that N is the set of natural numbers $\mathbb{N}$. On the other hand, convergence is a more delicate topic.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. We know from elementary real and complex analysis that $(x_n)_{n\in\mathbb{N}}$ *converges* to a point $x \in \mathbb{C}$, as $n \to \infty$, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon, \quad \forall n \geq N.$$

More generally, let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ of points in X is said to converge to $x \in X$ if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon, \quad \forall n \geq N.$$

In this case, we would write $\lim_{n\to\infty} x_n = x$. To develop a similar theory for general topological spaces, we will need to make use of the open sets in order to give a meaningful definition of convergence that agrees what we have written above for metric spaces.

Definition 1.11. Let $(X, \mathfrak{T})$ be a topological space, $(x_n)_{n\in\mathbb{N}}$ a sequence of points in X, and fix $x \in X$. We say that $(x_n)_{n\in\mathbb{N}}$ converges to x if for every neighbourhood U of x, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. In this case, we will write

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \xrightarrow{n\to\infty} x.$$

We have now come to a very subtle point. In $\mathbb{R}$ and in $\mathbb{C}$, we know that limits of sequences are unique, if they exist. That is, a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ or $\mathbb{C}$ can converge to at most one point. The same continues to hold in a metric space. Certainly, let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X converging to points $x, y \in X$. Given $\varepsilon > 0$, one can find $N \in \mathbb{N}$ so that

$$d(x_n, x) < \frac{\varepsilon}{2} \quad \text{and} \quad d(x_n, y) < \frac{\varepsilon}{2}$$

for all $n \geq N$. But then,

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get $d(x, y) = 0$, i.e. $x = y$. Unfortunately, this nice property does **not** hold in a general topological space, as we demonstrate below.

Example 1.3. Let X be any set containing at least two elements and endow X with the coarsest possible topology, $\mathfrak{T} := \{\varnothing, X\}$. Define a sequence $(x_n)_{n\in\mathbb{N}}$ in X by putting $x_n := x$, for a fixed choice of $x \in X$. For any $y \in X$, the only neighbourhood of y is X. Since $(x_n)_{n\in\mathbb{N}} \subseteq X$, we see that

$$\lim_{n\to\infty} x_n = y, \quad \forall y \in X.$$

In particular, $(x_n)_{n\in\mathbb{N}}$ converges to multiple distinct points in the space X. This means that the limit of a sequence need not be unique, in general.

1.5.1 The Hausdorff Axiom

The previous example has made it clear that one requires an additional axiom for "nice" convergence properties to hold. Perhaps the most popular and choice is the so-called *Hausdorff*[4] axiom which guarantees the existence of special disjoint neighbourhoods of points.

Definition 1.12. Let $(X, \mathfrak{T})$ be a topological space. The space X is said to be *Hausdorff* if, for any two distinct $x, y \in X$, there exists open sets $U_x \ni x$ and $U_y \ni y$ with $U_x \cap U_y \neq \varnothing$.

Hausdorff spaces have nicer and more familiar properties. For one, the Hausdorff axiom solves the problem regarding uniqueness of limits.

Theorem 1.18. *Let $(X, \mathfrak{T})$ be a Hausdorff topological space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If $(x_n)_{n\in\mathbb{N}}$ converges, then its limit is unique.*

Proof. By way of contradiction, suppose that $(x_n)_{n\in\mathbb{N}}$ converges to two distinct points x and y in X. We may choose two *disjoint* open sets U_x and U_y, containing x and y respectively. Now, since $(x_n)_n$ converges to x, there exists $N_1 \in \mathbb{N}$ such that $x_n \in U_x$ for all $n \geq N_1$. Similarly, one can choose $N_2 \in \mathbb{N}$ so that $x_n \in U_y$ for all $n \geq N_2$. Taking $n := \max(N_1, N_2)$, we have $x_n \in U_x \cap U_y$, which is a contradiction. □

In a similar spirit, Hausdorff spaces respect some of our intuition on finite sets. This is illustrated by the following proposition.

Proposition 1.19. *Let $(X, \mathfrak{T})$ be a Hausdorff topological space. All finite subsets of X are closed in X.*

[4] This is also called the T_2-axiom.

Proof. Since finite unions of closed sets are again closed, it suffices to show that any singleton is closed. Let $x \in X$ be given and consider $\{x\}$. Let y belong to the closure of $\{x\}$. By way of contradiction, suppose that $y \neq x$. By choosing two disjoint neighbourhoods of x and y, we may find an open set $V \ni y$ such that $x \notin V$. However, by Theorem 1.15, $y \in \mathrm{Cl}(\{x\})$ implies that $V \cap \{x\}$ which is a contradiction. We conclude that $y = x$ and hence obtain $\{x\} = \mathrm{Cl}(\{x\})$. □

Example 1.4. Suppose that X is an infinite set and endow X with the *finite complement topology* $\mathfrak{T}$ (see Example 1.2). We show that X is not Hausdorff. Let x, y be distinct points in X and suppose, by way of contradiction, that there exist disjoint open sets U_x and U_y containing x and y, respectively. Since U_x and U_y are both open, they each contain all but finitely many points in x. That is,

$$U_x = X \setminus \{x_1, \ldots, x_k\} \quad \text{and} \quad U_y = X \setminus \{y_1, \ldots, y_l\}$$

for the appropriate $x_i, y_j \in X$. Since X is infinite, we can choose a point $\xi \in X$ distinct from each x_i and every y_j. Then, $\xi \in U_x \cap U_y$–which is absurd.

Separated Points

In the previous example, we showed that every infinite set (e.g. $\mathbb{Z}$) with the finite complement topology is not Hausdorff. However, a closer inspection of our argument shows something significantly stronger. Our proof actually tells us that for *all* distinct points $x \neq y$ there cannot exist disjoint open sets containing x and y, respectively. In a sense, we could say that X is not Hausdorff at every point of X. We formalize this below.

Definition 1.13. Let $(X, \mathfrak{T})$ be a topological space. Two points x and y are said to be *separated* if there exist disjoint open sets containing x and y, respectively.

A space X is Hausdorff whenever all distinct points are separated. Our example shows that an infinite set with the finite complement topology has no separated points

1.6 Continuous Functions

Continuous functions are cornerstones of modern mathematics and arise in many different subfields of analysis. This section will not contain any results that will leave the reader surprised, but it will introduce the *abstract* topological definition of continuity which we have hinted at towards the beginning of the chapter.

Definition 1.14. Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces and consider a function $f : X \to Y$. We say that f is *continuous* if $f^{-1}(V)$ is open in X for every open set V in Y.

In practice, this condition is ridiculous to check. Luckily, if $\mathfrak{W}$ is generated by a basis $\mathcal{B}$, we will only need to verify that $f^{-1}(B)$ is open for every $B \in \mathcal{B} \subseteq \mathfrak{W}$. Indeed, we know from Proposition 1.1 that every element of $\mathfrak{W}$ is simply a union of elements from $\mathcal{B}$. The equivalence of these two criteria then follows from the identity

$$f^{-1}\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha f^{-1}(V_\alpha)$$

and the fact that unions of open sets are open. We summarize this discussion below.

Theorem 1.20. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces. Assume that $\mathfrak{W}$ is generated by a basis $\mathcal{B}$. A function $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open in X, for every $B \in \mathcal{B}$.*

As it is an elementary fact from real analysis, we will not prove that our definition of continuity is equivalent to the $\varepsilon - \delta$ definition used when working over a metric space.[5] However, we will offer four abstract characterizations of continuity that hold over any topological space.

Theorem 1.21. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces and fix a function $f : X \to Y$. The following statements are equivalent.*

(1) f is continuous.

(2) For every $A \subseteq X$, there holds $f(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(f(A))$.

(3) If B is closed in Y, then $f^{-1}(B)$ is closed in X.

(4) For every $x \in X$ and every neighbourhood V of $f(x)$ in Y, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Proof. We begin with the implication (1) $\implies$ (2). Let $A \subseteq X$ be given and suppose that $y \in f(\mathrm{Cl}(A))$. This means that there exists $x \in \mathrm{Cl}(A)$ such that $y = f(x)$. Let now V be a neighbourhood of $y = f(x)$ and notice that $f^{-1}(V)$ is a neighbourhood of x, by continuity of f. Since x belongs to the closure of A, this

[5] For a proof of this fact, we urge the reader to consult [DBMN].

means that $f^{-1}(V)$ intersects A. Given that $f(f^{-1}(V)) \subseteq V$, we conclude that V intersects $f(A)$. As V was taken arbitrarily, Theorem 1.15 gives that $f(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(f(A))$.

Let us now argue for (2) $\implies$ (3). Let $B \subseteq Y$ be a closed subset and let A denote $f^{-1}(B)$. We claim that $A = \mathrm{Cl}(A)$, and for this the only non-trivial inclusion is $\mathrm{Cl}(A) \subseteq A$. Let $x \in \mathrm{Cl}(A)$ so that, by hypothesis,

$$f(x) \in f(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(f(A)) \subseteq \mathrm{Cl}(B) = B.$$

This certainly gives $x \in f^{-1}(B) = A$. Since x was arbitrary, we conclude that $\mathrm{Cl}(A) \subseteq A$ as was required.

We show (3) $\implies$ (1). Let $B \subseteq Y$ be open, so that $Y \setminus B$ is closed. Then,

$$f^{-1}(Y \setminus B) = f^{-1}(B^{\mathrm{c}}) = f^{-1}(B)^{\mathrm{c}}$$

is closed. That is, $f^{-1}(B)$ is open in X.

It now only remains to check that (1) $\iff$ (4). First, assume that f is continuous and fix $x \in X$ and a neighbourhood V of $f(x)$. By continuity, $f^{-1}(V)$ is open and contains x. Thus, taking $U := f^{-1}(V)$ gives one implication. Conversely, fix an open set V in Y. For any $x \in f^{-1}(V)$, we can choose an open set U_x containing x such that $f(U_x) \subseteq V$. But, this gives $U_x \subseteq f^{-1}(V)$. Since $U_x \ni x$, it is then easy to see that

$$\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V).$$

Since unions of open sets are open, $f^{-1}(V)$ is open and this concludes the proof. □

1.6.1 Homeomorphisms

We now introduce a notion of "equivalence" for topological spaces. Suppose that $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ are two topological spaces. We ask ourselves what it could mean for these two spaces to be "isomorphic". This is made precise below.

Definition 1.15. Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces. A continuous bijection $f : X \to Y$ is called a *homeomorphism* if its inverse $f^{-1} : Y \to X$ is also known to be continuous. We will say that $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ are *homeomorphic* if there exists a homeomorphism $X \to Y$.

We now claim that being homeomorphic is an equivalence relation on topological spaces. However, this will require a little bit of work. We summarize the relevant results below in the form of a proposition.

Proposition 1.22. *Let X, Y and Z be topological spaces with underlying topologies. There hold the following.*

(1) If $y_0 \in Y$ is fixed, then the function $f : X \to Y$ given by $f(x) = y_0$ is continuous.

(2) Let A be a subspace of X. Then inclusion mapping $\iota : A \to X$ given by $a \mapsto a$ is continuous.

(3) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is the composite $f \circ g : X \to Z$.

(4) If $f : X \to Y$ is continuous and A is a subspace of X, the restriction of f to A is a continuous function $A \to Y$.

(5) If $f : X \to Y$ is continuous, then f is also continuous when viewed as a function $X \to f(X)$.

(6) If $f : X \to Y$ is continuous and Y is a subspace of Z, then f is continuous when viewed as a function $X \to Z$.

Proof. We will handle each portion of the proof separately.

(1) Let $V \subseteq Y$ be an open set, then

$$f^{-1}(V) = \begin{cases} X, & \text{if } y_0 \in V, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Regardless, $f^{-1}(V)$ is always open in X and hence f is continuous.

(2) Let $V \subseteq X$ be an open set and notice that

$$\iota^{-1}(V) = A \cap V.$$

Since the open sets in A are all of the form $A \cap V$, we see that $\iota^{-1}(V)$ is open in A. This means that ι is a continuous map.

(3) Let $W \subseteq Z$ be an open set, elementary set theory yields

$$(f \circ g)^{-1}(W) = f^{-1}\left(g^{-1}(W)\right).$$

Now, $g^{-1}(W)$ is open in Y by the continuity of g. But then, the set $f^{-1}(g^{-1}(W))$ is open in X by continuity of f. All in all, we get that $f \circ g$ is a continuous function.

(4) Let $f : X \to Y$ be continuous and let A be a subspace of X. Denote by g the restriction $f|_A$. If $V \subseteq Y$ is an open set, then

$$g^{-1}(V) = f^{-1}(V) \cap A$$

which is open in the subspace topology with which we endow A.

(5) For the fifth point, notice that the "new" function $f : X \to f(X)$ is well defined. A general open subset U of $f(X)$ will be of the form

$$U = V \cap f(X) \quad \text{where } V \text{ is open in } Y.$$

Elementary set theory then gives

$$f^{-1}(U) = f^{-1}(V) \cap f^{-1}(f(X)) = f^{-1}(V),$$

which is open in X by continuity from $X \to Y$.

(6) Finally, let W be open in Z. Then, since $f(X) \subseteq Y$,

$$\begin{aligned} f^{-1}(W) = \{x \in X : f(x) \in W\} &= \{x \in X : f(x) \in W \cap Y\} \\ &= f^{-1}(W \cap Y) \end{aligned}$$

where $W \cap Y$ is open in Y. Since f is continuous from $X \to Y$, we conclude that $f^{-1}(W)$ is open in X. The statement now follows.

□

With this out of the way, we can give the following promised result.

Corollary 1.23. *Being homeomorphic is an equivalence relation on topological spaces. If two spaces $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ are homeomorphic, we will write $X \cong Y$.*

Proof. First, fix a topological space $(X, \mathfrak{T})$. The identity map $\mathbf{1}_X$ is an inclusion $X \to X$, and is continuous by the previous theorem. Since $\mathbf{1}_X$ is its own inverse, we see that this is a homeomorphism $X \to X$. This gives reflexivity. If $(X, \mathfrak{T}) \cong (Y, \mathfrak{W})$, there exists a homeomorphism $X \to Y$. As the inverse of a homeomorphism is again a homeomorphism, we see that $(Y, \mathfrak{W}) \cong (X, \mathfrak{T})$. Finally, the composition of two homeomorphisms will again be a homeomorphism from which we draw the desired conclusion. □

A more interesting result is the following, which is sometimes dubbed the "Pasting lemma".

Theorem 1.24. *Let $(X, \mathfrak{T})$ be a topological space and suppose A and B are two closed subsets of X whose union is X. Let $(Y, \mathfrak{W})$ be a topological space and suppose*

$$f : A \to Y \quad \text{and} \quad g : B \to Y$$

are continuous relative to their respective subspace topologies. If $f \equiv g$ on $A \cap B$, then the function

$$h(x) := \begin{cases} f(x), & \text{if } x \in A, \\ g(x), & \text{if } x \in B \end{cases}$$

is a well defined continuous function $X \to Y$.

Proof. It is obvious that $h(x)$ is well defined. If $C \subseteq Y$ is a closed set, then

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

It follows from Theorem 1.21 that $f^{-1}(C)$ is a closed subset of A. Since A is closed in X, Proposition 1.12 implies that $f^{-1}(C)$ is closed in X. Likewise, $g^{-1}(C)$ is closed in X. Since the union of finitely many closed sets is again closed, $h^{-1}(C)$ is closed in X. Since C was arbitrary, Theorem 1.21 implies that h is continuous. □

1.7 The Product Topology

We have already briefly discussed a topology on a Cartesian product of the form $X \times Y$, and we now extend this discussion to arbitrary Cartesian products. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be any indexed family of sets. The Cartesian product of the X_λ, denoted $\prod_{\lambda \in \Lambda} X_\lambda$, is defined to be the set of all functions

$$\rho : \Lambda \to \bigcup_{\lambda \in \Lambda} X_\lambda$$

such that $\rho(\lambda) \in X_\lambda$ for all $\lambda \in \Lambda$. The functions $\rho \in \prod_{\lambda\in\Lambda} X_\lambda$ are sometimes referred to as *coordinate maps*. This notation is, however, not too convenient for our purposes. Instead of writing ρ, we will write something along the lines of $(x_\lambda)_{\lambda\in\Lambda}$ to mimic the usual tuple notation we use for finite Cartesian products (e.g. $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$).

Definition 1.16. Let I be set indexing a family of topological spaces $\{(X_i, \mathfrak{T}_i)\}_{i\in I}$. Consider the basis for a topology on $\prod_{i\in I} X_i$

$$\mathcal{B} := \left\{ \prod_{i\in I} U_i : U_i \text{ open in } (X_i, \mathfrak{T}_i) \right\}$$

The topology generated by $\mathcal{B}$ on $\prod_{i\in I} X_i$ is called the *box topology*.

Let us now verify that the family $\mathcal{B}$ described in the definition above is indeed a basis for a topology on $\prod_{i\in I} X_i$. First, any element of $\prod_{i\in I} X_i$ belongs to the basis element $\prod_{i\in I} X_i$. The second criterion is easily satisfied because if U_i, V_i are open in X_i then

$$\left(\prod_{i\in I} U_i \right) \cap \left(\prod_{i\in I} V_i \right) = \prod_{i\in I} U_i \cap V_i.$$

Let us now attempt to work with a subbasis, as the did when we were considering the simpler case $X \times Y$. This again requires some preliminary terminology. Let suppose we have a set I indexing a family of topological spaces $(X_i, \mathfrak{T}_i)$. For any index $j \in I$, we have a *projection mapping*

$$\pi_j : \prod_{i\in I} X_i \twoheadrightarrow X_j, \quad (x_i)_{i\in I} \mapsto x_j.$$

We are now prepared to define the more general *product topology*.

Definition 1.17. For $j \in I$, let $\mathcal{S}_j$ be the family

$$\mathcal{S}_j := \left\{ \pi_j^{-1}(U_j) : U_j \in \mathfrak{T}_j \right\}.$$

Define $\mathcal{S} := \bigcup_{j\in I} \mathcal{S}_j$. Clearly, $\mathcal{S}$ is a subbasis for a topology on $\prod_{i\in I} X_i$ as $\prod_{i\in I} X_i$ belongs to $\mathcal{S}$. The topology generated by $\mathcal{S}$ is then called the *product topology* on $\prod_{i\in I} X_i$.

1.7.1 The Structure of the General Product Topology

It is easy to understand the box topology. Indeed, its basis elements are the products of open sets $\prod_{i\in I} U_i$. Therefore, open sets in the box topology are unions of such elements. We now turn towards the product topology. Let $\mathcal{B}$ the basis that $\mathcal{S}$ generates (according to Lemma 1.4). For a fixed index j, if U_j, V_j are open in X_j, it is easily seen that

$$\pi_j^{-1}(U_j) \cap \pi_j^{-1}(V_j) = \pi_j^{-1}(U_j \cap V_j) \in \mathcal{S}_j.$$

Therefore, $\mathcal{S}_j$ is closed under finite intersections. Since $\mathcal{B}$ consists of finite intersections of elements in $\mathcal{S}$, it follows from the above that a general element B of $\mathcal{B}$ has the form

$$B = \bigcap_{k=1}^{n} \pi_{j_k}^{-1}(U_{j_k}), \quad U_{j_k} \in \mathfrak{T}_{j_k}.$$

This means that B is the set of all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ with the property that $x_{j_k} \in U_{j_k}$ for all $k = 1, \ldots, n$. Since this places no restriction on the other “coordinates”, we have that

$$B = \prod_{i\in I} A_i \quad \text{where } A_i := \begin{cases} U_{j_k}, & \text{if } i = j_k \text{ for some } k = 1, \ldots, n, \\ X_i, & \text{otherwise.} \end{cases}$$

This perfectly describes the box and product topologies on the Cartesian product $\prod_{i\in I} X_i$. Henceforth, when we discuss the product space $\prod_{i\in I} X_i$, it will always be assumed that the space is given the *product topology*.

1.7.2 Properties of Product Spaces

Having discussed both the box and product topologies, we consider how topological properties of the X_i carry over to the product $\prod X_i$. We partially answer this question in the following theorem.

Theorem 1.25. *Let I be an index set and $\{(X_i, \mathfrak{T}_i)\}_{i\in I}$ an indexed family of topological spaces. There hold the following.*

(1) If for each i the topology $\mathfrak{T}_i$ is generated by a basis $\mathcal{B}_i$, then the collection of all sets of the form

$$\prod_{i\in I} B_i, \quad B_i \in \mathcal{B}_i \subseteq \mathfrak{T}_i,$$

forms a basis for the box topology on $\prod_{i\in I} X_i$. The collection of all products having the form $\prod_{i\in I} B_i$ with $B_i \in \mathcal{B}_i$ for finitely many i and $B_i = X_i$ for all remaining i serves as a basis for the product topology on $\prod_{i\in I} X_i$.

(2) *If each space X_i is Hausdorff then so is $\prod_{i\in I} X_i$, for both the product and box topologies.*

Example 1.5. Consider $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. We endow $\mathbb{F}_2$ with the discrete topology and consider the induced product space $\mathbb{F}_2^\infty = \prod_1^\infty \mathbb{F}_2$ with both the box and product topologies. With the box topology, $\mathbb{F}_2^\infty$ inherits th discrete topology. To see this, simply note that any singleton in $\mathbb{F}_2^\infty$ can be expressed as the product of singletons in $\mathbb{F}_2^\infty$.

Clearly, this argument fails if we instead give $\mathbb{F}_2^\infty$ the product topology. In this case, the point

$$\{\mathbf{0}\} = \{(0, 0, \ldots, 0, \ldots,)\}$$

is **not** open in $\mathbb{F}_2^\infty$. If it were open, then we could find countably many open subsets $\{U_n\}_{n\in\mathbb{N}}$ of $\mathbb{F}_2$, all but finitely many equal to $\mathbb{F}_2$, such that

$$\mathbf{0} \in \prod_{n\in\mathbb{N}} U_n \subseteq \{\mathbf{0}\}.$$

The only possibility is then $U_n = \{0\}$ for every n. This however contradicts the choice of U_n, for all n sufficiently large.

1.7.3 A Criterion for Continuity

Let I be an index set and $\{X_\alpha\}_{\alpha\in I}$ an indexed family of topological spaces. Give $\prod_{\alpha\in I} X_\alpha$ the product topology and let $(X, \mathfrak{T})$ be a topological space. We have seen that the projection maps

$$\pi_\beta : \prod_{\alpha\in I} X_\alpha \to X_\beta$$

are continuous, for any $\beta \in I$. Now, consider a function

$$f : X \to \prod_{\alpha\in I} X_\alpha.$$

For each index $\beta \in I$, composition gives us a function which collapses the image of f to X_β:

$$f_\beta : X \xrightarrow{f} \prod_{\alpha\in I} X_\alpha \xrightarrow{\pi_\beta} X_\beta, \quad f_\beta(x) = \pi_\beta(f(x)). \tag{1.3}$$

It turns out that the continuity of f depends only on the behaviour of each f_β.

Proposition 1.26. *The function f is continuous if and only if f_β is continuous for every $\beta \in I$.*

Proof. If f is continuous, then f_β is automatically continuous by composition.

Conversely, assume that f_β is continuous $X \to X_\beta$ for every $\beta \in I$. It is enough to check that the pre-image under f of every basis element (for the product topology on $\prod_{\alpha\in I} X_\alpha$) is open in X. To this end, let $\prod_{\alpha\in I} U_\alpha$ be basis element. Then, $U_\alpha = X_\alpha$ for all but finitely many α; let us enumerate these α by $\alpha_1, \ldots, \alpha_l$. For each index α, the pre-image

$$f_\alpha^{-1}(U_\alpha) = f^{-1}(\pi_\alpha^{-1}(U_\alpha))$$

is open in X. The continuity of f then follows from the following:

$$f^{-1}\left(\prod_{\alpha\in I} U_\alpha\right) = \bigcap_{j=1}^{l} f_{\alpha_j}^{-1}(U_{\alpha_j}).$$

□

1.8 Metric Spaces

As natural generalizations of Euclidean space, metric spaces are a special class of topological spaces that arise frequently in analysis and physics. In this section, we introduce the concept of a metric space and relate it to our general definition of a topological space. Only the most rudimentary notions are explored in this section; in Chapter 3 we will explore these spaces in greater detail.

Definition 1.18. Let X be a non-empty set and $d : X \times X \to \mathbb{R}_{\geqslant 0}$ a function satisfying the following properties:

(1) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) For any $x, y, z \in X$ there holds

$$d(x, y) \leq d(x, z) + d(z, y).$$

This is known as the *triangle inequality.*

The function d is called a *metric* on the set X. The pair (X, d) shall be referred to as a *metric space*.

The metric d induces a familiar topology on X called the *metric topology*. If $x_0 \in X$ and $\delta > 0$ are given, the "open ball of radius δ centered at x_0" is the set

$$B(x_0, \delta) := \{x \in X : d(x, x_0) < \delta\}.$$

Consider the collection of all such balls

$$\mathcal{B} := \{B(x_0, \delta) : x_0 \in X,\ \delta > 0\}.$$

This set $\mathcal{B}$ is easily seen to be a basis for a topology on X. We then endow X with the topology generated by $\mathcal{B}$, and this topology is called the *metric topology* induced by d. This topology interacts nicely with the metric d.

Definition 1.19. Let $(X, \mathfrak{T})$ be a topological space. We say that X is metrizable if there exists a metric d on X such that the metric topology induced by d on X is precisely $\mathfrak{T}$.

We must be careful with metrizable spaces. The reader may be tempted to believe that a metrizable space is, in fact, a metric space. However, this is most certainly not the case (and the explanation is quite ontological). We have defined a metric space to be an ordered pair (X, d) where X is a *set* and d is a *function* on $X \times X$. On the other hand, a topological space is a pair $(X, \mathfrak{T})$ where X is a set and $\mathfrak{T}$ is a family of *subsets* of X. Even if $(X, \mathfrak{T})$ is metrizable with metric d, it does not make sense (formally) to write $(X, \mathfrak{T}) = (X, d)$. Moreover, much of what we touch upon in this text is purely topological and will not relate to the metric d itself. For these reasons, we will be careful to distinguish between metric spaces and metrizable spaces.

Remark 1.1. In practice, there is good reason to separate the two notions. If $(X, \mathfrak{T})$ is metrizable, then we know there exists *at least* one metric d on X inducing the topology $\mathfrak{T}$. However, there is nothing canonical about any such d, as there may even be infinitely many d having this property!

Since a metric space (X, d) has a canonical topology induced by the metric d, it makes sense to discuss the continuity of a function f defined on X. However, the metric d also allows us to extend the classical $\varepsilon - \delta$ definition of continuity, which we make precise below. Ideally, we would like this generalized $\varepsilon - \delta$ definition to agree with our open set notion of continuity. Of course, this will turn out to be the case.

Definition 1.20. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a function. We say that f is continuous at a point $x_0 \in X$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_Y\left(f(x), f(x_0)\right) < \varepsilon \quad \text{whenever } d_X(x, x_0) < \delta.$$

We say that f is *continuous* if it is continuous at all points $x_0 \in X$.

Let us now check that this definition of continuity agrees with our topological definition.

Theorem 1.27. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function. The following statements are equivalent:*

(1) $f : X \to Y$ is continuous in the $\varepsilon - \delta$ sense;

(2) for every open set $V \subseteq Y$, the pre-image $f^{-1}(V)$ is open in X.

Proof. First let us assume that f satisfies the $\varepsilon - \delta$ condition for continuity and let V be an open set in Y; we must show that $f^{-1}(V)$ is open in X. To this end, let $x \in X$ be given and consider $f(x) \in V$. Since V is open, we can find $\varepsilon > 0$ so small that $B(f(x), \varepsilon) \subseteq V$. For this $\varepsilon > 0$, there is some corresponding $\delta > 0$ having the property that

$$d_Y(f(z), f(x)) < \varepsilon$$

whenever $d_X(z, x) < \delta$ with $z \in X$. Now consider the open set $B(x, \delta)$ which obviously contains x. If we can show that $B(x, \delta) \subseteq f^{-1}(V)$, we will have established (2). Now, $z \in B(x, \delta)$ implies $d_X(z, x) < \delta$ whence $d_Y(f(z), f(x)) < \varepsilon$. That is, we have $f(z) \in B(f(x), \varepsilon) \subseteq V$ whenever $z \in B(x, \delta)$. Of course, this simply means that $B(x, \delta) \subseteq f^{-1}(V)$.

Conversely, assume that $f^{-1}(V)$ is open in X for all open sets $V \subseteq Y$. Fix any point $x_0 \in X$ and let $\varepsilon > 0$ be given. Putting $V := B(f(x_0), \varepsilon)$, the pre-image $f^{-1}(V) \ni x_0$ will be open in X. By definition, there is some $\delta > 0$ such that $B(x_0, \delta) \subseteq f^{-1}(V)$. This is equivalent to having $f(B(x_0, \delta)) \subseteq V = B(f(x_0), \varepsilon)$. Or, rather, this is equivalent to writing

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon.$$

This completes the proof. $\square$

We will not dwell further upon the elementary topological nature of metric spaces as these notes are intended for readers who have already had plenty of exposure to real and functional analysis (and abstract mathematics in general). If the reader has not studied metric spaces in great detail, we suggest they put this text aside to rapidly consult the short document [DBMN].

1.9 Exercises

The following exercises are intended to help the reader review the material we have covered within this chapter. Of course, the reader should also familiarize themselves with all the proofs *once again*. Solutions to the exercises are given in Appendix B.

For the most part, the exercises are taken from [MNKS]. Others are given to ensure that the reader understands key points and subtleties.

Problem 1.1. Let X be a topological space and $A \subseteq Y$ subspaces of X. Show that A inherits the same topology from Y as it does from X.

Problem 1.2. Let $\mathcal{B}$ be a basis for a topology on X. Show that the topology generated by $\mathcal{B}$ is precisely the intersection of all topologies *containing* $\mathcal{B}$. Prove the same when $\mathcal{B}$ is a *subbasis*.

Problem 1.3. Let X be a space and Y a subspace of X. If $A \subseteq Y$ is closed in X, then A is closed in Y.

Problem 1.4. Show that the *countable* family

$$\mathcal{B} := \{(a, b) : a < b,\ a, b \in \mathbb{Q}\}$$

is a basis for the standard topology on $\mathbb{R}$. Furthermore, show that

$$\mathcal{C} := \{[a, b) : a < b,\ a, b \in \mathbb{Q}\}$$

does *not* generate the topology of $\mathbb{R}_\ell$.

Problem 1.5. Let $\mathfrak{T}$ and $\mathfrak{T}'$ be topologies on a set X and assume that $\mathfrak{T}'$ is strictly finer than $\mathfrak{T}$. What can be said about the corresponding subspace topologies on a set $Y \subseteq X$?

Problem 1.6. Let X and Y be spaces and let $A \subseteq X$ be closed in X and $B \subseteq Y$ closed in Y. Show that $A \times B$ is closed in $X \times Y$.

Problem 1.7. Let A, B and $\{A_\alpha\}_{\alpha \in I}$ be subsets of a topological space $(X, \mathfrak{T})$. Show that the following hold:

(1) If $A \subseteq B$ then $\mathrm{Cl}\,(A) \subseteq \mathrm{Cl}\,(B)$;
(2) $\mathrm{Cl}\,(A \cup B) = \mathrm{Cl}\,(A) \cup \mathrm{Cl}\,(B)$;

(3) $\bigcup_{\alpha \in I} \mathrm{Cl}\,(A_\alpha) \subseteq \mathrm{Cl}\,(\bigcup_{\alpha \in I} A_\alpha)$ and provide an example where equality does not hold.

Problem 1.8. Let X,Y be topological spaces and $A \subseteq X$, $B \subseteq Y$. Prove that $\mathrm{Cl}\,(A \times B) = \mathrm{Cl}\,(A) \times \mathrm{Cl}\,(B)$.

Problem 1.9. Let X and Y be Hausdorff spaces. Prove that $X \times Y$ (with the product topology) is Hausdorff.

Problem 1.10. Let $(X, \mathfrak{T})$ be a Hausdorff space and Y a subspace of X. Prove that Y is itself a Hausdorff space.

Problem 1.11. Let $(X, \mathfrak{T})$ be a topological space and define $\Delta := \{(x, x) : x \in X\}$. Clearly, $\Delta \subseteq X \times X$. Prove that X is Hausdorff if and only if Δ is closed in $X \times X$.

Problem 1.12. Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. Prove that the equality $X \setminus \mathrm{Int}\,(A) = \mathrm{Cl}\,(A^c)$ holds.

Problem 1.13. Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. The boundary of A is defined via the set

$$\partial A := \mathrm{Cl}\,(A) \cap \mathrm{Cl}\,(A^c)\,.$$

(1) Prove that $\mathrm{Int}\,(A)$ and ∂A are disjoint and $\mathrm{Cl}\,(A) = \mathrm{Int}\,(A) \sqcup \partial A$.

(2) Show that ∂A is empty if and only if A is clopen.

(3) Prove that A is open if and only if $\partial A = \mathrm{Cl}\,(A) \setminus A$.

(4) If A is open, does it hold that $A = \mathrm{Int}\,(\mathrm{Cl}\,(A))$.

Problem 1.14. Let X and Y be topological spaces and $f : X \to Y$ continuous. If x is a limit point of $A \subseteq X$, it it true that $f(x)$ is a limit point of $f(A)$ in Y?

Problem 1.15. Let X and Y be topological spaces with Y Hausdorff. Suppose that $A \subseteq X$ and $f : A \to Y$ is continuous. Prove that if f may be extended to a continuous function $g : \mathrm{Cl}\,(A) \to Y$, then g is uniquely determined by f.

Problem 1.16. Let I be an index set and $\{(X_i, \mathfrak{T}_i)\}_{i \in I}$ an indexed family of topological spaces. There hold the following.

(1) If for each i the topology $\mathfrak{T}_i$ is generated by a basis $\mathcal{B}_i$, then the collection of all sets of the form

$$\prod_{i\in I} B_i, \quad B_i \in \mathcal{B}_i \subseteq \mathfrak{T}_i,$$

forms a basis for the box topology on $\prod_{i\in I} X_i$. The collection of all products having the form $\prod_{i\in I} B_i$ with $B_i \in \mathcal{B}_i$ for finitely many i and $B_i = X_i$ for all remaining i serves as a basis for the product topology on $\prod_{i\in I} X_i$.

(2) If each space X_i is Hausdorff then so is $\prod_{i\in I} X_i$, for both the product and box topologies.

Problem 1.17. Let $\{X_\alpha\}$ be a family of topological spaces and let $\{A_\alpha\}$ be a family of subspaces. Show that $\prod_\alpha A_\alpha$ is a subspace of $\prod_\alpha X_\alpha$.

The following problem deals with sequence spaces over $\mathbb{R}$. We denote by $\mathbb{R}^\omega$ the countable product space $\prod_1^\infty \mathbb{R}$ with either the product or box topology, depending on the context. Then, $\mathbb{R}^\infty$ is defined to be the subspace of $\mathbb{R}^\omega$ consisting of all sequences $(x_n)_{n=1}^\infty$ in $\mathbb{R}$ whose terms are "eventually zero".

Problem 1.18. Describe the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ when both are given the product topologies. Do the same when they are endowed with the box topology.

Problem 1.19. Let (X, d_X) and (Y, d_Y) be metric spaces and let $\Phi : X \to Y$ be an isometry[6]. Prove that Φ is an embedding (continuous and injective).

[6]That is, $d_Y(\Phi(x_1), \Phi(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

Chapter 2

Connectedness, Compactness, and Quotient Spaces

Let $(X, \mathfrak{T})$ be a topological space. Intuitively, we should say that $(X, \mathfrak{T})$ is connected if it cannot be broken down into two separate sets. Such a bland definition certainly cannot work from a set theoretic perspective. After all, any non-empty set can be partitioned. Thus, our notion of connectedness will have to rely upon the topology $\mathfrak{T}$. With this question in mind, the first part of the chapter rigorously studies what it means for a topological space to be either "connected" or "disconnected".

The second portion of this chapter introduces a very important topological property, called *compactness*. Loosely speaking, a compact space will behave almost like a finite dimensional space. This deep structural property lies at the very heart of many theorems from real analysis and is fundamental to analysis.

In the final section, we introduce quotient maps and saturated sets. This section may seem out of place, but the concepts will be of immeasurable value when we tackle topics in algebraic topology (see Part II). We also cover a "factorization theorem" that is highly reminiscent of the first isomorphism theorem for groups.

2.1 Connected Spaces

Definition 2.1. Let $(X, \mathfrak{T})$ be a topological space. We say that X, or $(X, \mathfrak{T})$, is *connected* if there does not exist two non-empty disjoint open sets $A, B \subseteq X$ with the property that $X = A \sqcup B$. Otherwise, we say that $(X, \mathfrak{T})$ is *disconnected.*

More terminology is in order. If one can find two non-empty disjoint open sets A and B in X such that $X = A \sqcup B$, the pair (A, B) is called a *separation* of the space X. A set $A \subseteq X$ is called *clopen* in X if it is both open and closed relative to $\mathfrak{T}$. With this, one can reformulate connectivity purely in terms of topological properties.

Proposition 2.1. *Let $(X, \mathfrak{T})$ be a topological space. Then X is connected if and only if the only clopen sets in X are X and $\varnothing$.*

Proof. Let $A \subseteq X$ be a clopen set that is neither empty nor the whole space X. Since A is closed, its complement A^c is open in X. Since A and A^c are disjoint, we have found a separation of the space X. This means that $(X, \mathfrak{T})$ is disconnected. Conversely, assume that $(X, \mathfrak{T})$ is disconnected. We may then choose non-empty disjoint open sets A and B whose union is X. This means that $A^c = B$ so that A is clopen. Since $A \neq X$ and $A \neq \varnothing$, the proof is complete. □

When discussing subspaces, we have another useful criterion.

Proposition 2.2. *Let Y be a subspace of a topological space $(X, \mathfrak{T})$. A separation of Y is a pair of non-empty disjoint sets, both not containing a limit point of the other (relative to X), whose union is Y.*

Proof. First, let (A, B) be a separation of Y. Then A and B are non-empty disjoint open sets (in Y) whose union is Y. Thus, both A and B are clopen relative to Y. Now, the closure of A in Y is the set $\mathrm{Cl}\,(A) \cap Y$. Since A is closed in Y, we get that

$$A = \mathrm{Cl}\,(A) \cap Y.$$

Using that A and B are disjoint, we get that

$$A \cap B = \mathrm{Cl}\,(A) \cap B \cap Y = \mathrm{Cl}\,(A) \cap B = \varnothing.$$

Since $\mathrm{Cl}\,(A)$ contains all limit points of A, we see that B does not contain any limit points from A. By symmetry, the same can be said for A and the limit points of B. Conversely, suppose that $Y = A \sqcup B$ for disjoint non-empty sets A and B, neither of which contains a limit point from the other. Of course, this means that

$$\mathrm{Cl}\,(A) \cap B = \varnothing \quad \text{and} \quad A \cap \mathrm{Cl}\,(B) = \varnothing.$$

It then follows that

$$\begin{aligned} \mathrm{Cl}\,(A) \cap Y = \mathrm{Cl}\,(A) \cap (A \sqcup B) &= (\mathrm{Cl}\,(A) \cap A) \cup (\mathrm{Cl}\,(A) \cap B) \\ &= \mathrm{Cl}\,(A) \cap A \\ &= A \end{aligned}$$

whence A is closed in Y. The same argument gives that B is closed in Y. Since they form a partition of Y, we conclude that $A = Y \setminus B$ and $B = Y \setminus A$ are both open in Y. This concludes the proof. □

Example 2.1. Let X be an infinite set and give X the finite complement topology (see Example 1.2). We claim that X is connected. By way of contradiction, suppose that X can be written as the disjoint union $A \sqcup B$, where A and B are both non-empty open sets. Since $A, B \neq X$, both A^c and B^c must be finite. Therefore,

$$A^c \cup B^c$$

forms a proper subset of X. Let now $\xi \in X$ be a point not belonging to $A^c \cup B^c$. Then $\xi \in A \cap B$, which contradicts the choice of A and B.

2.1.1 Construction of Connected Spaces

Here we ask ourselves how to construct connected spaces from a given family of connected topological spaces. This whole process begins with the following easy lemma.

Lemma 2.3. *Let $(X, \mathfrak{T})$ be a topological space and Y a connected subspace of X. If (A, B) forms a separation of X, then Y is contained in one of A or B.*

Proof. Let A and B be two non-empty disjoint open sets whose union is X. In the subspace topology of Y, the sets $A \cap Y$ and $B \cap Y$ are open. Of course,

$$(A \cap Y) \cup (B \cap Y) = (A \cup B) \cap Y = Y$$

and $(A \cap Y) \cap (B \cap Y) = \varnothing$ since A and B are disjoint. This means that $A \cap Y$ and $B \cap Y$ are disjoint open subsets of Y whose union is Y. Since Y is connected, one of these is empty. Without harm, assume that $B \cap Y$ is empty. This implies that $Y \subseteq A$, as was required. □

Example 2.2. Let $(X, \mathfrak{T})$ be a space and suppose $\{A_n\}_{n \in \mathbb{N}}$ is a countable family of connected subspaces of X such that $A_n \cap A_{n+1} \neq \varnothing$ for each $n \in \mathbb{N}$. We will show that $A := \bigcup_{n \in \mathbb{N}} A_n$ is again a connected subspace of X. By way of contradiction, suppose that (C, D) is a separation of A. By the lemma above, each A_n will be contained in exactly one of C or D. Without loss of generality, assume that $A_1 \subseteq C$. Using that $A_2 \cap A_1$ is non-empty, and A_2 is contained within exactly one of C or D, it follows that $A_2 \subseteq C$. Similarly, $A_3 \subseteq C$. In fact, repeating this argument inductively shows that $A_n \subseteq C$, for every $n \in \mathbb{N}$. Thus, $C \subseteq A = \bigcup_{n \in \mathbb{N}} A_n \subseteq C$ whence $D = \varnothing$.

Example 2.3. Let X be a topological space and A a connected subspace of X. Suppose that $\{A_\alpha\}_{\alpha\in I}$ is an indexed family of connected subspaces of X such that $A \cap A_\alpha$ is non-empty, for each index α. We claim that

$$Y := A \cup \bigcup_{\alpha\in I} A_\alpha$$

is a connected subspace of X. To see this, suppose that (C, D) is a separation of Y. Without loss of generality, we may again assume that $A \subseteq C$. Similarly, every A_α will be contained in either C or D. However, because C and D are disjoint and $A \cap A_\alpha \neq \emptyset$, the only option is to have $A_\alpha \subseteq C$ for every $\alpha \in I$. With this, we see that $Y \subseteq C$ and thus Y must be connected.

These examples give way to the following powerful theorem, which tells that the union of connected spaces is always connected provided they are "close enough".

Theorem 2.4. *Let $(X, \mathfrak{T})$ be a topological space and let $\{Y_\alpha\}_{\alpha\in I}$ be an indexed family of connected subspaces of X. If $\bigcap_{\alpha\in I} Y_\alpha$ is non-empty, then $\bigcup_{\alpha\in I} Y_\alpha$ is connected.*

Proof. The claim amounts to proving that $\bigcup_{\alpha\in I} Y_\alpha$ does not admit a separation. By way of contradiction, assume we can find two non-empty disjoint open sets A and B whose union is $\bigcup_{\alpha\in I} Y_\alpha$. Let p be a point belonging to $\bigcap_{\alpha\in I} Y_\alpha$. Without loss of generality, assume that $p \in A$. Since A and B are disjoint, $p \notin B$. Since every Y_α is connected, it must lie entirely within one of A or B, by the previous lemma. As $p \in A$, we get that $Y_\alpha \subseteq A$ for each $\alpha \in I$. That is, $\bigcup_{\alpha\in I} Y_\alpha \subseteq A$. This leaves us with $B = \emptyset$, which is a contradiction. □

Theorem 2.5. *Let $(X, \mathfrak{T})$ be a topological space and assume $A \subseteq X$ is a connected subspace. If B is a subspace of X with $A \subseteq B \subseteq \mathrm{Cl}(A)$, then B is also connected.*

Proof. We argue by contradiction. Let (C, D) be a separation of the subspace B in X. By Lemma 2.3, since A is a connected subspace of B, we get that $A \subseteq C$ or $A \subseteq D$. Without loss of generality, assume that $A \subseteq C$. Taking closures yields

$$\mathrm{Cl}(A) \subseteq \mathrm{Cl}(C) \subseteq \mathrm{Cl}(B) \subseteq \mathrm{Cl}(A)$$

whence $\mathrm{Cl}(A) = \mathrm{Cl}(C)$. It follows that $B \subseteq \mathrm{Cl}(C)$. On the other hand, Proposition 2.2 gives that D does not contain any limit points of C. From this, we conclude that $\mathrm{Cl}(C) \cap D = \emptyset$. In particular,

$$D = B \cap D \subseteq \mathrm{Cl}(C) \cap D = \emptyset.$$

This contradicts the fact that (C, D) was a separation of the subspace B. □

Theorem 2.6. *Let* $(X, \mathfrak{T})$ *and* $(Y, \mathfrak{W})$ *be topological spaces with* X *connected. If* $f : X \to Y$ *is a continuous map, then* $f(X)$ *is a connected subspace of* Y.

Proof. First, recall that f is continuous when viewed as a function $X \to f(X)$.

Let A and B be two disjoint open sets in $f(X)$ whose union is $f(X)$. By continuity, $f^{-1}(A)$ and $f^{-1}(B)$ are open subsets of X. Moreover,

$$X = f^{-1}(A) \cup f^{-1}(B) \quad \text{and} \quad f^{-1}(A) \cap f^{-1}(B) = \varnothing.$$

Since X is connected, we may assume that $f^{-1}(A)$ is empty. This implies that A is empty as it is a subset of $f(X)$. □

We conclude this section with a final result involving finite products of connected spaces.

Theorem 2.7. *Let* $X_1, \ldots, X_n$ *be connected topological spaces and let* X *denote the product* $X_1 \times \cdots \times X_n$ *with the product/box topology. Then,* X *is connected.*

Proof. By induction, it suffices to show that $X_1 \times X_2$ is connected whenever $X_{1,2}$ are themselves connected. To this end, let us first fix a point $(a, b) \in X_1 \times X_2$. The reader may easily verify that $X_1 \times \{b\}$ is homeomorphic to X. Similarly, for every $x \in X_1$ it can be shown that $\{x\} \times X_2 \cong X_2$. Thus, by the previous theorem, both $X_1 \times \{b\}$ and $\{x\} \times X_2$ will be connected, for each $x \in X_1$. Noticing that

$$(x, b) \in (X_1 \times \{b\}) \cap (\{x\} \times X_2),$$

we apply Theorem 2.4 to deduce that the "slice"

$$\Gamma_x := (X_1 \times \{b\}) \cup (\{x\} \times X_2)$$

is also connected. Finally, since $(a, b) \in \bigcap_{x \in X_1} \Gamma_x$, this same theorem implies that $\bigcup_{x \in X_1} \Gamma_x = X_1 \times X_2$ is connected. □

2.1.2 *Arbitrary Products of Connected Spaces*

Actually, the result given at the end of the previous subsection may be strengthened to show that arbitrary products of connected spaces are again connected (when given the product topology). To see this, we first start with an indexed family $\{X_\alpha\}_{\alpha \in I}$ of connected spaces and we define

$$X := \prod_{\alpha \in I} X_\alpha.$$

As a first step, let us fix a point $\mathbf{a} = (a_\alpha)_{\alpha \in I}$ in X. Keeping with the setup described above, we give the first of three lemmas.

Lemma 2.8. *If K is a finite subset of I, we define X_K to be the subspace of X consisting of all points $(x_\alpha)_{\alpha\in I} \in X$ such that $x_\alpha = a_\alpha$ for all $\alpha \notin K$. For any finite set $K \subseteq I$, the subspace X_K is connected.*

Proof. It is easily seen that X_K is homeomorphic to the space $\prod_{\alpha\in K} X_\alpha$. Since every X_α is connected, it readily follows from the previous theorem that the finite product $\prod_{\alpha\in K} X_\alpha$, and hence X_K, is connected. □

Lemma 2.9. *Keeping in line with the notation of the lemma above, the subspace $Y := \bigcup_K X_K$, where the union is taken over all finite subsets K of I, is connected.*

Proof. Since every X_K is connected (by the last lemma), it is enough to show that $\bigcap_K X_K$ is non-empty. Indeed, $\bigcap_K X_K$ contains the point $\mathbf{a}$. □

Viewing Y as a subspace of X, we will show that $\mathrm{Cl}\,(Y) = X$. To this end, let $(x_\alpha)_{\alpha\in I}$ be a point in X and choose a basis element $\prod_{\alpha\in I} U_\alpha$ for the product topology on X. Then, there exists a finite set $K \subseteq I$ such that $U_\alpha = X_\alpha$ for all $\alpha \notin K$. Then, $\prod_{\alpha\in I} U_\alpha$ has non-empty intersection with this X_K. In particular, $\prod_{\alpha\in I} U_\alpha$ intersects $\bigcup_K X_K$ whence the claim follows. Since the closure of a connected set is again connected, we conclude that X is connected. To summarize, we have the following theorem.

Theorem 2.10. *Let $\{X_\alpha\}_{\alpha\in I}$ be an indexed family of connected spaces. Then, the product space $\prod_{\alpha\in I} X_\alpha$ is connected when it is given the product topology.*

2.2 Connected Components and Local Connectedness

We recall some notions that are best taught in an analytic topology course. First, let L be a *simply ordered set* with order $\leq$. If L has more than a single element, then L is called a *linear continuum* provided each of the following hold true:

(1) L has the least upper bound property;

(2) If $x < y$, i.e. $x \leq y$ and $x \neq y$, there exists $z \in L$ with $x < z < y$.

Definition 2.2. Let $(X, <)$ be a simply ordered set with more than a single element. Let $\mathcal{B}$ be the collection of all the following:

(1) Intervals of the form $(a, b) := \{x \in X : a < x < b\}$, for $a, b \in X$ with $a < b$;

(2) Intervals of the form $[x_0, b) := \{x \in X : x < b\}$ **if** there exists an element x_0 of X such that $x_0 \leq x$ for all $x \in X$;

(3) Intervals of the form $(a, x_1] := \{x \in X : a < x\}$ **if** there exists an element $x_1 \in X$ such that $x \leq x_1$ for all $x \in X$.

Then, $\mathcal{B}$ is a basis for a topology on X and the topology it generates is called the *order topology* on X.

Currently, we must make sense of the above. That is, we check that $\mathcal{B}$ is indeed a basis on X. Note that every element of X lives in some member of $\mathcal{B}$: a least element belongs to a set looking like (2), a greatest element lives in some interval appearing in (3), and every other point of X is contained in some open interval (a, b) in (1). It can be manually verified that non-empty intersections of elements in $\mathcal{B}$ again live in $\mathcal{B}$. Checking this ensures that the order topology on $(X, <)$ is well defined.

Remark 2.1. Notice that we have only defined the order topology for simply ordered sets having at least two elements. When we speak of spaces having the order topology, it will henceforth be assumed that the underlying set has more than a single element.

Example 2.4. The order topology on $\mathbb{R}$ is the usual (Euclidean) topology on $\mathbb{R}$.

One can easily make sense of "open rays" of the form (a, ∞), $(-\infty, a)$, "closed rays" $[a, \infty)$, $(-\infty, a]$, and "closed intervals" of the form $[a, b]$. We leave it as an exercise to check that "open rays" are open and "closed intervals/rays" are closed in the order topology.

Example 2.5. A linear continuum L is connected in the order topology. So are the intervals and rays. The proof is choppy and hence omitted. We should note that, in particular, intervals and rays in $\mathbb{R}$ are connected (and so is the entire space).

Given our lengthy setup and familiar terminology, it isn't too surprising that the following generalization of a well known theorem holds true.

Theorem 2.11 (Intermediate Value Theorem). *Let $(X, \mathfrak{T})$ be a connected space and let Y be a simply ordered set with the order topology. Suppose $f : X \to Y$ is a continuous function. Let $a, b \in X$ and let $r \in Y$ be a point satisfying*

$$f(a) < r < f(b) \quad \text{or} \quad f(b) < r < f(a),$$

there exists $c \in X$ such that $f(c) = r$.

Proof. Consider the sets

$$A := f(X) \cap (-\infty, r) \quad \text{and} \quad f(X) \cap (r, \infty)$$

which are disjoint and open in the subspace $f(X)$, of Y. Moreover, they are both non-empty (one contains $f(a)$ whilst the other includes $f(b)$). Also, since X is connected and f is continuous, the image $f(X)$ is a connected subspace of Y. Arguing by contradiction, suppose that $f(x) \neq r$ for all $x \in X$. Then, $f(X) = A \sqcup B$; which contradicts the connectedness of $f(X)$. □

A stronger variant of connectivity exists.

Definition 2.3. Let $(X, \mathfrak{T})$ be a topological space. A *path* between two points x_0 and x_1 of X is a continuous function $f : [a, b] \subset \mathbb{R} \to X$ such that $f(a) = x_0$ and $f(b) = x_1$. Two points x_0 and x_1 in X are said to be *path connected* if there exists a path between x_0 and x_1. The space $(X, \mathfrak{T})$ is called *path connected* if all points in X are path connected.

First, we show that any path connected space is necessarily connected. To this end, suppose that $(X, \mathfrak{T})$ is path connected and let (A, B) be a separation of X. Choose $x_0 \in A$ and $x_1 \in B$. Since X is path connected, there exists a continuous function

$$\gamma : [a, b] \subset \mathbb{R} \to X, \quad \gamma(a) = x_0 \quad \text{and} \quad \gamma(b) = x_1.$$

Now, $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ will be non-empty disjoint open subsets of $[a, b]$. However, $\gamma^{-1}(A) \cup \gamma^{-1}(B) = \gamma^{-1}(X)$ will be the entirety of $[a, b]$. Since this interval is connected, we have attained a contradiction.

Remark 2.2. Any closed and bounded interval $[a, b]$ is homeomorphic to $[0, 1]$; to see this one needs only to consider the function

$$f : [0, 1] \to [a, b], \quad f(x) := (b - a)x + a.$$

Consequently, we may always assume that a path in a space X is given by a continuous function $f : [0, 1] \to X$. We will always make this assumption in the second part of the text.

2.2.1 Components

In this subsection, we fix a topological space $(X, \mathfrak{T})$. Given $x, y \in X$, we will write $x \sim y$ if there exists a connected subspace of X containing both x and y.

We now show that '$\sim$' is an equivalence relation on X. First, observe that $x \sim x$ since $\{x\}$ is always a connected subspace of X. It is clear that $x \sim y$ if and only if $y \sim x$. Suppose that $x \sim y$ and that $y \sim z$. Let U_1 and U_2 be connected subspaces of X containing x, y and y, z, respectively. Since $y \in U_1 \cap U_2$, we see that $U_1 \cup U_2$ is a connected subspace of X. Realizing that $x, z \in U_1 \cup U_2$, we get $x \sim z$.

Definition 2.4. An element of $X/\sim$ is called a *component* of the space X. Under the relation $\sim$, we denote the equivalence class of a point x by $[x]$. This set $[x]$ is called the component (or connected component) of x.

It is known from elementary set theory that $X/\sim$ forms a partition of the space X. We may further characterize the components of X in the following way.

Theorem 2.12. *The components of X are connected disjoint subspaces, whose union is the entire space X, such that every non-empty connected subspace of X intersects exactly one of them.*

Proof. Since $X/\sim$ forms a partition of X, the second statement is clear. Let U be a connected subspace of X. Since the components form a partition of the space X, U will intersect some component, say, $[x]$. Let then $t \in U$ be such that $t \sim x$. If $y \in U$, then $y \sim t$ so that $y \sim x$, by transitivity. We get that $U \subseteq [x]$. Since these equivalence classes are disjoint, the final point holds.

Let C be a component of X and give it the subspace topology. Fix a point x_0 of C. For every $x \in C$, we have $x \sim x_0$ by definition. Thus, there exists a connected subspace U_x of X containing x_0 and x. Since U_x intersects C, the result we have just proven gives $U_x \subseteq C$. It follows that

$$C = \bigcup_{x \in C} U_x.$$

Noticing that $x_0 \in \bigcap_{x \in C} U_x$, we see that C is a connected subspace of X by virtue of Theorem 2.4. ◻

In general, the components also enjoy nice topological properties. Let C be a component of a space X. Invoking Theorem 2.5, we see that $\mathrm{Cl}(C)$ is also a connected subspace of X. Now, $\mathrm{Cl}(C)$ will intersect the component C, and hence we must have $\mathrm{Cl}(C) \subseteq C$ by the theorem above. We therefore conclude the following.

Corollary 2.13. *The components of X are disjoint connected closed subspaces of X whose union is the entire space. If U is a connected subspace of X, there exists a unique component C containing U.*

2.2.2 Path Components

We continue to work in an arbitrary topological space $(X, \mathfrak{T})$. Let us momentarily forget about the connected components we treated in the previous subsection. For $x, y \in X$, we will write $x \sim y$ if they are *path connected*, i.e. if there exists a path from x to y in X. Let us now show that '$\sim$' is an equivalence relation on X. It is clear that $x \sim x$ for all $x \in X$, for the constant function $f : [0,1] \to X$ given by $f(t) \equiv x$ is continuous. If $x \sim y$, there exists by definition a continuous function

$$f : [0,1] \subset \mathbb{R} \to X, \quad f(0) = x \quad \text{and} \quad f(1) = y.$$

Then, the function

$$g : [0,1] \subset \mathbb{R} \to X, \quad g(t) := f(1-t)$$

is a path from y to x. It follows that $y \sim x$. Finally, assume $x \sim y$ and that $y \sim z$. We may pick two continuous maps

$$f, g : [0,1] \to X, \quad f(0) = x,\ f(1) = y, \quad g(0) = y,\ g(1) = z.$$

Then, the function

$$h(t) := \begin{cases} f(2t), & 0 \le t \le \frac{1}{2}, \\ g(2t-1), & \frac{1}{2} \le t \le 1 \end{cases} \tag{2.1}$$

is well defined and continuous (by the *pasting lemma*). Since $h(0) = f(0) = x$ and $h(1) = g(1) = z$, we conclude that $x \sim z$. This shows that '$\sim$' is an equivalence relation on X.

Definition 2.5. An element of $X/\sim$ is called a *path component* of X. The path components of X form a *partition* of X.

One can in general say less about path components than simple components. In fact, the following proposition is where we will draw the line.

Proposition 2.14. *The path components of a space X are path connected disjoint subspaces of X whose union is the entire space. Moreover, if U is a path connected subspace of X, then U will be contained in exactly one path component of X.*

Proof. Since the path components form a partition of X, we know they are disjoint and that their union is X. Let now U be a path connected subspace of X and suppose U intersects some path component C. Let $x \in U \cap C$. Then, $C = [x]$

as is easy to check[1]. If $u \in U$, then x and u are connected by a path so that $u \sim x$. This gives $U \subseteq [x] = C$.

For the second part, let C be a path component of X. Fix a representative x of the class C, i.e. $C = [x]$. If $y, z \in C$ then we can choose paths γ_1 and γ_2 connecting y to x, and x to z. It is easy to see that every point along $\gamma_{1,2}$ will be connected to x, as restrictions of continuous maps are continuous. Gluing these two paths together as in (2.1), we obtain a path connecting y and z that is contained within the component C. □

2.3 Locally Connected Spaces

In the previous sections we introduced connected and path connected spaces. Often times, it is more realistic to work in spaces that satisfy a weaker condition known as *local connectedness*.

Definition 2.6 (Locally Connected Space)**.** A topological space $(X, \mathfrak{T})$ is said to be *locally connected* if for every point $x \in X$ and every neighbourhood U of x, there is a connected neighbourhood V of x contained within U.

We give a similar definition for locally path connected spaces.

Definition 2.7. A space X is called *locally path connected* if for every point $x \in X$ and every neighbourhood U of x, there is a path connected neighbourhood V of x contained within U.

A priori, it is not clear that connected spaces are themselves locally connected. This is for good reason, as not every connected space is locally connected. One can consult [MNKS] for counter examples, i.e. connected spaces that are not locally connected and locally connected spaces that are not themselves connected.

Let us now offer a nice and useful characterization of the spaces which are locally connected.

Theorem 2.15. *Let $(X, \mathfrak{T})$ be a topological space. The following statements are equivalent:*

(1) X is locally connected.

[1] Here we momentarily abusing notation. For this proof, we let $[x]$ denote the equivalence class of x under the path connected relation '$\sim$'.

(2) *For every open set $U \subseteq X$, the components of U are open in X.*

Proof. First suppose that X is locally connected and consider an open subset U of X. Fix now a component C of U. If $x \in C$, then we can choose a connected subspace V of X containing x having the property that $V \subseteq U$. Actually, we note that $V \cap U = V$ is a connected subspace of U. Since V intersects C, Theorem 2.12 implies that $V \subseteq C$. It follows that every component C is open in X.

Conversely, we prove that X is locally connected. Let $x \in X$ be given and choose a neighbourhood U of x. Let C be the component of U containing x. Then, C is a connected subspace of U, and hence of X, by virtue of Corollary 2.13. By hypothesis, C is open in X and thus is the connected neighbourhood we seek. □

A similar proof yields the following analogous theorem.

Theorem 2.16. *Let $(X, \mathfrak{T})$ be a topological space. The following statements are equivalent.*

(1) *X is locally path connected.*

(2) *For every open subset U of X, the path components of U are open in X.*

2.4 Compactness

The concept of a compact space is an elusive one. Despite being the property behind many fundamental analytic properties (e.g. the uniform continuity theorem), it was centuries before mathematicians could formulate compactness in its most general and applicable form. Despite its analytic origins, compactness is truly a topological property that makes a topological space "almost finite dimensional". Nonetheless, compactness finds itself at the heart of many problems in mathematical analysis.

Definition 2.8. Let $(X, \mathfrak{T})$ be a topological space and let K be a subset of X. A covering of K is a family of subsets of X whose union contains K. An *open cover* of K is a family $\mathcal{U}$ of open sets in X such that $K \subseteq \bigcup_{U \in \mathcal{U}} U$.

We now come to the definition of compactness.

Definition 2.9. Let $(X, \mathfrak{T})$ be a topological space and K a subset of X. The set K is called *compact* in X if for every open cover $\mathcal{U}$ of K by open subsets of X, there exists a finite sub-collection of $\mathcal{U}$ whose union contains K. When the context is understood, we will simply say that K is compact.

A topological space $(X, \mathfrak{T})$ is called compact if X is compact as a subset of X. In this case, we would simply say that X is a compact topological space (or subspace). Equivalently, we have the following.

Definition 2.10. Let $(X, \mathfrak{T})$ be a topological space. We say that X is a compact space if for every open cover $\mathcal{U}$ of X, by elements of $\mathfrak{T}$, there exists a finite sub-collection of $\mathcal{U}$ whose union is equal to X.

A priori, this causes some confusion. Must we now carefully distinguish between compact sets and compact subspaces? The following proposition answers this annoying question.

Theorem 2.17. *Let $(X, \mathfrak{T})$ be a topological space and Y a subspace of X. Then Y is a compact space if and only if Y is compact (as a set) in X.*

Proof. Let Y be a compact space, with the subspace topology. We claim that Y is compact in X. To this end, let $\mathcal{U}$ be an open covering of Y by elements of $\mathfrak{T}$. For every element $U \in \mathcal{U}$, the set $U \cap Y$ will be open in Y. Since $\mathcal{U}$ is a cover of Y, the collection

$$\{U \cap Y : U \in \mathcal{U}\}$$

is an open covering of Y, *in* Y. Since Y is compact as a space, we can extract a finite collection, say, $\{U_1 \cap Y, \ldots, U_n \cap Y\}$ whose union is equal to Y. But then,

$$Y \subseteq U_1 \cup \cdots \cup U_n$$

where $U_j \in \mathcal{U}$. We have therefore found a finite sub-collection of $\mathcal{U}$ covering Y. As $\mathcal{U}$ was arbitrary, we see that Y is compact in X. Conversely, suppose that Y is compact in X and let $\mathcal{U}$ be an open cover of Y, by elements open in Y. Every element $U \in \mathcal{U}$ can be represented as

$$U = V \cap Y, \quad V \text{ open in } X.$$

The collection of all such V forms an open covering of Y viewed as a subset of X. By compactness, we can find a sub-collection of these V, say, $V_1, \ldots, V_m$ whose union *contains* Y. For each $j = 1, \ldots, m$ let

$$U_j := V_j \cap Y$$

so that $U_j \in \mathcal{U}$. It is then easy to see that $Y = \bigcup_1^m U_j$. As $\mathcal{U}$ was arbitrary, this means that Y is a compact space which concludes the proof. □

As in $\mathbb{R}^n$, several nice properties continue to hold regarding subsets of compact spaces. We illustrate a few of these below.

Theorem 2.18. *Let $(X, \mathfrak{T})$ be a compact topological space. If $F \subseteq X$ is closed, then F is compact in X. Hence, F is a compact subspace of X.*

Proof. Let $\mathcal{U}$ be an open covering of F by subsets of X. Since F^{c} is open in X, the family $\mathcal{U} \cup \{F^{\mathsf{c}}\}$ is itself an open covering of X, by subsets of X. Using that $(X, \mathfrak{T})$ is compact, we may extract a finite sub-collection

$$U_1, \ldots, U_n, F^{\mathsf{c}}, \quad \text{with } U_j \in \mathcal{U},$$

such that $F^{\mathsf{c}} \cup \bigcup_1^n U_n$ contains X, and hence F. Since F and F^{c} are disjoint, we get that $F \subseteq \bigcup_1^n U_j$. This means that we have found a finite sub-collection of $\mathcal{U}$ that covers F. Since $\mathcal{U}$ was arbitrary, we conclude that F is compact in X. □

Theorem 2.19. *Let $(X, \mathfrak{T})$ be a Hausdorff topological space. If $K \subseteq X$ is compact, then K is closed in X.*

Proof. We will prove that $K^{\mathsf{c}} = X \setminus K$ is open in X. To this end, let us first fix a point $x_0 \in K^{\mathsf{c}}$. For each $y \in K$, we may choose disjoint open sets $U_y \ni y$ and $V_y \ni x_0$ in X. Notice then that the collection

$$\mathcal{U} := \left\{U_y : y \in K\right\}$$

is an open cover of K in X. By compactness, may choose finitely many points $y_1, \ldots, y_n$ in K so that $K \subseteq \bigcup_1^n U_{y_j}$. Now, the set

$$V_{x_0} := V_{y_1} \cap \cdots \cap V_{y_n}$$

is open in X and contains x_0. By construction, V_{x_0} does not intersect $\bigcup_1^n U_j$. Since $\bigcup_1^n U_j \supseteq K$, we have found an open set V containing x_0 with the property thats $V_{x_0} \subseteq K^{\mathsf{c}}$. But then, $K^{\mathsf{c}} = \bigcup_{x_0 \in K^{\mathsf{c}}} V_{x_0}$ whence K^{c} is open. □

We now prove a result that is a significant generalization of the "nested interval theorem" familiar from real analysis.

Theorem 2.20. *Let $(X, \mathfrak{T})$ be a compact topological space and let $\mathcal{C}$ be a collection of closed subsets of X such that $\bigcap_1^n C_n \neq \varnothing$ for all finite sub-collections $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$. Then, $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$.*

Proof. We argue by contradiction. Suppose that $\bigcap_{C\in\mathcal{C}} C$ is empty. Then, its complement is the entire space X, i.e. $\bigcup_{C\in\mathcal{C}} C^{c} = X$. But then, the collection of complements in $\mathcal{C}$ is an open covering of the compact space X. One can therefore extract a finite sub-collection $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$ such that $X = \bigcup_1^n C_j^{c}$. This implies that $\bigcap_1^n C_j$ is empty, which contradicts our assumption. □

First, notice that the theorem above continues to hold if we are merely working in a compact subspace of a general space. Indeed, let $(X, \mathfrak{T})$ a topological space and $K \subseteq X$ compact. If $C \subseteq K$ is closed in X then $X \setminus C$ is open in X whence $(X \setminus C) \cap K$ is open in K. Then,

$$K \setminus [(X \setminus C) \cap K] = K \setminus (X \setminus C) = C$$

is closed in K. Second, the theorem we have just proven admits a converse. Stating this converse elegantly requires a definition, which we now give.

Definition 2.11. A topological space $(X, \mathfrak{T})$ is said to have the *finite intersection property* if every family $\mathcal{C}$ of closed subsets of X having the property that

$$C_1 \cap C_2 \cap \cdots \cap C_n \neq \varnothing,$$

for all finite sub-collections $\{C_1, \ldots, C_n\} \subseteq \mathcal{C}$, also satisfies $\bigcap_{C\in\mathcal{C}} C \neq \varnothing$.

We have just shown that compact spaces have this property. We now prove that all topological spaces having this property are, in fact, compact.

Theorem 2.21. *Let $(X, \mathfrak{T})$ be a topological space having the finite intersection property. Then, $(X, \mathfrak{T})$ is compact.*

Proof. We shall prove the contrapositive. If $(X, \mathfrak{T})$ is not compact, then one can find an open covering $\mathcal{U}$ of X having no finite sub-covering of X. Consider now the set

$$\mathcal{C} := \{X \setminus U : U \in \mathcal{U}\}$$

which is a family of closed subsets in X. Since no finite sub-collection of $\mathcal{U}$ can cover X, finite intersections of elements in $\mathcal{C}$ are non-empty. However, $\bigcup_{U\in\mathcal{U}} U = X$ which means that $\bigcap_{C\in\mathcal{C}} C$ is empty. Hence, $(X, \mathfrak{T})$ does not have the finite intersection property. □

2.4.1 Continuous Functions and Compacta

In this subsection, we explore how compact sets behave under the action of continuous functions. Over the real or complex numbers, we know that a continuous function over a closed and bounded set will be bounded. As we shall see, this continues to be true in general for compact sets.

Theorem 2.22. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces. Let $K \subseteq X$ be compact and suppose that $f : K \to Y$ is continuous. Then, the image $f(K)$ is compact in Y.*

Proof. First, as $K \subseteq X$ is compact, K is a compact space when given the subspace topology. Let now $\mathcal{U}$ be an open cover of $f(K)$ in Y, by open subsets of Y. For every $U \in \mathcal{U}$, by continuity of f, the pre-image $f^{-1}(U)$ is an open subset of K. Therefore, the collection

$$\left\{f^{-1}(U)\right\}_{U\in\mathcal{U}}$$

is an open cover of the compact space K. Since K is compact, we can extract finitely many $U_1, \ldots, U_n$ from $\mathcal{U}$ so that $\bigcup_1^n f^{-1}(U_j) = K$. Elementary set theory then gives

$$f(K) \subseteq \bigcup_{j=1}^{n} U_j.$$

Thus, we have found a finite sub-collection of $\mathcal{U}$ covering $f(K)$. Since $\mathcal{U}$ was arbitrary, we conclude that $f(K)$ is compact in Y. □

In any decent real analysis stream, it will be proven that compact subsets of $\mathbb{R}$ or $\mathbb{C}$ are bounded. Actually, it is very easy to show that this is the case for a general metric space. Let (X, d) be a metric space and $K \subseteq X$ compact. Fix $x_0 \in X$ and consider the collection

$$\mathcal{U} := \{B(x_0, n) : n \in \mathbb{N}\}$$

which is obviously an open covering of K in X. Since K is compact, we may contain K within a finite collection of balls

$$K \subseteq \bigcup_{n=1}^{N} B(x_0, n) \subseteq B(x_0, N)$$

whence K is bounded. Combining this with the previous theorem yields the following well known result.

Corollary 2.23. *Let $(X, \mathfrak{T})$ be a compact topological space and let $f : X \to \mathbb{C}$ be continuous. Then f is bounded.*

Proposition 2.24. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces. Assume further that X is compact and Y is Hausdorff. Any continuous function $X \to Y$ is closed.*

Proof. Let f be a continuous function $X \to Y$ and fix a closed subset A of X. Since X is compact, A is itself compact. By continuity, the image $f(A)$ is compact in Y. Using that Y Hausdorff, we see that $f(A)$ is closed in Y. □

A Homeomorphism Criterion

This short part of the text is devoted to the proof of a very surprising result which can often be used to prove a bijective function is, in fact, a homeomorphism (i.e. isomorphism of topological spaces).

Theorem 2.25. *Let $(X, \mathfrak{T})$ be a compact topological space and $(Y, \mathfrak{W})$ a Hausdorff topological space. Let $f : X \to Y$ be a bijective continuous function. Then, f is a homeomorphism.*

Proof. The only non-trivial property to prove is the continuity of f^{-1}. By Theorem 1.21, this is equivalent to showing that f maps closed sets to closed sets (that is, f is a closed map). Let $A \subseteq X$ be a closed set. Theorem 2.22 states that $f(A)$ is compact in Y. Since Y is Hausdorff, $f(A)$ is closed in Y and the proof is complete. □

2.5 Locally Compact Spaces

Topological spaces that are locally compact frequently appear in mathematical analysis. In fact, familiar vector spaces such that $\mathbb{R}^m$ and $\mathbb{C}^n$ are locally compact (in fact, they are the prototypes). Studying these spaces is a must, especially since beasts such as *locally compact Hausdorff spaces* appear frequently in measure theory and in functional analysis. Moreover, they are also of algebraic interest.

Definition 2.12. Let $(X, \mathfrak{T})$ be a topological space. We will say that X is *locally compact* if for every point $x \in X$, there exist a compact set C in X containing a neighbourhood of x.

A very simple example comes to mind.

Example 2.6. Any compact space is locally compact.

As mentioned above, we will be greatly interested in topological spaces that are both Hausdorff and locally compact. These are called *locally compact Hausdorff spaces* and are often abbreviated by 'LCH spaces'. We will begin our analysis of such spaces with the following theorem.

Theorem 2.26. *Let $(X, \mathfrak{T})$ be a topological space. Then X is LCH if and only if there exists a topological space $(Y, \mathfrak{W})$ having the following properties:*

(1) Y is a compact Hausdorff space containing X as a subspace;

(2) $Y \setminus X$ is a singleton.

In this case, if Y and Y' both satisfy the above criteria, there exists a homeomorphism

$$\psi : Y \longrightarrow Y', \quad \psi\,|_X \equiv \mathbf{1}_X.$$

We shall not prove this theorem, as the proof is quite involved and technical. The result is Theorem 29.1 in [MNKS] and the curious reader may consult the proof given there. Despite this, there are some important aspects to the proof of this theorem, which we now outline.

Surprisingly enough, the proof of Theorem 2.26 is for the most part constructive; we have an explicit description of the topology on Y in terms of $\mathfrak{T}$. If we denote by $\mathfrak{Y}$ the topology on Y, then the proof in [MNKS] tells us that

$$\mathfrak{Y} := \{U \text{ open in } X\} \cup \{Y \setminus K : K \text{ compact in } X\}. \tag{2.2}$$

In light of the theorem above, we give the subsequent definition.

Definition 2.13. Let $(Y, \mathfrak{W})$ be a compact Hausdorff space and X a proper subspace of Y. If $\mathrm{Cl}(X) = Y$, where this closure is taken in Y, we say that Y is a *compactification* of the space X. If $Y \setminus X$ is a singleton, then Y is called the *one-point compactification* of X.

Theorem 2.26 then states that a topological space $(X, \mathfrak{T})$ has a one-point compactification if and only if X is a non-compact locally compact Hausdorff space. Indeed, if $(X, \mathfrak{T})$ has a one-point compactification Y then the theorem gives that X is LCH. However, one also has $\mathrm{Cl}(X) = Y \neq X$ whence X is not closed. In particular, X cannot be compact since a compact subset of a Hausdorff space is closed. Conversely, suppose that X is a non-compact LCH space. By Theorem

2.26, we can view X as a subspace of a unique compact Hausdorff space Y such that $Y \setminus X$ is a singleton. Since X is not compact in Y, it is not closed in Y. This means that $\mathrm{Cl}(X) \neq X$ whence $\mathrm{Cl}(X) = Y$. That is, X has a one-point compactification. In this case, the uniqueness property of the theorem certainly allows us to speak of *the* one-point compactification.

2.6 Compactness in $\mathbb{R}$

On the real line one has a very nice classification of compact sets. This is often called the "Heine-Borel theorem" which states that a set $K \subseteq \mathbb{R}$ is compact if and only if K is both closed and bounded. This coincides with our intuition, and is not too surprising. After all, continuous functions on closed and bounded sets enjoy regularity properties that continuous functions on unbounded or open sets do not (e.g. boundedness, uniform continuity, etc.). This correspondence continues to hold over $\mathbb{R}^m$ and $\mathbb{C}^n$ for $n, m \geq 1$, but we will only prove it for $\mathbb{R}$ as the proof for higher dimensional spaces belongs in an analysis course.

Summarizing the discussion above, we devote this section to the proof of the following theorem due to Heine and Borel.

Theorem 2.27 (Heine-Borel)**.** *Endow $\mathbb{R}$ with the usual topology. A set $K \subseteq \mathbb{R}$ is compact if and only if K is both closed and bounded.*

We will prove this result by way of two lemmas. The first, establishes that compact subsets of $\mathbb{R}$ are both closed and bounded.

Lemma 2.28. *Consider $\mathbb{R}$ with the standard topology. If $K \subseteq \mathbb{R}$ is compact, then K is closed and bounded.*

Proof. Obviously, $\mathbb{R}$ is locally compact Hausdorff. In any case, this means that all compact sets are closed in $\mathbb{R}$. It therefore remains only to establish the boundedness of K. Consider the family of open intervals

$$\mathcal{U} := \{(-n, n) : n \in \mathbb{N}\}.$$

Obviously, $\bigcup_{n=1}^{\infty}(-n, n) = \mathbb{R}$ and so $\mathcal{U}$ is an open cover of the compact set K. We can therefore extract finitely many open intervals from our collection such that

$$K \subseteq \bigcup_{n=1}^{N}(-n, n) \subseteq (-N, N).$$

This implies that K is bounded with respect to the absolute value metric $|\cdot|$, as was required. ◻

We now prove the converse direction, which will be more analytic than topological. The truth of the matter is that our work up until here significantly shortened the proof of Lemma 2.28. Unfortunately, our topological advances will not help with the following:

Lemma 2.29. *Let $K \subset \mathbb{R}$ be closed and bounded. Then K is compact.*

Proof. We will argue by contradiction. Assume that K is not compact, then one can find an open cover $\mathcal{U}$ of K such that no finite sub-collection of $\mathcal{U}$ covers the set K. Since K is bounded, we can find $R > 0$ such that $K \subseteq [-R, R]$. Let I_1 denote this interval. We partition I_1 into two closed intervals

$$I_1' := [-R, 0] \quad \text{and} \quad I_1'' := [0, R].$$

Now, one of $K \cap I_1'$ or $K \cap I_1''$ is non-empty and not contained in any finite sub-collection of $\mathcal{U}$. Let I_2 be any of the I_1' or I_1'' for which this condition holds. We now repeat this procedure, partitioning I_2 into two intervals I_2' and I_2'' and picking an interval I_3 so that $K \cap I_3$ is non-empty and not covered by any finite sub-collection of $\mathcal{U}$. This grants us a sequence of nested closed and bounded intervals

$$I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

in $\mathbb{R}$. It is known from elementary real analysis that $\bigcap_1^\infty I_n$ is non-empty. Let ξ be chosen from this intersection. We now claim that $\xi \in K$. To see this, notice that $\xi \in I_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ pick x_n from $K \cap I_n$. Then,

$$|x_n - \xi| \leq \operatorname{diam}(I_n) \xrightarrow{n \to \infty} 0.$$

Since K is closed, we get that $\xi \in K$. Since $\mathcal{U}$ is an open covering of K, there exists $U_i \in \mathcal{U}$ such that $\xi \in U_i$. Since $\mathcal{U}_i$ is open, there exists $\delta > 0$ having the property that

$$B(\xi, \delta) \subseteq U_i.$$

Choose now $N \in \mathbb{N}$ such that $\operatorname{diam}(I_N) < \delta$. If $x \in K \cap I_N$, then

$$|x - \xi| < \delta$$

whence $K \cap I_N \subseteq U_i$ which contradicts our assumption. Thus, we see that K is compact. ◻

Combining Lemmas 2.28 and 2.29 proves the Heine-Borel Theorem. We now move on to an unrelated topic that will be of great use in the algebraic portion of the text.

2.7 Metrizability and Compactness

There are several different notions of compactness that do not (directly) involve open coverings of a space. Unfortunately, these are not always equivalent to the our definition of compactness (the most applicable and common definition). However, many of these *do* turn out to be equivalent if we restrict ourselves to topological spaces that are metrizable. In this section, we will focus on proving this fact. First, let us present two new notions of compactness.

Definition 2.14. Let $(X, \mathfrak{T})$ be a topological space.

(1) We say that X is *sequentially compact* if every sequence in X has a subsequence that converges in X.

(2) We call X *limit point compact* if every infinite subset of X has a limit point in X.

Ultimately, our goal in this section is to establish the following theorem.

Theorem 2.30 (Metrizable Compact Spaces Equivalence Theorem)**.** *Let* $(X, \mathfrak{T})$ *be a metrizable space. The following statements are equivalent.*

(1) X *is compact;*

(2) X *is limit point compact;*

(3) X *is sequentially compact.*

It turns out that the proof is far from straightforward. For this reason, we will have to present a new definition and several lemmas before even starting the proof. This begins with the so-called *Lebesgue number* property.

2.7.1 The Lebesgue Number Property

A metric space $(X, \mathfrak{T})$ is said to have the *Lebesgue number property* if, for every open cover $\mathcal{U}$ of X, there exists $\delta > 0$ such that each ball $B(x, \delta)$ in X is contained within some open set $U \in \mathcal{U}$. If such a δ exists, it is called a *Lebesgue number* for the cover $\mathcal{U}$.

Lemma 2.31. *Every compact metric space has the Lebesgue number property.*

Proof. Let X be a compact metric space and suppose that $\mathcal{U}$ is an open cover of X. For every $x \in X$ there exists $U_x \in \mathcal{U}$ containing the point x. Let $r_x > 0$ be such that $B(x, 2r_x) \subseteq U_x \in \mathcal{U}$. Then consider the family

$$\mathcal{G} := \{B(x, r_x) : x \in X\}.$$

Clearly, $\mathcal{G}$ is also an open cover for X. By compactness, we extract a finite set $\Sigma \subseteq X$ such that

$$X = \bigcup_{x \in \Sigma} B(x, r_x).$$

Define $\delta > 0$ to be the minimum of the r_x, where x ranges in Σ. Let $y \in X$ be given and consider the open ball $B(y, \delta)$. Clearly, $y \in B(x, r_x)$ for some $x \in \Sigma$. Then, $B(y, \delta) \subseteq B(y, r_x) \subseteq B(x, 2r_x)$ so that $B(y, \delta) \subseteq U_x \in \mathcal{U}$. □

Next, we check that the Lebesgue number property holds for metrizable spaces that are sequentially compact. The proof here is slightly less elegant, but nonetheless strategic.

Lemma 2.32. *Let $(X, \mathfrak{T})$ be a sequentially compact metric space. Then, X has the Lebesgue number property.*

Proof. Let $\mathcal{U}$ be an open covering of the space X. Assume, by way of contradiction, that for all $\delta > 0$ there exists a point $x \in X$ such that $B(x, \delta)$ is not contained within any element of $\mathcal{U}$. Thus, for every $n \in \mathbb{N}$, there exists $x_n \in X$ such that $B(x_n, 1/n)$ is not contained within any element of $\mathcal{U}$. This gives us a sequence $(x_n)_{n \in \mathbb{N}}$ in X. Choose a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ that converges to a point $x \in X$. Then, let $U \in \mathcal{U}$ be an open set which contains the point x. There exists $\varepsilon > 0$ so small that $B(x, \varepsilon) \subseteq U$. Since $x_{n_k} \to x$, we can find $K \in \mathbb{N}$ such that

$$d(x_{n_K}, x) < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{n_K} < \frac{\varepsilon}{2}.$$

But then,

$$B\left(x_{n_K}, \frac{1}{n_K}\right) \subseteq B(x,\varepsilon) \subseteq U;$$

which is a contradiction □

Before diving into the proof of Theorem 2.30, we only require one additional preliminary result.

Proposition 2.33. *Let (X,d) be a metric space and let A be an infinite subset of X. If x is a limit point of A, then every open ball $B(x,\varepsilon)$ contains infinitely many points in A different from x.*

Proof. Since x is a limit point of A, we know that $B(x,\varepsilon)$ must intersect A at a point other than x. Suppose that this intersection is finite, and choose distinct points $x_1,\ldots,x_m \in A$ (different from x) such that

$$B(x,\varepsilon) \cap (A \setminus \{x\}) = \{x_1,\ldots,x_k\}.$$

Then, let $\gamma > 0$ be the minimum of all the $d(x,x_j)$. Again, $B(x,\gamma)$ intersects A at a point other than x; choose such a point u. Then, $d(x,u) < \gamma$ whence $u \neq x_j$ for any $j = 1,\ldots,k$. But, $u \in B(x,\varepsilon)$. This contradiction shows that $B(x,\varepsilon)$ must intersect A at infinitely many points different from x. □

2.7.2 The Proof of Theorem 2.30

Let $(X,\mathfrak{T})$ be a metrizable space and let d be a metric on X which generates the topology $\mathfrak{T}$. Assume that X is compact; we will show that X is limit point compact. This amounts to showing that every infinite subset of X has a limit point in X. Equivalently, we can show that a set without any limit points must be finite. To this end, let $A \subseteq X$ be a set with no limit points. Vacuously, A contains all its limit points and must therefore be closed. Since X is compact, A is itself a compact space.

Fix $a \in A$ and note that there must exist a neighbourhood U_a of a that intersects A only at a. The collection $\{U_a\}_{a\in A}$ then forms an open cover of A in X. By compactness, we can extract a finite sub-cover $U_{a_1},\ldots,U_{a_n}$ for A. But then,

$$A = A \cap \bigcup_{k=1}^{n} U_{a_k} = \bigcup_{k=1}^{n} U_{a_k} \cap A = \bigcup_{k=1}^{n} \{a_k\}.$$

Thus, A must be finite. This proves the first implication.

Suppose now that X is limit point compact; we will show that X is sequentially compact. Let $(x_n)_{n\in\mathbb{N}}$ be a given sequence in X and consider the set

$$A := \{x_n : n \in \mathbb{N}\}.$$

If A is finite, then the sequence $(x_n)_{n\in\mathbb{N}}$ must have a constant subsequence which trivially converges in X. Otherwise, the set A is infinite and has a limit point $x \in X$. By definition, $x \in \mathrm{Cl}(A \setminus \{x\})$. For every $k \in \mathbb{N}$, the open ball $B(x, 1/k)$ intersects A at infinitely many points other than x. Choose any point x_{n_1} from $B(x, 1) \cap A$. Subsequently, for $k > 1$, choose $x_{n_k} \in B(x, 1/k) \cap A$ such that $n_k > n_{k-1}$. This gives us a subsequence that converges to x as $n_k \to \infty$.

We now show that sequential compactness implies compactness Let X be sequentially compact; in particular, it satisfies the Lebesgue number property. Let $\varepsilon > 0$ be given, we claim that there exist finitely many points $x_1, \ldots, x_m$ in X such that

$$X = \bigcup_{j=1}^{m} B(x_j, \varepsilon).$$

Suppose not, then for all $x_1, \ldots, x_n \in X$ there exists a point $x \notin \bigcup_{j=1}^{n} B(x_j, \varepsilon)$. Therefore, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $x_{n+1} \notin \bigcup_{j=1}^{n} B(x_j, \varepsilon)$ for all $n \in \mathbb{N}$. Since X is sequentially compact, we are free to extract a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to some $x \in X$. In particular, this subsequence is Cauchy. Thus, we can find $K \in \mathbb{N}$ so large that

$$d(x_{n_k}, x_{n_l}) < \varepsilon$$

whenever $n_k > n_l \geq K$. But this means that $x_{n_k} \in B(x_{n_l}, \varepsilon)$, which defies our construction. Thus, for every $\varepsilon > 0$, we can find finitely many open balls $B(x_1, \varepsilon), \ldots, B(x_n, \varepsilon)$ whose union is X.[2] Let now $\mathcal{U}$ be an open cover of X and choose a Lebesgue number $\delta > 0$ for this cover. By the above, we can find finitely many points $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{k=1}^{n} B(x_k, \delta)$. But, being a Lebesgue number, each $B(x_k, \delta)$ is contained within a single element of $\mathcal{U}$. Thus, $\mathcal{U}$ has a finite sub-covering of X. With this, the proof is complete.

[2] A metric space with this property is called *totally bounded*

2.8 The Quotient Topology

We have come to a topic that will be of considerable use within the algebraic and geometric parts of these notes. The concept of a quotient topology is not something one will have encountered in analysis or calculus as it is a purely topological notion. Despite this, it is rather easy to motivate the ideas. Loosely speaking, quotient topologies formalize the geometric concept of "gluing" two objects together.

Definition 2.15. Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces. A surjection

$$\rho : X \twoheadrightarrow Y$$

is called a *quotient map* if it satisfies the property that $p^{-1}(U)$ is open in X if and only if U is open in Y. Notice that a quotient is automatically continuous. Some mathematicians instead say that ρ is "strongly continuous".

A useful observation follows. Assume that $\rho : X \twoheadrightarrow Y$ is a surjective map. Then ρ is a quotient map if and only if, for all $B \subseteq Y$,

$$\rho^{-1}(B) \text{ is closed in } X \iff B \text{ is closed in } Y.$$

Indeed, this fact follows from the identity $\rho^{-1}(U^{\mathsf{c}}) = \rho^{-1}(U)^{\mathsf{c}}$.

Definition 2.16. Let X and Y be spaces with underlying topologies and consider a surjective function $\rho : X \twoheadrightarrow Y$. A subset $C \subseteq X$ is said to be *saturated* with respect to ρ if

$$\forall y \in Y,\ \rho^{-1}(\{y\}) \cap C \neq \varnothing \implies \rho^{-1}(\{y\}) \subseteq C.$$

Said differently, the set C is saturated with respect to ρ if it contains all the fibers it intersects.

The following characterization of quotient maps is sometimes useful when proving that certain functions are quotient maps.

Lemma 2.34. *Let X and Y be spaces and $\rho : X \twoheadrightarrow Y$ a surjective function. Then ρ is a quotient map if and only if the following hold true:*

(1) ρ is continuous;

(2) ρ maps saturated open subsets of X to open subsets of Y.

Proof. Suppose first that ρ is a quotient map. Let $U \subseteq X$ be open and saturated with respect to ρ and note that $U \subseteq \rho^{-1}(\rho(U))$. We claim that equality holds. Let $x \in \rho^{-1}(\rho(U))$ so that $y = \rho(x) \in \rho(U)$. Then, $\rho^{-1}(\{y\})$ intersects U, and is therefore contained in U. In particular, $x \in U$ whence $U = \rho^{-1}(\rho(U))$. Since U is open and ρ is a quotient map, we see that $\rho(U)$ is open in Y.

Conversely, we suppose ρ satisfies the criteria and let V be a subset of Y such that $\rho^{-1}(V)$ is open in X. Since $\rho(\rho^{-1}(V)) = V$, it suffices to show that $\rho^{-1}(V)$ is saturated with respect to ρ. Suppose $y \in Y$ is such that $\rho^{-1}(\{y\})$ intersects $\rho^{-1}(V)$. Let then $x \in \rho^{-1}(\{y\}) \cap \rho^{-1}(V)$. Obviously, $\rho(x) = y \in V$ so that $\rho^{-1}(\{y\}) \subseteq \rho^{-1}(V)$ by extension. The lemma follows. □

A general function between topological spaces is called an *open map* if it takes open sets to open sets. Likewise, this function is called a *closed map* if it maps closed sets to closed sets.

Definition 2.17. Let $(X, \mathfrak{T})$ be a topological space and A be *any set.* Suppose we are given a surjective function $\rho : X \twoheadrightarrow A$. There exists a unique topology $\mathfrak{A}$ on A that makes ρ into a quotient map. The set $\mathfrak{A}$ is then called the *quotient topology induced by* ρ.

We must make sure that the above is well defined. Naturally, we should let

$$\mathfrak{A} := \left\{ U \subseteq Y : \rho^{-1}(U) \in \mathfrak{T} \right\}.$$

If we can ensure that $\mathfrak{A}$ is a topology on A, then it will obviously be the unique topology making ρ a quotient map. Clearly, $\varnothing, A \in \mathfrak{A}$. If $\{U_\alpha\}_{\alpha \in I}$ is an indexed family of sets belonging to $\mathfrak{A}$, then

$$\rho^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_{\alpha \in I} \rho^{-1}(U_\alpha)$$

belongs to $\mathfrak{T}$ whence $\bigcup_\alpha U_\alpha \in \mathfrak{A}$. Similarly, one shows easily that $\mathfrak{A}$ is closed under finitely many intersections. We conclude that $\mathfrak{A}$ is indeed a topology on the set A and that ρ is a quotient map $X \to A$.

Definition 2.18. Let $(X, \mathfrak{T})$ be a topological space and let X^* be a partition[3] of the space X. Consider the unique surjective map $\rho : X \twoheadrightarrow X^*$ defined by taking $x \in X$ to the unique element of X^* which contains x. Let $\mathfrak{T}^*$ be the quotient topology induced by ρ on X^*. The pair $(X^*, \mathfrak{T}^*)$ is called a *quotient space* of X.

[3] A partition of a non-empty set X is a collection of non-empty disjoint subsets whose union is the whole of X.

The construction above shows that one can always obtain a topological space on equivalence classes of X. In general, one will defined an equivalence relation $\sim$ on elements of X and then give the quotient topology to $X/\sim$. Doing so places elements of X into disjoint "boxes" according to some well defined classifying property.

Theorem 2.35. *Let $(X, \mathfrak{T})$, $(Y, \mathfrak{W})$ be topological spaces and let $\rho : X \twoheadrightarrow Y$ be a quotient map. Let A be a subspace of X that is saturated with respect to ρ and let*

$$\varrho : A \to \rho(A)$$

be the map obtained by restricting ρ to A.

(1) If A is either open or closed in X, then ϱ is a quotient map.

(2) If ρ is either an open or a closed map, then ϱ is also a quotient map.

Proof. We will only handle the cases where A is open or ρ is an open map. A symmetric argument applies when A is closed or ρ is a closed map. We decompose the proof into two steps.

STEP 1. The following equalities hold true:

$$\begin{cases} \varrho^{-1}(V) = \rho^{-1}(V), & \text{if } V \subseteq \rho(A), \\ \rho(U \cap A) = \rho(U) \cap \rho(A), & \text{for } U \subseteq X. \end{cases} \tag{2.3}$$

We now commence with the proof of (2.3). First, suppose that $V \subseteq \rho(A)$. It is immediate from the definitions that $\varrho^{-1}(V) \subseteq \rho^{-1}(V)$, as ϱ is merely a restriction of ρ. It is also clear that $\rho^{-1}(V) = \bigcup_{y \in V} \rho^{-1}(\{y\})$. Since A is saturated with respect to ρ and $y \in V \subseteq \rho(A)$, it follows that $\rho^{-1}(V) \subseteq A$. This means that $\rho^{-1}(V)$ consists only of the points in A mapping to V under ρ. It follows from this observation that $\rho^{-1}(V) = \varrho^{-1}(V)$.

We now handle the second equality. Let $U \subseteq X$. It always holds that

$$\rho(U \cap A) \subseteq \rho(U) \cap \rho(A),$$

as it easily verified by elementary set theory. For the reverse inclusion, we fix $y \in \rho(U) \cap \rho(A)$. Pick $a \in A$ and $u \in U$ such that $y = \rho(a) = \rho(u)$. Then, $\rho^{-1}(\{y\})$ intersects A whence $\rho^{-1}(\{y\}) \subseteq A$. We see that $u \in A$ and so $u \in U \cap A$. Since $y = \rho(u)$, it follows that $y \in \rho(U \cap A)$. This proves the first step.

Let us now take a step back and make a few important observations. By definition, ρ is continuous (it is a quotient map). By Proposition 1.22, the restriction ϱ is continuous as a map $A \to Y$. By this same Proposition, ϱ is also continuous as a function $A \to \rho(A)$. By definition, this means that $\varrho^{-1}(V)$ is open in A whenever V is open in $\rho(A)$. Thus, to show that ϱ is a quotient map (it is surjective by definition), we need only show that $\varrho^{-1}(V)$ being open implies that V is open in $\rho(A)$. This is our last step.

Step 2. Suppose A is open or that ρ is an open map. Then, ϱ is a quotient map $A \to \rho(A)$.

First assume that A is open in X. Let $V \subseteq \rho(A)$ and assume that $\varrho^{-1}(V)$ is open in A. Step 1 gives $\rho^{-1}(V) = \varrho^{-1}(V) \subseteq A$ whence $\rho^{-1}(V)$ is open in A, and hence in the whole of X. Since ρ is a quotient map, the set V is open in Y. Recalling that $V \subseteq \rho(A)$ gives that V is open in $\rho(A)$.

Suppose instead that ρ is an open map and let $V \subseteq \rho(A)$ be such that $\varrho^{-1}(V)$ is open in $\rho(A)$. Again, we have $\rho^{-1}(V) = \varrho^{-1}(V)$ whence $\rho^{-1}(V)$ is open in A as well. By definition of the subspace topology, there exists an open set U of X having the property that

$$\rho^{-1}(V) = U \cap X.$$

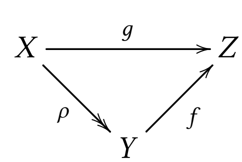
Applying the surjective map ρ to both sides yields

$$V = \rho(\rho^{-1}(V)) = \rho(U \cap A) = \rho(U) \cap \rho(A).$$

Since ρ is an open map, the image $\rho(U)$ is open in X and therefore V is open in $f(A)$ by the above. This completes the proof, □

We now show how one can "factor through" quotient maps. The reader should observe that this result is very similar to the famous first isomorphism theorem for groups.

Theorem 2.36 (Factorization Theorem)**.** *Let X,Y and Z be topological spaces and $\rho : X \twoheadrightarrow Y$ a quotient map. Assume $g : X \to Z$ is a map that is constant on every fiber $\rho^{-1}(\{y\})$ of X. There exists a map $f : Y \to Z$ having the property that $g = f \circ \rho$. As a diagram,*



(2.4)

That is, the function g can be "factored" through the quotient map ρ. Furthermore,

(1) f is continuous if and only if g is continuous;

(2) f is a quotient map if and only if g is a quotient map.

Proof. We begin by constructing the function f. For every $y \in Y$, the map g is constant on the fiber $\rho^{-1}(\{y\})$. For any $x \in \rho^{-1}(\{y\})$, we define $f(y) := g(x)$. Clearly, this gives a well defined map $f : Y \to Z$. If $x \in X$, then $y = \rho(x)$ belongs to Y. Since g is constant along $\rho^{-1}(\{y\})$, we get that $f(\rho(x)) = g(x)$ as was required. It now remains only to prove the two other conclusions.

(1) First, suppose that f is continuous. Then, $f^{-1}(V)$ is open in Y for all V open in Z. Therefore, if V is open in Z,

$$g^{-1}(V) = (f \circ \rho)^{-1}(V) = \rho^{-1}(f^{-1}(V)) \tag{2.5}$$

which is open in Y since ρ is continuous. Conversely, suppose that g is continuous. If V open in Z, (2.5) shows that $f^{-1}(V)$ is open in Y since ρ is a quotient map. Thus, f is continuous.

(2) Suppose that f is a quotient map. Since f is continuous, the previous part shows that g is also continuous. If $V \subseteq Z$ and $g^{-1}(V)$ is open in X, (2.3) implies that $f^{-1}(V)$ is open in Y, since ρ is a quotient map. Since f is also a quotient map, it follows that V is open in Z. Thus, g is a quotient map. Conversely, suppose that g is a quotient map so that f is continuous (at the very least). Let $V \subseteq Z$ be such that $f^{-1}(V)$ is open. It then follows from (2.5) that V is open in Z.

This completes the proof. □

The theorem above gives way to what can be considered the topological first isomorphism theorem. As a setup, consider two spaces X and Y with underlying topologies. Let $g : X \twoheadrightarrow Y$ be a continuous function. We define an equivalence relation on X by saying that $x_1 \sim x_2$ if, and only if, $g(x_1) = g(x_2)$. It is very easy to check that $\sim$ is an equivalence relation on X. Define now

$$X^* := X/\sim .$$

Then, X^* is a partition of the space X. The equivalence class of $x \in X$ relative to the equivalence relation $\sim$ is simply

$$[x] = \{t \in X : g(t) = g(x)\} = g^{-1}(g(x)).$$

Therefore, the canonical projection map

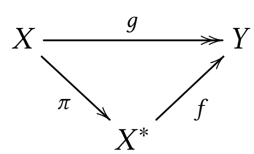
$$\pi : X \to X^*, \quad x \mapsto [x]$$

is surjective. In fact, π is the only surjective map which takes $x \in X$ to the element of X^* containing x. This means that π induces a quotient topology on X^* which makes $\pi : X \twoheadrightarrow X^*$ a quotient map. Now, if $[x]$ is an element of X^*, a moment's consideration gives

$$\pi^{-1}\left(\{[x]\}\right) = \{t \in X : \pi(t) = [x]\} = \{t \in X : g(t) = g(x)\}\,.$$

In short, g is constant along each fiber of X^*. We are now prepared to prove the following theorem.

Theorem 2.37. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces. Let $g : X \twoheadrightarrow Y$ be a surjective continuous map. Then, the map π satisfies the commutative diagram*



where f is a continuous bijection obtained through the previous theorem. The map f is a homeomorphism if and only if g is a quotient map. If Y is Hausdorff, then so is X^*, relative to its quotient topology.

Proof. First, the map f obtained through Theorem 2.36 is defined by taking an equivalence class $[x]$ in X^* to $g(x)$. It is also clear that the diagram above is satisfied. Obviously, f is surjective since g is. Moreover, X^* is obtained by taking X modulo the pointwise values of $g(x)$. Hence, f is also injective. Given that g is continuous, Theorem 2.36 also ensures that f is a continuous map.

Suppose that f is a homeomorphism. We claim that f is a quotient map. Indeed, let $V \subseteq Y$ be such that $f^{-1}(V)$ is open in X^*. Since f^{-1} is a continuous map $Y \to X^*$, we see that

$$\left[f^{-1}\right]^{-1}\left(f^{-1}(V)\right) = f(f^{-1}(V)) = V$$

is open in Y. Thus, f is a quotient map. Since $g = f \circ \pi$ is the composition of quotient maps, we get that g is a quotient map. Conversely, Theorem 2.36 states that f is a quotient map whenever g is itself a quotient map. Thus, if U is open in X^*, then $f^{-1}(f(U)) = U$ is open in X^*. As f is a quotient map, we get that $f(U)$ is open in Y. Consequently, f is a homeomorphism.

Let Y be Hausdorff and choose two distinct points $[x_1]$ and $[x_2]$ from X^*. Since f is injective, $f([x_1])$ and $f([x_2])$ are distinct points in Y. Being Hausdorff, choose disjoint neighbourhoods U and V of $f([x_1])$ and $f([x_2])$, respectively. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighbourhoods of $[x_1]$ and $[x_2]$. □

2.9 Collapsing and Wedging Spaces

In this section, we introduce three new operations on topological spaces. First, we show how one can *collapse* a subspace to a single point. Then, we explain how one can "glue" together a given collection of topological spaces. Using these notions, we will construct "bouquet of spaces".

Collapsing a Subspace to a point. Let $(X, \mathfrak{T})$ be a topological space and consider a subspace A of X. We define the "quotient" X/A to be the following family of subsets of X:

$$X/A := \{A\} \sqcup \bigsqcup_{x \notin A} \{x\}.$$

Clearly, X/A is a *partition* of the space X. Thus, every point $x \in X$ belongs to a unique element of X/A, which we denote by $[x]$. This fact gives rise to a canonical projection map

$$\pi : X \to X/A, \quad x \mapsto [x].$$

We choose to topologize X/A by declaring a set $V \subseteq X/A$ to be open if and only if $\pi^{-1}(U)$ is open in X. One can easily check that

$$\left\{V \subseteq X/A : \pi^{-1}(V) \text{ is open in } X\right\}$$

is indeed a topology on X/A. In fact, it is the finest topology with respect to which π is continuous. Note that the subspace A of X corresponds to a point in X/A (hence the term *collapse*).

Topological Sums. Next, we discuss how to join together two (possibly identical) topological spaces. Obviously, simply taking a union cannot work, as then the "sum" of two circles would return a single circle. Thus, we need an operator that does not forget about copies of a space.

Let $I \neq \emptyset$ be an indexed set and $\{(X_\alpha, \mathfrak{T}_\alpha)\}_{\alpha \in I}$ an indexed family of topological spaces. For each index $\alpha \in I$, define

$$W_\alpha := X_\alpha \times \{\alpha\},$$

where $\{\alpha\}$ is treated as a point. We then define the *topological sum* of the X_α, denoted $\coprod_{\alpha \in I} X_\alpha$, according to the following:

$$\coprod_{\alpha \in I} X_\alpha := \bigcup_{\alpha \in I} W_i = \bigsqcup_{\alpha \in I} (X_\alpha \times \{\alpha\}).$$

It remains to topologize $\coprod_{\alpha \in I} X_\alpha$. For every index $\alpha \in I$, there is a natural inclusion map

$$\sigma_\alpha : X_\alpha \to \coprod_{\alpha \in I} X_\alpha, \quad x \mapsto (x, \alpha).$$

We then declare a set $V \subseteq \coprod_{\alpha \in I} X_\alpha$ to be *open* if and only if $\sigma_\alpha^{-1}(V)$ is open in X_α, for every $\alpha \in I$. Again, it can be readily checked that the family

$$\left\{ V \subseteq \coprod_{\alpha \in I} X_\alpha : \sigma_\alpha^{-1}(V) \in \mathfrak{T}_\alpha, \ \forall \alpha \in I \right\}$$

is indeed a topology on $\coprod_{\alpha \in I} X_\alpha$.

Proposition 2.38. *Let $\{(X_\alpha, \mathfrak{T}_\alpha)\}_{\alpha \in I}, Y$ be topological spaces. Let*

$$f : \coprod_{\alpha \in I} X_\alpha \to Y$$

be any function. Then, f is continuous if and only if

$$f \circ \sigma_\alpha : X_\alpha \to Y$$

is continuous for every index $\alpha \in I$.

Proof. Suppose that f is continuous and fix an index $\alpha \in I$. Given an open set $V \subseteq Y$, we must show that

$$(f \circ \sigma_\alpha)^{-1}(V) = \sigma_\alpha^{-1}(f^{-1}(V))$$

is open in X_α. By continuity, $f^{-1}(V)$ is open in $\coprod_{\alpha \in I} X_\alpha$. By definition of the summation topology, $\sigma_\alpha^{-1}(f^{-1}(V))$ must also be open.

Conversely, suppose that each $f \circ \sigma_\alpha$ is continuous. If $V \subseteq Y$ is open, then

$$(f \circ \sigma_\alpha)^{-1}(V) = \sigma_\alpha^{-1}(f^{-1}(V))$$

is open in X_α for each $\alpha \in I$. By definition, this means that $f^{-1}(V)$ is open in $\coprod_{\alpha \in I} X_\alpha$. □

Wedging Spaces. Suppose that you are given two unit disks in $\mathbb{R}^2$, and wanted to glue these two disks together at a single boundary point to obtain a "figure 8" in the plane. One way to achieve this would be to look at the topological sum of two disks. We would then have to identify these two (given) points with each other. That is, we would collapse these two points to a single point.

Let us now make the above precise. Let $\{(X_\alpha, \mathfrak{T}_\alpha)\}_{\alpha \in I}$ be a family of topological spaces and suppose we are given a point x_α from each X_α. Clearly, every X_α be identified with a point belonging to the topological sum

$$\coprod_{\alpha \in I} X_\alpha.$$

The *wedge product* of these X_α (at the points x_α) is then defined as:

$$\bigvee_{\alpha \in I} X_\alpha := \left(\coprod_{\alpha \in I} X_\alpha \right) / \{x_\alpha : \alpha \in I\}.$$

2.10 Exercises

Problem 2.1. Let $(X, <)$ be a simply ordered set with the order topology. Prove that one always has $\mathrm{Cl}((a, b)) \subseteq [a, b]$.

Problem 2.2. A space X is called *totally disconnected* if its only connected subspaces are singletons. Show that if X has the discrete topology, then it is totally disconnected. Show that the converse does not hold.

Problem 2.3. Determine whether or not the space $\mathbb{R}_\ell$ is connected.

Problem 2.4. Let X, Y be spaces, with Y connected, and suppose $\rho : X \twoheadrightarrow Y$ is a quotient map. Prove that if each fiber $\rho^{-1}(\{y\})$ is connected, then so is X.

Problem 2.5. Let $f : \mathbb{S}^1 \to \mathbb{R}$ be a continuous map. Prove that there exists a point x on $\mathbb{S}^1$ having the property that $f(x) = f(-x)$.[4]

Problem 2.6. Let $f : [0,1] \to [0,1]$ be continuous. Prove that there exists a point $x \in [0,1]$ such that $f(x) = x$. Does this still hold for the intervals $(0,1)$ and $[0,1)$?

Problem 2.7. Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$, for all $n > 1$.

Problem 2.8. Let X be a space and Y a connected subspace of X. Will $\operatorname{Int}(Y)$ and ∂Y necessarily be connected? Need the converse also hold true?

Problem 2.9. Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be connected topological spaces. Let A, B be proper non-empty subspaces of X and Y, respectively. Prove that

$$(X \times Y) \setminus (A \times B)$$

is a connected subspace of $X \times Y$.

Problem 2.10. Let $A \subseteq X$ and assume that C is a connected subspace of X that intersects both A and A^c. Prove that C also intersects ∂A.

Problem 2.11. Let X and Y be connected, with Y a subspace of X. If (A, B) forms a separation of $X \setminus Y$, show that $Y \cup A$ and $Y \cup B$ are connected subspaces.

Problem 2.12. Let $(X, \mathfrak{T})$ be a locally path connected space[5]. Prove that every connected open set in X must be path connected.

Problem 2.13. Let X, Y be topological spaces and $A \subseteq X$.

(1) Let $\rho : X \to Y$ be continuous and assume there exists a continuous map $f : Y \to X$ such that $\rho \circ f \equiv \mathbf{1}_Y$. Prove that ρ is a quotient map.

(2) A **retraction** onto A is a continuous map $\mathfrak{r} : X \to A$ such that $\mathfrak{r}(a) = a$ for all $a \in A$. Show that a retraction is a quotient map.

Problem 2.14. Let $\pi : \mathbb{R} \times \mathbb{R} \twoheadrightarrow \mathbb{R}$ be a projection onto the first coordinate. Define

$$A := \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0 \text{ or } y = 0\} = (\mathbb{R}_{\geqslant 0} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$$

and give A the subspace topology. Let $q : A \to \mathbb{R}$ be the restriction of π to A. Show that q is a quotient map that is neither open nor closed.

[4] Here, $\mathbb{S}^n := \left\{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\right\}$.

[5] The space X is called *locally path connected* if for every $x \in X$ and every neighbourhood U of x, there is a path connected neighbourhood V of x with $V \subseteq U$.

Problem 2.15. Let $A \subset \mathbb{R}^2$ be countable (e.g. $\mathbb{Q} \times \mathbb{Q}$). Show that $\mathbb{R}^2 \setminus A$ is path connected.

Problem 2.16. Let $\rho : X \twoheadrightarrow Y$ be a quotient map and assume that X is locally connected. Prove that Y is locally connected. *Hint*: Given a component C of an open set $U \subseteq Y$, show that $\rho^{-1}(C)$ is the disjoint union of components from $\rho^{-1}(U)$.

Chapter 3

Countability, Separation, & Topological Vector Spaces

In this chapter, we move away from general topological spaces and focus on special cases. In the context of metric spaces, one has a beautiful connection between topological and analytic theories that general spaces lack. For example, if (X, d) is a metric space, then a set $F \subseteq X$ is closed if and only if every convergent sequence in F converges *in* F. This is not true for general topological spaces, as sequences do not wholly capture the analytic nature of every space. To rectify such issues, we will introduce additional conditions (the countability and separation axioms) giving the topological space a more analytic structure.

Alternatively, one can introduce *nets*, a generalization of the familiar sequence. In most cases, nets completely capture the weak analytic structure of a topological space and thus serve as a nice replacement for sequences. In the final portion of this chapter, we will briefly study these nets.

3.1 First and Second Countable Spaces

Definition 3.1. Let $(X, \mathfrak{T})$ be a topological space. We say that X is *first countable* at a point x if there exists a countable family of open sets $\mathcal{B}$, each containing x, such that every neighbourhood of x contains at least one element from $\mathcal{B}$. Such a family is called a *countable basis* for the point x. The space X is called first countable if it is first countable at every point.

First countable spaces enjoy many reminiscent properties that general topological spaces do not. An important instance of such a property is shown below.

Theorem 3.1. *Let $(X, \mathfrak{T})$ be a first countable space and let $F \subseteq X$. The following statements are equivalent.*

(1) F is closed.

(2) If $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements in F converging to $x \in X$, then $x \in F$.

Proof. Suppose that F is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in F and assume that x_n converges to $x \in X$, as $n \to \infty$. Suppose for a contradiction that $x \notin F$. Equivalently, $x \in F^{\mathsf{c}}$ (which is open in X). In more familiar terminology, F^{c} is a neighbourhood of x. Since x_n converges to x, there exists $N \in \mathbb{N}$ so large that $x_N \in F^{\mathsf{c}}$, which contradicts the assumption that $x_n \in F$ for all n. Notice that we have not yet used the countability assumption on X.

Conversely, we argue by contradiction. Assume that F is not closed, so that F^{c} is not open. Consequently, there exists a point $x \in F^{\mathsf{c}}$ such that no neighbourhood of x is contained in F^{c}. That is, every neighbourhood of x intersects F. Let now $\mathcal{B}$ be a countable basis at x. We may enumerate this family as

$$\mathcal{B} = \{B_1, B_2, \dots, B_n, \dots\}$$

where each B_k is open in X (and contains x). Let $C_1 = B_1$, and for $n \geq 1$ set

$$C_{n+1} := C_n \cap B_{n+1} = \bigcap_{j=1}^{n+1} B_j.$$

Since finite intersections of open sets are open, every C_k is a neighbourhood of the point x (and hence intersects F). Also, $C_n \supseteq C_{n+1}$ for all indices n. For each $n \in \mathbb{N}$, let x_n be any point belonging to the intersection of F and C_n. If $U \subseteq X$ is a neighbourhood of x, then it contains some B_N and, hence, C_N. Thus, $x_n \in U$ for all $n \geq N$. Since U was arbitrary, we conclude that

$$\lim_{n\to\infty} x_n = x.$$

On the other hand, $(x_n)_{n\in\mathbb{N}}$ is a sequence living in F. Our assumption thus gives $x \in F$, which is absurd. This completes the proof. □

The argument used in the first implication grants us the following important observation.

Corollary 3.2. *Let $(X, \mathfrak{T})$ be a topological space and $F \subseteq X$ closed. If $(x_n)_{n\in\mathbb{N}}$ is a sequence of points in F converging to $x \in X$, then $x \in F$.*

First countable spaces enjoy many more analytic properties reminiscent of $\mathbb{R}$ (and, more generally, of metric spaces). For instance, when the domain is first countable, we can characterize the continuity of a function in terms of sequences. This is described within the following theorem.

Theorem 3.3. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces and let $f : X \to Y$.*

(1) Let f be continuous. If $(x_n)_{n\in\mathbb{N}}$ is a sequence converging to x in X, then $f(x_n)$ converges to $f(x)$ in Y.

(2) Suppose that for every sequence $(x_n)_{n\in\mathbb{N}}$ in X converging to some $x \in X$, one has that $\lim_{n\to\infty} f(x_n) = f(x)$. If $(X, \mathfrak{T})$ is first countable, then f is continuous.

Proof. We begin with the first claim. Suppose f is continuous and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging to a point $x \in X$. We show that $f(x_n)$ tends to $f(x)$, as $n \to \infty$. Let $V \subseteq Y$ be a neighbourhood of $f(x)$, so that $f^{-1}(V)$ is a neighbourhood of x. Since $\lim_{n\to\infty} x_n = x$, one can find $N \in \mathbb{N}$ having the property that $x_n \in f^{-1}(V)$ for all $n \geq N$. Thus, $f(x_n) \in V$ for all $n \geq N$. This establishes that

$$\lim_{n\to\infty} f(x_n) = f(x).$$

We now establish (2). Let us argue by contradiction. If f is not continuous, there exists an open set $V \subseteq Y$ so that $f^{-1}(V)$ is not open in X. Since $f^{-1}(V)$ is not open in X, there exists a point $x \in f^{-1}(V)$ so that no neighbourhood of x is contained in $f^{-1}(V)$. Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable basis at x and define for $n \in \mathbb{N}$:

$$C_n := \bigcap_{k=1}^{n} B_k, \quad \text{which is a neighbourhood of } x.$$

Obviously, $C_n \supseteq C_{n+1}$ for all $n \in \mathbb{N}$. Since every C_n is a neighbourhood of x, it intersects the set $f^{-1}(V)^c = f^{-1}(V^c)$. For each $n \in \mathbb{N}$, choose $x_n \in C_n$ with $x_n \notin f^{-1}(V)$. If U is a neighbourhood of x, then U contains some $C_N \subseteq B_N$ whence it follows that x_n converges to x in X. By hypothesis on f, we get that

$$f(x) = \lim_{n\to\infty} f(x_n).$$

For every $n \in \mathbb{N}$, we have $f(x_n) \in V^c$ since $x_n \in f^{-1}(V^c)$. Since V^c is closed, we apply the previous theorem to deduce that $f(x) = \lim_{n\to\infty} f(x_n)$ belongs to V^c. This means that $x \in f^{-1}(V^c)$ whence $x \notin f^{-1}(V)$. This contradiction shows that f must be continuous. □

Finally, we give a nice characterization for the closure of a set within a first countable space.

Theorem 3.4. *Let $(X, \mathfrak{T})$ be a topological space and let $A \subseteq X$. If there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in A converging to $x \in X$, then $x \in \mathrm{Cl}(A)$. If $(X, \mathfrak{T})$ is first countable, then*

$$\mathrm{Cl}(A) := \left\{ x \in X : \exists\, (x_n)_{n\in\mathbb{N}} \subseteq A \text{ such that } \lim_{n\to\infty} x_n = x \right\}.$$

Proof. The first part is immediate from Theorem 3.1. Let now $(X, \mathfrak{T})$ be first countable, it remains to show that every element of $\mathrm{Cl}(A)$ is the limit of a sequence in A. Fix $x \in \mathrm{Cl}(A)$ and recall Theorem 1.15 which states that every neighbourhood of x intersects A. Let $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$ be a countable basis at x. By this observation, every element of $\mathcal{B}$ will intersect A. Once again, define for $n \in \mathbb{N}$ the set $C_n := \bigcap_1^n B_n$. This generates a decreasing family $\{C_n\}_{n\in\mathbb{N}}$ of open sets, each containing x. As a result, every C_n contains an element x_n from A. If U is a neighbourhood of x, then it contains some $B_N \in \mathcal{B}$ and therefore all C_n, for $n \geq N$. Since this gives $x_n \in U$ for all $n \geq N$, we conclude that

$$\lim_{n\to\infty} x_n = x.$$

Since $x_n \in A$ for all n, the proof is complete. □

3.1.1 Metrizable Spaces

We will now take a moment to discuss how the topics covered thus far pertain to metrizable spaces. We recall that a topological $(X, \mathfrak{T})$ is called *metrizable* if there exists a metric on X inducing the topology $\mathfrak{T}$. We have just proven very practical theorems (of an analytic nature) applying to all first countable spaces and would like to show they also apply to the metrizable spaces. This is easily achieved by way of the following proposition.

Proposition 3.5. *Let $(X, \mathfrak{T})$ be a metrizable topological space. Then it is both first countable and Hausdorff.*

Proof. Let d be a metric generating the topology $\mathfrak{T}$. We first check that X is Hausdorff. If x and y are distinct points in X, then let $\gamma := d(x, y) > 0$. The two open balls $B(x, \gamma/2)$ and $B(y, \gamma/2)$ are open with respect to d, and hence with respect to $\mathfrak{T}$. Suppose that $z \in B(x, \gamma/2) \cap B(y, \gamma/2)$. Then,

$$d(x, y) \leq d(x, z) + d(z, y) < \gamma = d(x, y)$$

which is absurd. Hence, $B(x, \gamma/2)$ and $B(y, \gamma/2)$ are disjoint. This proves that X is Hausdorff. If $x \in X$ is fixed consider the family

$$\mathcal{B} := \left\{B(x, r) : r \in \mathbb{Q}_{>0}\right\}$$

of open sets in X (with respect to both the metric and the topology). If U is a neighbourhood of the point x it will contain a ball $B(x, \varepsilon)$, for some $\varepsilon > 0$. Taking a positive rational r with $r < \varepsilon$ gives $B(x, r) \subseteq B(x, \varepsilon) \subseteq U$. We conclude that $\mathcal{B}$ is a countable basis at x. □

3.1.2 Second Countable Spaces

This subsection introduces a stronger variant of the first countability axiom, suitably dubbed the *second* countability axiom.

Definition 3.2. A topological space having a countable basis for its topology is called *second countable.*

Obviously, any second countable space is first countable. Moreover, it is not hard to see that every metric space is first countable. On the other hand, not every metric space is second countable! This testifies to the strength of the assumption within second countability. Nonetheless, there are some very nice second countable spaces that we are familiar with (e.g. $\mathbb{R}^m$ and $\mathbb{C}^n$).

Proposition 3.6. *A subspace of a first countable space is first countable. Similarly, a subspace of a second countable space is again second countable. The countable product of second countable spaces is also second countable.*

Proof. Let $(X, \mathfrak{T})$ be a topological space and fix a subspace Y of X. If $(X, \mathfrak{T})$ is first countable, we fix $y \in Y$ and choose a countable basis $\mathcal{B}$ at y. Then, the family $\{B \cap Y : B \in \mathcal{B}\}$ is easily seen to be a countable basis at y, in Y. One handles second countability similarly. Let $(X_n, \mathfrak{T}_n)$, for $n \in \mathbb{N}$, be a countable family of second countable spaces. For each $n \in \mathbb{N}$, let $\mathcal{B}_n$ be a countable basis for $\mathfrak{T}_n$. The *countable* set

$$C := \left\{ \prod_{n\in\mathbb{N}} U_n, U_n \in \mathcal{B}_n, \text{ all but finitely many } U_n \neq X_n \right\}$$

is known to be a basis for the product topology on $\prod_1^\infty X_n$. □

Definition 3.3. If X is a topological space and $A \subseteq X$, we say that A is *dense* in X if $\mathrm{Cl}(A) = X$. The space X is called *separable* if it has a countable dense subset.

For instance, $\mathbb{Q}$ is dense in $\mathbb{R}$. We now give a result demonstrating just how strong second countability is.

Theorem 3.7. *Let $(X, \mathfrak{T})$ be a second countable topological space.*

(1) $(X, \mathfrak{T})$ is a Lindelöf space, i.e. every open cover $\mathcal{U}$ of X has a countable sub-covering.

(2) $(X, \mathfrak{T})$ is separable.

Proof. First, let $\mathcal{B} := \{B_n\}_{n\in\mathbb{N}}$ be a basis for $\mathfrak{T}$ on X. Let $\mathcal{U}$ be an open covering of the space X. For each $n \in \mathbb{N}$, if there exists a set $U \in \mathcal{U}$ containing the basis element B_n, choose any such U and label it U_n. If no such U exists, put $U_n := \varnothing$ (or simply skip this set). Let $\mathcal{V}$ be the resulting collection $\{U_n\}_{n\in\mathbb{N}}$. Clearly, this family is countable sub-collection of $\mathcal{U}$ and satisfies $\bigcup_1^\infty U_n \subseteq X$. Let now $x \in X$ be given; since $\mathcal{U}$ covers X, the point x belongs to some $U \in \mathcal{U}$. Since U is open and $\mathcal{B}$ generates $\mathfrak{T}$, there exists $B_k \in \mathcal{B}$ having the property that $x \in B_k \subseteq U$. Thus, B_k (and hence x) will be contained within one of the elements of our list $\mathcal{V}$. This shows that $\mathcal{V}$ covers the entire space X.

For each $n \in \mathbb{N}$ we choose a point x_n from the basis element B_n. Define now $Q := \{x_n\}_{n=1}^\infty$. We claim that $\mathrm{Cl}(Q) = X$. Certainly, let $x \in X$ and fix a neighbourhood U of x. This neighbourhood U will contain an element of $\mathcal{B}$ and will therefore intersect Q. By Theorem 1.15, Q is dense in X. □

For metrizable spaces, a partial converse holds true.

Proposition 3.8. *Every separable metrizable space is second countable.*

Proof. Let X be a separable space whose topology is generated by the metric d. Let $\{s_n\}_{n\in\mathbb{N}}$ be a countable dense subset of X. Clearly, the family

$$\mathscr{B} := \{B(s_n, r) : r \in \mathbb{Q},\ n \in \mathbb{N}\}$$

is a countable collection of open subsets of X. We will show that $\mathscr{B}$ is a basis for the topology on X. Let $U \ni x$ be an open set and choose $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Let $r \in \mathbb{Q}$ be such that $0 < r < \frac{\varepsilon}{2}$ and fix s_n with $d(x, s_n) < r$. Clearly, $x \in B(s_n, r) \in \mathscr{B}$. If $y \in B(s_n, r)$, then

$$d(y, x) \le d(y, s_n) + d(s_n, x) < 2r < \varepsilon.$$

Hence, $x \in B(s_n, r) \subseteq B(x, \varepsilon)$. This shows that $\mathscr{B}$ is a basis. □

3.2 Nets

This section is devoted to studying a generalization of sequences. We have seen previously that one can capture the analytic structure of a first countable space using sequences, but not that of a general space. One solution to this problem is to instead work with what we call *nets*.

Definition 3.4. A directed set is a non-empty set Λ equipped with a binary relation $\lesssim$ satisfying the following conditions:

(1) $\lambda \lesssim \lambda$ for all $\lambda \in \Lambda$;

(2) If $\alpha \lesssim \beta$ and $\beta \lesssim \gamma$ then $\alpha \lesssim \gamma$, for all $\alpha, \beta, \gamma \in \Lambda$;

(3) For any $\alpha, \beta \in \Lambda$ there exists $\lambda \in \Lambda$ having the property that $\alpha \lesssim \lambda$ and $\beta \lesssim \lambda$.

If $(X, \mathfrak{T})$ is a topological space, a *net* on X is a function $x : \Lambda \to X$, often denoted by $(x_\lambda)_{\lambda\in\Lambda}$, where x_λ stands for $x(\lambda)$.

We have analogous notions of convergence for nets. A net $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point $x \in X$ if, for every neighbourhood U of X, there exists $\eta \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \gtrsim \eta$. Clearly, a net indexed by $\mathbb{N}$ may be realized as a sequence.

As we shall shortly see, nets fully capture the analytic property of *any* topological space; a feat which cannot be accomplished by using only sequences. By using nets, many "methods of proof" from the study of metric spaces carry over to the general topological setting.

Theorem 3.9. *Let $(X, \mathfrak{T})$ be a topological space and $F \subseteq X$. The following statements are equivalent.*

(1) F is closed.

(2) For every net $(x_\lambda)_{\lambda\in\Lambda}$ in F converging to $x \in X$, one has $x \in F$.

Proof. The first implication $(1 \implies 2)$ is the same as in the proof of Theorem 3.1 and its proof is left as an exercise to the reader.

Conversely, suppose (2) holds true but that F is not closed. Then, F^c is not open. Again, this means that there exists a point $x \in F^\mathsf{c}$ such that every neighbourhood of x intersects F. Let N be the collection of all neighbourhoods of x, directed via reverse inclusion[1]. From each $U \in N$, we choose an element $x_U \in U \cap F$. This gives us a net $(x_U)_{U\in N}$ in F. It is not hard to check that x_U converges to x. Since $x \notin F$, this is a contradiction. We conclude that F is closed. □

Lemma 3.10. *Let $(X, \mathfrak{T})$ be a topological space and A a subset of X. Then, $x \in$ $\mathrm{Cl}\,(A)$ if and only if there exists a net of elements in A converging to x.*

Proof. Suppose first that there exists a net $(x_\lambda)_{\lambda\in\Lambda}$ of elements in A converging to $x \in X$. This is also a net belonging to the closed set $\mathrm{Cl}\,(A)$. By Theorem 3.9, the limit x must belong to $\mathrm{Cl}\,(A)$.

Conversely, suppose that $x \in \mathrm{Cl}\,(A)$; we must find a net in A converging to x. Let N be the collection of all neighbourhoods of x indexed by *reverse inclusion.* From each $U \in N$ we choose $x_U \in U \cap A$. Clearly, (x_U) converges to x. This concludes the proof. □

Our final result generalizes the sequential criterion for continuity given in Theorem 3.3 for first countable spaces.

Theorem 3.11. *Let $(X, \mathfrak{T})$ be a topological space and $f : X \to Y$ a function. The following statements are equivalent.*

(1) f is continuous;

(2) For every net $(x_\lambda)_{\lambda\in\Lambda}$ converging to $x \in X$, there holds $\lim f(x_\lambda) = f(x)$.

Proof. Again, the proof of the first implication is the same as in Theorem 3.3 and is thus left as an exercise to the reader.

[1] We write $E \lesssim F$ if $F \subseteq E$.

Conversely, suppose that (2) holds. We now argue by contradiction. If f is not continuous, then there exists an open set $V \subseteq Y$ having the property that $f^{-1}(V)$ is *not* open in X. Especially, $f^{-1}(V)^c = f^{-1}(V^c)$ is not closed. Invoking Theorem 3.9, there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ in $f^{-1}(V^c)$ converging to some $x \notin f^{-1}(V^c)$, i.e. $x \in f^{-1}(V)$. For every $\lambda \in \Lambda$ one has $f(x_\lambda) \in V^c$, which is closed in Y. By hypothesis on f, we get also that

$$\lim f(x_\lambda) = f(x).$$

Since V^c is closed, we again apply Theorem 3.9 to obtain $f(x) \in V^c$. However, this contradicts the fact that $x \in f^{-1}(V)$. We conclude that f is continuous. □

3.3 Separation Axioms

The Hausdorff axiom is the first of three separation axioms. Before we formulate the remaining two axioms, we introduce the simple concept of a T_1-space. We do this because the remaining axioms involve T_1-spaces.

Definition 3.5. A topological space X is called a T_1-space if for any two distinct points $x, y \in X$, there is an open set containing x but not y.

Before stating the separation axioms, we only require a more concrete characterization of T_1-spaces.

Proposition 3.12. *Let X be a topological space. Then X is T_1 if and only if every singleton $\{x\}$ is closed in X.*

Proof. Suppose that X is a T_1-space and fix x. For every $y \neq x$ one can find a neighbourhood U_y of y that does not contain x. We get $\bigcup_{y \neq x} U_y = X \setminus \{x\}$. Since $\bigcup_{y \neq x} U_y$ is open, we conclude that $\{x\}$ is closed.

Conversely, assume that all singletons are closed in X. If $x \neq y$ then $X \setminus \{x\}$ is an open set containing y but not x. This shows that X is T_1. □

As a corollary, we obtain the following.

Corollary 3.13. *A Hausdorff space is a T_1-space.*

We now state the three separation axioms, in order of increasing "strength".

Definition 3.6 (Separation Axioms)**.** Let $(X, \mathfrak{T})$ be a T_1-space.

(1) The space X is called *Hausdorff* if for every two points $x \neq y$ there exist disjoint neighbourhoods of x and y, respectively. (We are reiterating a definition here.)

(2) The space X is called *regular* if for every x and every closed set B **not** containing x there exist disjoint open sets containing x and B, respectively.

(3) The space X is called *normal* if for every two disjoint closed sets A and B, there exist two disjoint open sets containing A and B, respectively.

Obviously, every normal space is regular (singletons are closed since X is T_1), and every regular space is Hausdorff (again, singletons are closed). Without the requirement that X be T_1, we could not guarantee the inclusion

$$\{\text{normal spaces}\} \subseteq \{\text{regular spaces}\} \subseteq \{\text{Hausdorff spaces}\} .$$

We offer a more concrete characterization of regular spaces.

Proposition 3.14. *Let X be a T_1-space. The following statements are equivalent.*

(1) X is regular.

(2) For every point x and every neighbourhood U of x, there exists an open set V containing x having the property that $\mathrm{Cl}(V) \subseteq U$.

Proof. First suppose that X is regular and fix a pair (x, U), where x is a point and U is a neighbourhood of x. The set $B := U^{\mathsf{c}}$ is closed in X. Since the space is regular, we can choose disjoint open sets V and W containing x and B, respectively. Let now $y \in \mathrm{Cl}(V)$. If $y \notin U$, then $y \in B \subseteq W$. But then, W is a neighbourhood of y whence W intersects V. This shows that $\mathrm{Cl}(V) \subseteq U$.

Conversely, let $x \in X$ be given and fix a closed set B not containing x. Then B^{c} is a neighbourhood of x. By assumption, there exists an open set $V \ni x$ such that $\mathrm{Cl}(V) \subseteq B^{\mathsf{c}}$. Therefore, $X \setminus \mathrm{Cl}(V)$ is an open set containing B. Since V and $X \setminus \mathrm{Cl}(V)$ are disjoint, we see that X is regular. □

In a similar vein, we have the following.

Proposition 3.15. *Let X be a T_1-space. The following statements are equivalent.*

(1) X is normal.

(2) For closed set A and every open set U containing A, there exists an open set V containing A having the property that $\mathrm{Cl}(V) \subseteq U$.

Proof. Assume that X is normal and let A be a closed set contained within an open set U. Then, U^{c} is closed and does not intersect A. Therefore, we can find disjoint open sets $V \supseteq A$ and $W \supseteq U^{\mathsf{c}}$. Let $y \in \mathrm{Cl}(V)$ and suppose for a contradiction that $y \notin U$. Then, $y \in U^{\mathsf{c}} \subseteq W$ which makes W a neighbourhood of y. It follows that V intersects W, which is absurd. Hence, $\mathrm{Cl}(V) \subseteq U$.

Conversely, let A and B be disjoint closed sets in X. Then, B^{c} is an open set containing A whence one can find an open set $V \supseteq A$ having the property that $\mathrm{Cl}(V) \subseteq B^{\mathsf{c}}$. Clearly, B is contained in $X \setminus \mathrm{Cl}(V)$ and, moreover, this last set is disjoint from V. This shows that X is normal. □

Example 3.1. Let $(X, \mathfrak{T})$ be a regular space and let $x, y \in X$ be distinct points. Since $X \setminus \{y\}$ is an open set containing x, we can find an open set $U \ni x$ such that $\mathrm{Cl}(U) \subseteq X \setminus \{y\}$. Now, $\mathrm{Cl}(U)^{\mathsf{c}}$ is an open set containing y whence there exists a neighbourhood V of y with the property that $y \in V \subseteq \mathrm{Cl}(V) \subseteq X \setminus \mathrm{Cl}(U)$. Thus, x and y have neighbourhoods U and V with disjoint closures.

We provide further treatment of Hausdorff spaces.

Theorem 3.16. *A subspace of a Hausdorff space is Hausdorff. Similarly, the product of Hausdorff spaces is again Hausdorff.*

Proof. Let X be Hausdorff and Y a subspace of X. For any two distinct points $y_1, y_2 \in Y$, one can choose disjoint open sets in X, say U_1 and U_2, containing y_1 and y_2 respectively. Then $U_1 \cap Y$ and $U_2 \cap Y$ are disjoint open sets in Y containing y_1 and y_2, respectively.

Similarly, let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of Hausdorff spaces and give $\prod_\alpha X_\alpha$ the product topology. Let (x_α) and (y_α) be distinct points in $\prod_\alpha X_\alpha$. There then exists a coordinate β such that $x_\beta \neq y_\beta$. Choose now disjoint open sets U_β and V_β in X_β containing x_β and y_β, respectively. Then, the products

$$\prod_{\alpha \in I} W_\alpha \quad \text{and} \quad \prod_{\alpha \in I} W'_\alpha,$$

where

$$W_\alpha := \begin{cases} U_\beta, & \alpha = \beta, \\ X_\alpha, & \alpha \neq \beta \end{cases} \quad \text{and} \quad W'_\alpha := \begin{cases} V_\beta, & \alpha = \beta, \\ X_\alpha, & \alpha \neq \beta \end{cases},$$

are disjoint *open* sets in $\prod_\alpha X_\alpha$ containing (x_α) and (y_α), respectively. We conclude that $\prod_\alpha X_\alpha$ is Hausdorff with the product topology. □

A similar statement holds when we consider regular spaces. First, we check that subspaces of T_1-spaces are again T_1.

Lemma 3.17. *Let $(X, \mathfrak{T})$ be a T_1-space and Y a subspace of X. With the subspace topology, Y is also T_1.*

Proof. Let $\{y\}$ be a singleton in Y. Since X is T_1, the set $\{y\}$ is closed in X. Thus, $X \setminus \{y\}$ is open in Y which means that

$$Y \setminus \{y\} = Y \cap (X \setminus \{y\})$$

is open in Y. Since $\{y\}$ was taken arbitrarily, we conclude that Y is T_1. □

This simple sanity check will allow us to state the following fundamental result without fear.

Theorem 3.18. *A subspace of a regular space is regular.*

Proof. Let X be a regular space and Y a subspace of X. By the previous lemma, Y will be a T_1-space. Let x be a point of Y and let B be a closed subset of Y that does not contain x. Letting $\mathrm{Cl}\,(B)$ denote the closure of B in X, we have that $B = \mathrm{Cl}\,(B) \cap Y$. Hence, $x \notin \mathrm{Cl}\,(B)$. By the regularity of the larger space X, one can find an open sets $U \ni x$ and $V \supseteq \mathrm{Cl}\,(B)$ such that $U \cap V = \varnothing$. Then, $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y such that

$$U \cap Y \ni x \quad \text{and} \quad V \cap Y \supseteq \mathrm{Cl}\,(B) \cap Y = B.$$

This shows that Y is regular. □

We now give an alternative, and direct, proof of the same result.

Alternative Proof of Theorem 3.18. Let X be a regular space and Y a subspace. Let $y \in Y$ and suppose F is a closed subset of Y not containing y. There exists a closed set E (relative to X) such that $F = Y \cap E$. Since $y \notin F$, we cannot have $y \in E$. Using that X is regular, we choose disjoint open sets U_1 and U_2 containing y and E, respectively. Then

$$y \in Y \cap U_1 \quad \text{and} \quad F = Y \cap E \subseteq Y \cap U_2$$

are disjoint open subsets of Y. This concludes the proof. □

A similar argument also gives us the following.

Theorem 3.19. *The product of regular spaces is a regular space.*

Proof. Let $\{X_\alpha\}$ be a family of regular spaces. In particular, every X_α is Hausdorff so that $\prod_\alpha X_\alpha$ is itself a Hausdorff space (also, a T_1 space). To show that $\prod_\alpha X_\alpha$ is regular, we shall invoke Proposition 3.14.

Let (x_α) be a point in $\prod_\alpha X_\alpha$ and let U be a neighbourhood of (x_α). Choose a basis element $\prod_\alpha U_\alpha \subseteq U$ containing the point α; here U_α is open in X_α and equal to X_α for all but finitely many α. For every α, choose a neighbourhood V_α of x_α such that $\mathrm{Cl}\,(V_\alpha) \subseteq U_\alpha$. Whenever $U_\alpha = X_\alpha$, take $V_\alpha = X_\alpha$. Then, the product $V := \prod_\alpha U_\alpha$ is open in $\prod_\alpha X_\alpha$ and contains (x_α). It is not difficult to see that

$$\mathrm{Cl}\left(\prod_\alpha V_\alpha\right) = \prod_\alpha \mathrm{Cl}\,(V_\alpha) \subseteq \prod_\alpha U_\alpha \subseteq U.$$

This shows that $\prod_\alpha X_\alpha$ is regular. □

3.3.1 Normal Spaces

Let us now consider in greater depth the concept of a normal space. For this entire subsection, $(X, \mathfrak{T})$ will always denote a T_1-space. We begin with an important theorem whose proof I have chosen to omit for the sake of simplicity.

Theorem 3.20. *Every regular space whose topology is generated by a countable basis is normal.*

The curious reader may find the proof in [MNKS], where this is Theorem 32.1. I do not think the proof is particularly instructive or elegant, and it is certainly a pain to write out.

Example 3.2. We modify an argument made in Example 3.1. Let A and B be disjoint closed sets in a normal space X. Since $X \setminus B$ is a neighbourhood of A, we can find an open set U, containing A, such that $\mathrm{Cl}\,(U) \subseteq X \setminus B$. On the other hand, $X \setminus \mathrm{Cl}\,(U)$ is a neighbourhood of the closed set B whence we can find an open set $V \supseteq B$ with $\mathrm{Cl}\,(V) \subseteq X \setminus \mathrm{Cl}\,(U)$. Thus, we have found open sets containing A and B, respectively, whose closures are disjoint.

Let us now give the following useful lemma.

Lemma 3.21. *Let (X, d) be a metric space. Suppose that A and B are non-empty disjoint closed subsets of X. For each $a \in A$, there exists $\varepsilon > 0$ such that the open ball $B(a, \varepsilon)$ does not intersect B.*

Proof. We argue by contradiction. Suppose that there exists $a \in A$ such that $B(a, \varepsilon)$ intersects B for all $\varepsilon > 0$. Let $n \in \mathbb{N}$ be given and take $\varepsilon := 1/n$; we choose a point $b_n \in B(a, \varepsilon) \cap B$. This gives us a sequence $(b_n)_{n\in\mathbb{N}}$ of points living in the closed set B. For every index n, we have

$$d(a, b_n) < \frac{1}{n}$$

whence $\lim_{n\to\infty} b_n = a$. Since B is closed in X, we get $a \in B$. Since A and B are disjoint, this is a contradiction. □

Using this easy result, we obtain the following fundamental fact.

Theorem 3.22. *A metrizable topological space is normal.*

Proof. First, recall that a metrizable space is Hausdorff and hence T_1. Therefore, the statement of this theorem makes sense. Let now A and B be two disjoint closed subsets of X. We seek disjoint open sets U and V containing A and B, respectively. If $A = \varnothing$ or $B = \varnothing$, then we are done. Henceforth, assume A and B are both non-empty. For every $a \in A$, choose $\varepsilon_a > 0$ such that the open ball $B(a, \varepsilon_a)$ does not intersect B, and do the same for each $b \in B$. Put

$$U := \bigcup_{a\in A} B\left(a, \frac{\varepsilon_a}{2}\right) \quad \text{and} \quad V := \bigcup_{b\in B} B\left(b, \frac{\varepsilon_b}{2}\right);$$

which are both open subsets of X. By construction, $A \subseteq U$ and $B \subseteq V$. It remains to show that U and V are disjoint. By way of contradiction, suppose that there exists $x \in U \cap V$. Then, for some $a \in A$ and $b \in B$ there holds

$$x \in B\left(a, \frac{\varepsilon_a}{2}\right) \cap B\left(b, \frac{\varepsilon_b}{2}\right).$$

Denote by d the metric generating the topology on X. By the triangle inequality,

$$d(a, b) \leq d(a, x) + d(x, b) < \frac{\varepsilon_a + \varepsilon_b}{2}. \tag{3.1}$$

We now consider the only two possible cases.

(1) Case $\varepsilon_a \leq \varepsilon_b$. Then (3.1) gives $d(a, b) < \varepsilon_b$, whence $a \in B(b, \varepsilon_b)$ which contradicts the choice of ε_b.

(2) Case $\varepsilon_b \leq \varepsilon_a$; a similar argument shows that $b \in B(a, \varepsilon_a)$ thereby contradicting the choice of ε_a.

In either case, we have a contradiction. Thus, U and V are disjoint which concludes the proof. □

This is all very interesting, but metrizable spaces are not what an analyst would consider the most important class of topological spaces. This right is, perhaps, reserved for *locally compact Hausdorff spaces*. By the end of this section, we hope to show that all locally compact Hausdorff spaces are regular. This important fact will follow nicely from the following fundamental property of *compact* Hausdorff spaces.

Theorem 3.23. *Every compact Hausdorff space is normal.*

The proof of this theorem will hinge upon a *very* useful lemma. We urge the reader to memorize the statement of this result, as it can greatly simplify many arguments.

Lemma 3.24. *Let X be a Hausdorff space and let Y be a compact subspace of X. If $x \notin Y$, there exists a neighbourhood U of x and an open set V containing Y such that $U \cap V = \varnothing$.*

Proof of Lemma. For every $y \in Y$ we may choose disjoint open subsets U_y and V_y of X, containing x and y respectively. Then, the collection of all V_y's forms an open covering of Y by subsets of X. Since Y is compact in X, we may choose finitely many $y_1, \ldots, y_n$ in Y such that $Y \subseteq \bigcup_1^n V_{y_j}$. Define

$$U := \bigcap_{j=1}^{n} U_{y_j} \quad \text{and} \quad V := \bigcup_{j=1}^{n} V_{y_j}.$$

Clearly, U is a neighbourhood of x and V is an open set containing the subspace Y. Since U and V are obviously disjoint, the proof is complete. □

This gives way to the following.

Proof of Theorem 3.23. Let X be a compact Hausdorff space. Suppose that A and B are disjoint closed subsets of X. Since X is compact, both A and B are compact. By the previous lemma, for every $a \in A$ we may choose disjoint open sets U_a and V_a containing a and B, respectively. The family $\{U_a\}_{a\in A}$ is then an open covering of the compact set A. As a result, we may extract finitely many $a_1, \ldots, a_n$ from the set A so that $A \subseteq U_{a_1} \cup \cdots \cup U_{a_n}$. If we define $U := \bigcup_1^n U_{a_j}$, then we obtain an open set containing A. On the other hand, $V := \bigcap_1^n V_{a_j}$ will be an open set containing the set B. Since U and V are disjoint, we have that X is regular. □

We may now deduce the following promised result.

Corollary 3.25. *Every locally compact Hausdorff space is regular.*

Proof. Let X be a locally compact Hausdorff space. Invoking Theorem 2.26 yields a compact Hausdorff space Y containing X as a subspace. In parallel, Theorem 3.23 ensures that Y is normal, and hence regular. Finally, Theorem 3.18 states that X must then be regular. ◻

3.3.2 More on Locally Compact Hausdorff Spaces

We return to a topic we briefly discussed in the previous chapter, *locally compact spaces*. In the most recent theorem, we proved that all compact Hausdorff spaces are normal. This result will allow us to give a more convenient (not to mention more natural) characterization of locally compact Hausdorff spaces that is long overdue.

Theorem 3.26. *Let $(X, \mathfrak{T})$ be a Hausdorff space. The following statements are equivalent.*

(1) X is locally compact Hausdorff;

(2) Given $x \in X$ and a neighbourhood U of x, there exists an open set $V \ni x$ such that $\mathrm{Cl}\,(V)$ is compact and contained in U.

Proof. Clearly, if (2) holds then the space X must be both locally compact and Hausdorff. Therefore, we need only show that any locally compact Hausdorff space satisfies (2). To this end, let X be a locally compact Hausdorff space and fix a point $x \in X$. Suppose that U is a neighbourhood of x. According to Theorem 2.26, there exists a *compact* Hausdorff space Y containing X as a subspace. In fact, thanks to (2.2), we have an explicit description of the topology on Y.

By Theorem 3.23, the space Y is normal – and hence regular. By construction, the set U is open in Y. Invoking Proposition 3.14, we may choose an open set V (relative to Y) such that

$$x \in V \subseteq \bar{V} \subseteq U.$$

Here, $\bar{V}$ denotes the closure of V with respect to Y. On the other hand, the above tells us that $\bar{V}$ and V are subsets of X. In particular, an examination of the topology in (2.2) shows that V must be an open subset of X. Proposition 1.14 implies that the closure of V in X, written $\mathrm{Cl}\,(V)$, is equal to

$$\mathrm{Cl}\,(V) = \bar{V} \cap X = \bar{V} \subseteq U.$$

Since Y is a compact Hausdorff space, the set $\bar{V}$ is a compact subspace of Y, and hence of X. This completes the proof. □

With the help of this theorem, we establish a type of "subspace transitivity" for local compactness.

Corollary 3.27. *Let* $(X, \mathfrak{T})$ *be a locally compact Hausdorff space. If* $A \subseteq X$ *is either closed or open in* X*, then* A *is locally compact Hausdorff when given the subspace topology.*

Proof. Regardless of whether or not A is closed (or open), the space A *will* be Hausdorff. Henceforth, we distinguish the two cases.

(1) Assume that A is a closed subset of X. Fix a point $x \in A$; because X is locally compact, we may choose an open subset U of X containing x and a compact set K in X containing U. Since X is Hausdorff, K is closed in X. By Theorem 1.11, $K \cap A$ is a closed subspace of K. In particular, $K \cap A$ is a compact space. Now, $U \cap A$ is an open subset of A containing x. Finally, observe that
$$x \in U \cap A \subseteq K \cap A \subseteq A.$$

(2) Now, we handle the case where A is open. Given $x \in A$, we invoke the previous theorem to choose an open set $V \ni x$ such that $\mathrm{Cl}(V) \subseteq A$ is compact. In particular, $V \subseteq A$ so that V is a open in A. Now, $\mathrm{Cl}(V) \subseteq A$ also means that $\mathrm{Cl}(V)$ is a compact subset of A. This completes the proof.

□

3.3.3 σ-Compactness

It is sometimes of considerable use to weaken the notion of compactness. For instance, it is useful in measure theory to write a set as a countable union of compact sets. At the very least, compact sets have strong topological properties that partially carry over to countable unions of compact sets.

Definition 3.7. Let $(X, \mathfrak{T})$ be a topological space. A subset $\Sigma \subseteq X$ is called *σ-compact* if it can be written as the countable union of compact subsets of X.

A trivial example is $\mathbb{R}$; one can write $\mathbb{R} = \bigcup_{n=1}^{\infty}[-n, n]$ where each $[-n, n]$ is compact. Similarly, one can easily see that $\mathbb{R}^n$ and $\mathbb{C}$ are always σ-compact. We now see how σ-compact sets inherit a type of "weak compactness".

Proposition 3.28. *Any σ-compact space is Lindelöf.*

Proof. Let $(X, \mathfrak{T})$ be a σ-compact space and fix an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of X. We must show that there exists a countable sub-covering of $\mathcal{U}$ that covers X. Let us write $X = \bigcup_{n=1}^{\infty} K_n$, where each K_n is a compact subset of X. Clearly, $\mathcal{U}$ is also an open cover of each K_n. Thus, for every $n \in \mathbb{N}$, we may choose finitely many elements, say $\{U_n^{(j)}\}_j$, of $\mathcal{U}$ that cover K_n. It is then easy to see that

$$X = \bigcup_{n=1}^{\infty} K_n \subseteq \bigcup_{n,j} U_n^{(j)}.$$

Since this union is countable, we see that X is Lindelöf. □

Proposition 3.29. *An open subset of a second countable locally compact Hausdorff space is necessarily σ-compact.*

Proof. Let $\mathscr{B} = \{B_n\}_n$ be a countable basis for the topology on the space X. Suppose in addition that X is locally compact Hausdorff and fix a non-empty open set $O \subseteq X$. For each $x \in O$, let V be a neighbourhood of x such that $\operatorname{Cl}(V) \subseteq O$ is compact. Because $\mathscr{B}$ is a basis for the topology on X, there exists some $B_k \in \mathscr{B}$ with the property that $x \in B_k \subseteq V$. Thus, $x \in \operatorname{Cl}(B_k) \subseteq \operatorname{Cl}(V) \subseteq O$. Since $\operatorname{Cl}(V)$ is compact, we see that $\operatorname{Cl}(B_k)$ must also be compact. The proof is complete once we observe that $O = \bigcup_k \operatorname{Cl}(B_k)$. □

3.4 Metric Spaces

We now return to the concept of a metric space, which differs significantly from that of a metrizable space. Despite this difference, every result established for metrizable spaces holds when discussing metric spaces. Before proceeding, let us reiterate some previously established results that apply, in particular, to metric spaces.

Theorem 3.30. *Let (X, d) be a metric space.*

(1) If $A \subseteq X$, then $x \in \operatorname{Cl}(A)$ if and only if for each $\varepsilon > 0$:

$$B(x, \varepsilon) \cap A \neq \varnothing.$$

(2) Every metric space is normal, and in particular Hausdorff. Especially, sequences and nets have at most a single limit.

(3) *A metric space is metrizable and hence first countable. In particular, closed sets and continuity can be characterized via sequences.*

We should also point out that a metric d completely determines the behaviour of a sequence (x_n) as $n \to \infty$. Namely, a sequence (x_n) in a metric space (X, d) converges to $x \in X$ if and only if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

To see this, first assume that $x_n \to x$ in X. Given $\varepsilon > 0$, the "ball" $B(x, \varepsilon)$ is a neighbourhood of x and thus there exists $N \in \mathbb{N}$ such that $x_n \in B(x, \varepsilon)$ whenever $n \geq N$. That is, $d(x_n, x) < \varepsilon$ whenever $n \geq N$. Conversely, suppose that $d(x_n, x) \to 0$ in $\mathbb{R}$ as $n \to \infty$. For any open set $V \ni x$, we can find a basis element $B(x, \varepsilon) \subseteq V$. Since $d(x_n, x) \to 0$, there is $N \in \mathbb{N}$ with the property that $d(x_n, x) < \varepsilon$ for all $n \geq N$. Hence, $x_n \in B(x, \varepsilon) \subseteq V$ for all $n \geq N$.

With this technical detail out of the way, we can now discuss properties that are "unique" to metric spaces and are not necessarily true for metrizable spaces. The obvious difference here is that many of the upcoming results do not just depend on the topology with which X is endowed, but rather on the metric d itself.

Definition 3.8. Let (X, d) be a metric space. A sequence (x_n) in X is said to be *Cauchy* if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon$$

whenever $n, m \geq N$.

Definition 3.9. Let (X, d) be a metric space and let $A \subseteq X$. We say that A is *bounded* in X if there exists $x_0 \in X$ and $M > 0$ such that $d(x, x_0) \leq M$ for all $x \in A$. A sequence (x_n) is called bounded if $\{x_n : n \in \mathbb{N}\}$ is bounded.

As on $\mathbb{R}$, we have the following easy result.

Proposition 3.31. *Let (X, d) be a metric space. Any Cauchy sequence is bounded, and every convergent sequence is Cauchy.*

Proof. First let (x_n) be Cauchy in X. For $\varepsilon := 1$, we can find $N \in \mathbb{N}$ so large that $d(x_n, x_N) < 1$ for all $n \geq N$. Next, let us define

$$M := \max\{d(x_1, x_N), \ldots, d(x_{N-1}, x_N), 1\}.$$

Then, $d(x_n, x_N) \leq M$ for all $n \in \mathbb{N}$. This shows that (x_n) is bounded. If (y_n) is a convergent sequence in X, it converges to a unique point $y \in X$. By earlier remarks, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(y_n, y) < \frac{\varepsilon}{2}, \quad \forall n \geq N.$$

Therefore, if $n, m \geq N$:

$$d(y_n, y_m) \leq d(y_n, y) + d(y, y_m) < \varepsilon$$

whence (x_n) is Cauchy. □

It is well known that a sequence (x_n) in $\mathbb{R}$ converges (with respect to the standard topology) if and only it is Cauchy. This is not necessarily true in a metric space. For a counter example, consider $\mathbb{Q}$ with the metric

$$d(x, y) := |x - y|.$$

Let (x_n) be a sequence in $\mathbb{Q}$ that converges to $\sqrt{2}$ in $\mathbb{R}$. Then, (x_n) is Cauchy in $\mathbb{R}$, and thus in $\mathbb{Q}$. However, (x_n) cannot converge in $\mathbb{Q}$ as its unique limit in $\mathbb{R}$ is not an element of $\mathbb{Q}$! With this counter example in mind, the following special class of metric spaces is well motivated.

Definition 3.10. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Although Cauchy sequences may not be convergent, they are convergent if they contain at least one convergent subsequence. This is verified by the following lemma.

Lemma 3.32. *Let (X, d) be a metric space (not necessarily complete) and let (x_n) be a Cauchy sequence in X. If (x_{n_k}) is a subsequence converging to x, then x_n converges to x.*

Proof. Let $\varepsilon > 0$ be given and let $N_1 \in \mathbb{N}$ be such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ whenever $n, m \geq N$. Let $N_2 \in \mathbb{N}$ be so large that $d(x_{n_k}, x) < \frac{\varepsilon}{2}$ for all $k \geq N_2$. Put now $K := \max(N_1, n_2)$ and let $n \geq K$. Clearly,

$$d(x_n, x) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x) < \varepsilon$$

which proves the assertion. □

Clearly, $\mathbb{R}^n$ and $\mathbb{C}^m$ are complete for all $n, m \in \mathbb{N}$. Non-trivial examples of complete metric spaces include the following:

- Let $C([0,1])$ denote the $\mathbb{R}$-vector space of all continuous functions $[0,1] \to \mathbb{R}$. Then, $C([0,1])$ is complete with respect to the norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f| < \infty, \quad f \in C([0,1]).$$

- Let $(X, \mathfrak{M}, \mu)$, be a measure space, $1 \leq p < \infty$, and let $L^p(X)$ be the $\mathbb{C}$-vector space of all measurable functions $f : X \to \mathbb{C}$, identified up to almost everywhere equivalence, such that

$$\int_X |f|^p \mathrm{d}\mu < \infty.$$

  Then, $L^p(X)$ is complete when endowed with the metric

$$d(f,g) := \left(\int_X |f-g|^p \,\mathrm{d}\mu\right)^{1/p}.$$

  These L^p-spaces are called *Lebesgue spaces* and are of great interest in functional analysis.

- Any compact metric space is complete. Indeed, let (X, d) be a compact metric space and let (x_n) be a Cauchy sequence in X. Since X is metrizable, it is sequentially compact. Thus, (x_n) has a convergent subsequence. By our last lemma, we see that (x_n) is convergent.

3.5 Topological Vector Spaces

We now explore the concept of a topological vector space. First, we define these spaces in full generality and deduce basic results. We shall then show how a given family of semi-norms on a vector space V can be used to construct a "meaningful" topology for V. By "meaningful", we simply mean a topology that respects the given notions of addition and scalar multiplication. For this last part, we will closely follow Folland's *Real Analysis: Modern Techniques and Applications.*

Throughout this section, we will denote by V a vector space over a ground field $\mathbb{K}$. Here, $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. This vector space V comes equipped with two (addition and scalar multiplication, respectively) operations:

$$\oplus : V \times V \longrightarrow V, \quad (x, y) \mapsto x + y,$$
$$\odot : \mathbb{K} \times V \longrightarrow V, \quad (\alpha, x) \mapsto \alpha x.$$

We will of course give $\mathbb{K}$ the standard topology.

Definition 3.11. Let V be a non-zero vector space over $\mathbb{K}$ and let $\mathfrak{T}$ be a topology on V. We say that V is a *topological vector space* if both $\oplus$ and $\odot$ are continuous functions with respect to $\mathfrak{T}$. Here, we give $V \times V$ and $\mathbb{K} \times V$ their respective product topologies.

It will not always be possible to explicitly describe this topology $\mathfrak{T}$. Hence, we will need to determine more accessible ways to construct topologies for V. As mentioned previously, one way will be semi-norms.

Topological vector spaces are not frequently studied in full generality, just in the same way that one does not typically initiate a deep study of point-set topology. In practice, one often deals with topological vector spaces that are *locally convex* and Hausdorff.

Definition 3.12. Let V be a topological vector space with topology $\mathfrak{T}$. We say that V is *locally convex* if there exists a basis $\mathcal{B}$ for $\mathfrak{T}$ consisting of *convex* sets.

For the sake of completeness, let us recall that a set $A \subseteq V$ is called *convex* if, for each $x, y \in A$ and all $t \in [0, 1]$,

$$tx + (1 - t)y \in A.$$

3.5.1 Consequences of the Definitions

In this subsection, we will closely stick to the material covered in [SEMS]. Let V be a topological vector space over the field $\mathbb{K}$. We know from our discussion in Chapter 1 that the identity map

$$\mathbf{1}_V : V \to V, \quad v \mapsto v$$

is continuous. In fact, it is a *homeomorphism* of V. Fix a point $v_0 \in V$ and consider the *translation* mapping

$$\tau_{v_0} : V \to V, \quad v \mapsto v + v_0.$$

Clearly, this map is bijective with inverse

$$\tau_{v_0}^{-1} : V \to V, \quad w \mapsto w - v_0.$$

Moreover, τ_{v_0} can be obtained via composition:

$$V \longrightarrow V \times V \longrightarrow V, \quad v \mapsto (v, v_0) \mapsto v + v_0.$$

Since the map $v \mapsto (v, v_0)$ is a continuous map $V \to V \times V$, composition tells us that τ_{v_0} is continuous. Similarly, one can easily check that its inverse is also continuous. Thus, τ_{v_0} defines a *homeomorphism* of V. Likewise, if $\alpha \neq 0$ is a scalar, the *dilation* map

$$\delta_\alpha : V \longrightarrow V, \quad v \mapsto \alpha v$$

is a homeomorphism of V.[2]

Lemma 3.33. *Let V be a topological vector space and assume U is an open subset of V, containing the point 0. There exists an open set $U' \ni 0$ such that*

$$U' + U' = \{v + w : v, w \in U'\} \subseteq U.$$

In particular, $U' \subseteq U$.

Proof. By continuity, the set $O := \oplus^{-1}(U)$ is open in $V \times V$. It is also clear that $(0,0) \in O$. Choose open sets $O_1, O_2 \subseteq V$ such that

$$(0,0) \in O_1 \times O_2 \subseteq O.$$

Such sets can be chosen because the product topology on $V \times V$ has the family

$$\{W_1 \times W_2 : W_1, W_2 \text{ open in } V\}$$

as a basis. Finally, put

$$U' := (O_1 \cap O_2);$$

this set is open in V and also contains the point 0. Also,

$$U' + U' \subseteq \oplus(O_1 \times O_2) \subseteq \oplus(O) \subseteq U.$$

This completes the proof. □

[2] The proof of this fact is left as an exercise to the reader.

Let A be an arbitrary subset of V and assume U is non-empty and open in V. For each $a \in A$, we know that the canonical map

$$\tau_a : V \longrightarrow V, \quad x \mapsto x + a$$

is a homeomorphism of V. In particular, τ_a is an open map. This tells us that $A + U$ is open because we can write

$$A + U = \bigcup_{a \in A} \tau_a(U).$$

Suppose additionally that $0 \in U$. We can make similar remarks regarding the scalar multiplication operator $\odot$. This is what we accomplish below.

Lemma 3.34. *Let V be a topological vector space and U an open set containing 0. There exists a real number $r > 0$ and an open set $U' \subseteq U$, containing 0, such that*

$$\alpha U' := \{\alpha x : x \in U'\} \subseteq U'.$$

for all $\alpha \in \mathbb{K}$ with $|\alpha| < r$.

Proof. By continuity, $\odot^{-1}(U)$ is an open subset of $\mathbb{K} \times V$ containing the "origin" $(0, 0)$. By definition of the product topology, we choose an open ball $B(0, r) \subset \mathbb{K}$ and an open set $W \subset V$ such that

$$(0, 0) \in B(0, r) \times W \subseteq \odot^{-1}(U).$$

Let $U' := W$ and note that U' is open (it contains 0) and non-empty. Let $x \in U'$ and assume $\alpha \in \mathbb{K}$ satisfies $|\alpha| < r$; we get

$$\alpha x \in \odot(B(0, r) \times U') = \odot(B(0, r) \times W) \subseteq U.$$

This completes the proof. ◻

Remark 3.1. As before, there is something more to be said here. Let U be a non-empty open subset of V containing 0. Fix a vector $x \in V$ and note that $0x \in U$. By continuity, we can choose an open ball $B(0, r) \subset \mathbb{K}$ and an open set $W \subseteq V$ with the property that

$$(0, x) \in B(0, r) \times W \subseteq \oplus^{-1}(U).$$

Thus, $\alpha x \in U$ whenever $|\alpha| < r$. In particular, $\frac{1}{n}x$ belongs to U for some $n \in \mathbb{N}$. It follows that

$$V = \bigcup_{n \in \mathbb{N}} nU.$$

Suppose that X, X' and Y, Y' are topological spaces. Assume the functions $f : X \to X'$ and $g : Y \to Y'$ are continuous. Then, the function

$$h : X \times Y \longrightarrow X' \times Y', \quad (x, y) \mapsto (f(x), g(y))$$

is continuous with respect to the product topology. Certainly, let $A \times B$ be a basis element for the topology on $X' \times Y'$. Then, A, B are open in X', Y', respectively. Moreover,

$$h^{-1}(A \times B) = f^{-1}(A) \times g^{-1}(B).$$

It follows that h is continuous. This allows us to show that the "subtraction map"

$$\ominus : V \times V \longrightarrow V, \quad (x, y) \mapsto x - y$$

is continuous. Certainly, $\ominus$ can be obtained via composition:

$$(x, y) \xrightarrow{(1_V, \delta_{-1})} (x, -y) \xrightarrow{\oplus} x - y.$$

With this in mind, the following lemma is natural.

Lemma 3.35. *Let V be a topological vector space and assume $U \ni 0$ is open in V. There exists an open set $U' \subseteq U$, containing 0, where the following holds:*

$$U' - U' = \{x - y : x, y \in U'\} \subseteq U.$$

Proof. The proof is similar to that of Lemma 3.33. Since U is an open set containing 0, the continuity of $\ominus$ grants us open sets $O_1, O_2 \subseteq V$, each containing 0, such that

$$O_1 \times O_2 \subseteq \ominus^{-1}(U).$$

Indeed, this follows from the fact that $(0, 0) \in \ominus^{-1}(U)$. Now, define $U' := O_1 \cap O_2$; this is a non-empty open set containing 0. Moreover, $U' - U' \subseteq \ominus(O_1 \times O_2) \subseteq U$. □

Let V be a topological vector space and consider a subset $U \subseteq V$. We say that U is *symmetric* if $-U = U$.

Proposition 3.36. *Let V be a topological vector space and $U \ni 0$ an open set. There exists a symmetric neighbourhood of 0 contained in U.*

Proof. Recall that the inversion map $\delta_{-1} : V \to V$ is a homeomorphism. The set $O := \delta_{-1}^{-1}(U)$ is open and contains the origin. Define $W := O \cap U$, which is both open and non-empty. We claim that $-w \in W$ for each $w \in W$. If $w \in W$:

(1) $-w = \delta_{-1}(w) \in U$,

(2) $-(-w) = w \in U$ whence $-w \in f^{-1}(U) = O$.

It follows that $-W \subseteq W$. This implies that $W = -W$. □

With these preliminary results, our first useful/major result is finally within reach.

Theorem 3.37. *Let V be a topological vector space. If the singleton $\{0\}$ is closed, then V is Hausdorff.*

Proof. We will proceed in two steps. First, let $x \neq 0$ be a point in X. Since translations are homeomorphisms, $\{x\}$ is closed in V. Hence, $U := V \setminus \{x\}$ is open in V. By Lemma 3.35, we can select an open set $U' \ni 0$, contained in U, such that $U' - U' \subseteq U$. Since translation is an open map, $U' + x$ is open in V and contains x because $0 \in U'$. We now claim that $U' \cap (U' + x)$ is empty. Assume there exists $u, v \in U'$ such that $u = v + x$. Then, $u - v = x$ would live in $U' - U' \subseteq U$, which is impossible.

We now relax the assumption that $x \neq 0$. Simply let $x \neq y$ be points in V. Then, $x - y \neq 0$. By the first part, we can choose disjoint open sets U_1 and U_2 in V, containing 0 and $x - y$ respectively. Then, $U_1 + y$ and $U_2 + y$ are open sets containing y and x, respectively. By way of contradiction, suppose there exists $u_1 \in U_1$ and $u_2 \in U_2$ such that $u_1 + y = u_2 + y$. This gives $u_1 = u_2$ whence $U_1 \cap U_2 \neq \varnothing$. We conclude that $U_1 + y$ and $U_2 + y$ are disjoint. □

Corollary 3.38. *A T_1-topological vector space is Hausdorff.*

3.5.2 Semi-Norms

We now discuss how to construct meaningful topologies for a given vector space V over $\mathbb{K}$. In order to do this, we will cover the material from §5.4 of [FLND], perhaps giving more detail in the proofs. Again, we will omit the case $V = \{0\}$ since there can only be a single (indiscrete) topology for the zero vector space. Let V be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Recall that a *norm* on V is a function

$$\|\cdot\| : V \longrightarrow [0, \infty), \quad x \mapsto \|x\|$$

such that, for each $\alpha \in \mathbb{K}$ and every $x, y \in V$:

(1) $\|\alpha x\| = |\alpha| \, \|x\|$;

(2) $\|x + y\| \leq \|x\| + \|y\|$;

(3) $\|x\| = 0$ if and only if $x = 0$.

If $\|\cdot\|$ is a norm on V, then V is traditionally given the metric topology induced by the metric

$$d(x, y) := \|x - y\|.$$

Sometimes, the "norm" which best respects the algebraic and analytic structure of V and its elements is not at all a norm. What should one do in this case? Possibly, one should give up. But usually, a mathematician can find a way for things to work out. This is what the following definition attempts to do.

Definition 3.13 (Semi-norm)**.** Let V be a vector space over $\mathbb{K}$. A function

$$\rho : V \longrightarrow [0, \infty)$$

is called a *semi-norm* on V if, for every $\alpha \in \mathbb{K}$ and each $x, y \in V$:

(1) $\rho(\alpha x) = |\alpha| \rho(x)$;

(2) $\rho(x + y) \leq \rho(x) + \rho(y)$.

In particular, $\rho(0) = 0$ and $\rho(-x) = \rho(x)$.

Note that the function $d(x, y) := \rho(x - y)$ need not be a metric on V, since $\rho(v) = 0$ does not necessarily imply that $v = 0$. Nonetheless, with this map ρ one can prescribe a topology to V that views ρ as a "weak" distance function. In fact, one can do so with any given family of semi-norms.

Definition 3.14. Let V be a vector space over $\mathbb{K}$ and assume $\{\rho_\alpha\}_{\alpha \in \mathscr{A}}$ is a non-empty indexed family of semi-norms on V. Given $\alpha \in \mathscr{A}$, $\varepsilon > 0$, and $x \in V$ we define

$$B(x, \varepsilon; \alpha) := \{y \in V : \rho_\alpha(x - y) < \varepsilon\}.$$

Subsequently, let

$$\mathcal{S} := \Big\{B(x, \varepsilon; \alpha) : x \in V,\ \varepsilon > 0,\ \alpha \in \mathscr{A}\Big\}. \tag{3.2}$$

This set $\mathcal{S}$ is a sub-basis for a topology $\mathfrak{T}$ on V. We then say that the family $\{\rho_\alpha\}_{\alpha \in \mathscr{A}}$ induces the topology $\mathfrak{T}$ on V.

For the above to be well defined, we must check that $\mathcal{S}$ is indeed a sub-basis for a topology on V. To see that this is the case, we fix $x \in V$. For any $\alpha \in \mathscr{A}$, we have $x \in B(x, 1; \alpha)$ since $\rho_\alpha(x - x) = 0$. It follows that $\bigcup_{B \in \mathcal{S}} B = V$ and we conclude that $\mathcal{S}$ is a sub-basis for a topology on V. The work done in Chapter 1 also gives us a convenient way to describe the induced topology $\mathfrak{T}$.

Theorem 3.39. *Let V be a vector space over $\mathbb{K}$ and assume $\{\rho_\alpha\}_{\alpha \in \mathscr{A}}$ is a non-empty family of semi-norms. Let $\mathcal{B}$ be collection of all finite intersections of elements in $\mathcal{S}$ (defined as in (3.2)). Then, $\mathcal{B}$ is a basis for a topology on V.*

The above holds by definition of the topology induced by a sub-basis. Consequently, we see that the topology induced by a family of semi-norms consists of all unions of finite intersections of the $B(x, \varepsilon; \alpha)$.

Remark 3.2. So far, given a family $\{\rho_\alpha\}_{\alpha \in \mathscr{A}}$ of semi-norms on a vector space V, we have seen how to construct a topology $\mathfrak{T}$ on V. We have not *yet* shown that this topology makes V into a topological vector space. For this, we require that the operators $\oplus$ and $\odot$ be continuous with respect to $\mathfrak{T}$.

Theorem 3.40. *Let V be a vector space over $\mathbb{K}$ and let $\{\rho_\alpha\}_{\alpha \in \mathscr{A}}$ be a family of semi-norms on V. Equip V with the topology generated by this family of semi-norms.*

(1) For each $x \in V$, let $\mathcal{B}_x$ be the collection consisting of all finite intersections of the sets $B(x, \varepsilon; \alpha)$, for $\varepsilon > 0$ and $\alpha \in \mathscr{A}$. This is a basis[3] at x.

(2) Let $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ be a net in V. Then,

$$\lim_{\lambda \in \Lambda} x_\lambda = x$$

if and only if

$$\lim_{\lambda \in \Lambda} \rho_\alpha(x_\lambda - x) = 0$$

for every index $\alpha \in \mathscr{A}$.

(3) V is a locally convex topological vector space.

Proof. Let us begin with 1. Clearly, every element of $\mathcal{B}_x$ is open in V. Suppose U is a neighbourhood of the point x. Then, choose "open balls" such that

$$x \in B(y_1, \varepsilon_1; \alpha_1) \cap \cdots \cap B(y_l, \varepsilon_l, \alpha_l) \subseteq U.$$

[3] A basis at x is a family of open sets $\mathcal{F}$, each containing x, such that every neighbourhood of x contains at least one member of $\mathcal{F}$.

Fix any index $j = 1, \ldots, l$. Define

$$\delta_j := \varepsilon_j - \rho_{\alpha_j}(y_j - x) > 0.$$

If $z \in B(x, \delta_j; \alpha_j)$ the triangle inequality gives

$$\begin{aligned}\rho_{\alpha_j}(y_j - z) &\le \rho_{\alpha_j}(y_j - x) + \rho_{\alpha_j}(x - z) \\ &< \rho_{\alpha_j}(y_j - x) + \varepsilon_j - \rho_{\alpha_j}(y_j - x) \\ &= \varepsilon_j.\end{aligned}$$

Hence, we see that $B(x, \delta_j; \alpha_j) \subseteq B(y_j, \varepsilon_j; \alpha_j)$. Especially,

$$x \in \bigcap_{j=1}^{l} B(x, \delta_j; \alpha_j) \subseteq \bigcap_{j=1}^{l} B(y_j, \varepsilon_j; \alpha_j) \subseteq U.$$

This proves the first point.

For 2. we argue directly. Let $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ be a net in V converging to a point x. Fix $\alpha \in \mathscr{A}$ and let $\varepsilon > 0$ be given. Since $B(x, \varepsilon, \alpha)$ is a neighbourhood of x, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in B(x, \varepsilon; \alpha)$ for all $\lambda \gtrsim \lambda_0$. In particular,

$$\rho_\alpha(x_\lambda - x) < \varepsilon, \quad \forall \lambda \gtrsim \lambda_0.$$

It follows that $\rho_\alpha(x_\lambda - x) \to 0$. Conversely, let U be a neighbourhood of x. Select open balls with the property that

$$x \in B(y_1, \delta_1; \alpha_1) \cap \cdots \cap B(y_l, \delta_l; \alpha_l) \subseteq U.$$

For each index $j = 1, \ldots, l$ we have $\rho_{\alpha_l}(x_\lambda - x) \to 0$. Hence, for each index j we can choose $\lambda_j \in \Lambda$ such that

$$\rho_{\alpha_j}(x_\lambda - x) < \delta_j \quad \text{whenever } \lambda \gtrsim \lambda_j.$$

Choose $\lambda_0 \in \Lambda$ so that $\lambda_0 \gtrsim \lambda_j$ for all $j = 1, \ldots, l$. Then, if $\lambda \gtrsim \lambda_0$, there holds for each j:

$$\rho_{\alpha_j}(x_\lambda - x) < \delta_j.$$

It follows that $x_\lambda \in U$, for all $\lambda \gtrsim \lambda_0$. This proves the second point.

Finally, we show that V is a locally convex topological vector space. First, we show that $\oplus$ is continuous. Fix a point $(x, y) \in V \times V$ and assume that

$$\langle x_\lambda, y_\lambda \rangle_{\lambda \in \Lambda}$$

is a net in $V \times V$ converging to (x, y). By definition of the product topology, it follows that $x_\lambda \to x$ and $y_\lambda \to y$. For every $\alpha \in \mathscr{A}$, the quantity

$$\rho_\alpha(x_\lambda + y_\lambda - (x + y)) \leq \rho_\alpha(x_\lambda - x) + \rho_\alpha(y_\lambda - y)$$

converges to 0 by the second part. In fact, since α was arbitrary, we see that

$$\oplus (x_\lambda + y_\lambda) \to \oplus(x, y).$$

It follows that $\oplus$ is continuous at all points $(x, y) \in V \times V$. One argues similarly for $\odot$, but uses that a convergent sequence in $\mathbb{K}$ is bounded. Hence, we may treat V as a topological vector space.

Now, we prove that V is *locally convex*. Let $B(z, \varepsilon; \alpha)$ be any ball in V and fix $x, y \in B(z, \varepsilon; \alpha)$. For every $t \in [0, 1]$ we find that

$$\begin{aligned}
\rho_\alpha\left(tx + (1-t)y - z\right) &= \rho_\alpha\left(tx + (1-t)y - z + tz - tz\right) \\
&= \rho_\alpha\left(tx - tz + (1-t)y - (1-t)z\right) \\
&\leq t\rho_\alpha(x - z) + (1-t)\rho_\alpha(y - z) \\
&< t\varepsilon + (1-t)\varepsilon = \varepsilon.
\end{aligned}$$

This shows that every open ball $B(z, \varepsilon; \alpha)$ is convex. Since a basis for the topology on V is given by finite intersections of these convex sets, it follows that V is locally convex. □

When working within a normed vector space, one often enjoys regularity properties arising from the induced metric. For instance, continuity and closure and both be characterized by sequences, whence nets are unnecessary. Below, we give a partial analogue for topological vector spaces.

Proposition 3.41. *Let V be a topological vector space whose topology is induced by a semi-norm ρ. Then, V is first countable.*

Proof. Fix a point $x \in V$ and let U be a neighbourhood of x. Then, select open balls such that $x \in \bigcap_{j=1}^k B(x_j, \varepsilon_j)$. Here, we do not need to write $B(x, \varepsilon; \rho)$ as there is only a single choice of semi-norm. Define

$$\varepsilon := \min_{1 \leq j \leq k} \left\{ \varepsilon_j - \rho(x_j - x) \right\} > 0$$

and consider the open set $B(x,\varepsilon)$. If $y \in B(x,\varepsilon)$, then for every index $j = 1,\dots,k$ one has

$$\begin{aligned}\rho(y-x_j) \le \rho(y-x) + \rho(x-x_j) &< \varepsilon + \rho(x-x_j) \\ &\le \varepsilon_j - \rho(x_j - x) + \rho(x_j - x) \\ &= \varepsilon_j.\end{aligned}$$

Thus, $B(x,\varepsilon) \subseteq \bigcap_{j=1}^k B(x_j,\varepsilon_j)$. Choosing $n \in \mathbb{N}$ so large that $\frac{1}{n} < \varepsilon$, we obtain an open ball $B(x,\frac{1}{n})$ also contained within $\bigcap_{j=1}^k B(x_j,\varepsilon_j) \subseteq U$. It follows that the *countable* collection $\left\{B(x,\frac{1}{n})\right\}_{n=1}^{\infty}$ is a local basis at x. □

With Banach spaces, we have a useful characterization of continuity for linear maps (in terms of bounded operators). This result can be partially extended to maps between topological vector spaces whose topologies are generated by semi-norms.

Theorem 3.42. *Let V and W be topological vector spaces whose topologies are generated by the semi-norms $\{\rho_\alpha\}_\alpha$ and $\{\varrho_\beta\}_\beta$, respectively. Let $T : V \to W$ be a linear map. The following statements are equivalent.*

(1) T is continuous.

(2) For each β there exist indices $\alpha_1,\dots,\alpha_k$ and a constant $C > 0$ such that

$$\varrho_\beta(Tx) \le C \sum_{j=1}^{k} \rho_{\alpha_j}(x),$$

for all $x \in V$.

Proof. We follow the proof from Folland (see [FLND, CH-5]). Suppose that T is continuous and fix an index β. By continuity,

$$T^{-1}\left(B(0,1;\beta)\right) \ni 0$$

is open in V. Hence, there exists a neighbourhood U of 0 in V such that

$$\varrho_\beta(Tx) < 1$$

for all $x \in U$. By Theorem 3.40, we may assume that U has the form $\bigcap_1^k B(0,\varepsilon_j;\alpha_j)$ for some $\varepsilon_j > 0$ and indices α_j. Define $\varepsilon > 0$ to *strictly smaller* that the minimum of the ε_j. Then, $\varrho_\beta(Tx) < 1$ whenever $\rho_{\alpha_j}(x) \le \varepsilon$ for all j. Now, let $x \in V$ be given; there are two cases to distinguish.

(i) Assume there exists at least one $j = 1, \ldots, k$ such that $\rho_{\alpha_j}(x) > 0$. Define

$$y := \frac{\varepsilon x}{\sum_{j=1}^{k} \rho_{\alpha_j}(x)}$$

so that $\rho_{\alpha_j}(y) \leq \varepsilon$ for all $j = 1, \ldots, k$. By linearity,

$$Tx = \sum_{j=1}^{k} \varepsilon^{-1} \rho_{\alpha_j}(x) Ty$$

whence

$$\varrho_\beta(Tx) = \sum_{j=1}^{k} \varepsilon^{-1} \rho_{\alpha_j}(x) \varrho_\beta(Ty) \leq \varepsilon^{-1} \sum_{j=1}^{k} \rho_{\alpha_j}(x).$$

(ii) Suppose that $\rho_{\alpha_j}(x) = 0$ for all indices j. Then, $\rho_{\alpha_j}(rx) = 0$ for each j and every $r > 0$. Especially, for each $r > 0$, there holds

$$r\varrho_\beta(Tx) = \varrho_\beta(T(rx)) < 1$$

since $rx \in U$. Therefore, $\varrho_\beta(Tx) = 0$ whence the estimate from the previous case continuous to hold trivially.

Conversely, suppose that 2. holds and fix $x \in V$. Let $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ be a net in V converging to x. For every index β, we may select indices $\alpha_1, \ldots, \alpha_n$ and constant $C > 0$ such that

$$\varrho_\beta(Tv) \leq C \sum_{j=1}^{n} \rho_{\alpha_j}(v), \quad \forall v \in V.$$

Since $x_\lambda \to x$, Theorem 3.40 tells us that $\rho_\alpha(x_\lambda - x) \to 0$ for every index α. In particular,

$$\varrho_\beta(Tx_\lambda - Tx) = \varrho_\beta(T(x_\lambda - x)) \leq C \sum_{j=1}^{n} \rho_{\alpha_j}(x_\lambda - x) \xrightarrow{\lambda \in \Lambda} 0.$$

Since β was arbitrary, Theorem 3.40 also implies that Tx_λ converges to Tx in W. This means that T is continuous. □

We conclude this section with the following useful criterion for determining whether a space is Hausdorff.

Proposition 3.43. *Let V be a topological vector space whose topology is induced by a family $\{\rho_\alpha\}_{\alpha\in\mathscr{A}}$ of semi-norms. V is Hausdorff if and only if for each $x \neq 0$ there exists $\alpha \in \mathscr{A}$ with $\rho_\alpha(x) \neq 0$.*

Proof. Suppose that V is Hausdorff and choose $x \neq 0$. Let U_0 and U_x be disjoint open sets containing 0 and x, respectively. As before, we may assume that U_0 is of the form $\bigcap_1^k B(0, \varepsilon_j; \alpha_j)$. If $\rho_{\alpha_j}(x) = 0$ for all $j = 1, \ldots, k$, we would have $x \in U_0$–a contradiction. Hence, there exists some index j such that $\rho_{\alpha_j}(x) \neq 0$.

Conversely, let x and y be distinct points in V so that $x - y \neq 0$. Let $\alpha \in \mathscr{A}$ be such that $\rho_\alpha(x-y) \neq 0$. Define

$$r := \frac{\rho_\alpha(x-y)}{2} > 0$$

and note that $B(0, r; \alpha) \cap B(x-y, r, \alpha) = \varnothing$. Because translations are open maps, we see that the sets

$$W_1 := B(0, r; \alpha) + y \quad \text{and} \quad W_2 := B(x-y; r; \alpha) + y$$

are open subsets of V containing y and x, respectively. Since these are also disjoint, we are done. ◻

3.5.3 Further Properties

In this final section, we prove fundamental results that help to illustrate the underlying analytic structure of topological vector spaces. We begin by demonstrating the fact that all topological vector spaces are *path connected*. Again, we do not consider the zero vector space.

Unless stated otherwise, V is assumed to denote a non-zero topological vector space over the field $\mathbb{K}$, which denotes either $\mathbb{R}$ or $\mathbb{C}$.

Theorem 3.44. *A topological vector space is path connected.*

Proof. Fix two point $x, y \in V$ and consider the function

$$\varphi : [0,1] \longrightarrow V, \quad t \mapsto (1-t)x + ty.$$

Clearly, $\varphi(0) = x$ and $\varphi(1) = y$. The only property left to verify is continuity. To this end, suppose that $f(t)$ is a continuous function $[0,1] \to \mathbb{K}$ and fix $u \in V$. It is easy to see that the map

$$\phi : [0,1] \longrightarrow \mathbb{K} \times V, \quad t \mapsto (f(t), u)$$

is continuous. Hence, the function $t \mapsto f(t)u$ is continuous by composition:

$$[0,1] \xrightarrow{\phi} \mathbb{K} \times V \xrightarrow{\odot} V, \quad t \mapsto (f(t), u) \mapsto f(t)u.$$

In particular, both the functions $t \mapsto (1-t)x$ and $t \mapsto ty$ are continuous. If we can show that finite sums of continuous maps $[0,1] \to V$ are continuous, it will follow that φ is a path. Thus, let $f, g : [0,1] \to V$ be continuous and observe that $f + g$ can also be obtained via composition

$$t \xrightarrow{(f(t),g(t))} V \times V \xrightarrow{\oplus} f(t) + g(t).$$

Finally, $t \mapsto (f(t), g(t))$ is easily seen to be continuous. □

Theorem 3.45. *Let V be a topological vector space and let $x \in V$. If F is a closed set not containing x, one can find two disjoint open sets in V containing x and F, respectively.*

We leave the proof as an exercise to the reader. Of course, this means that the proof can be found in B. We do urge the reader to try this out themselves, as it will be the only exercise involving topological vector spaces.

Corollary 3.46. *A T_1-topological vector space is regular.*

In fact, we can conclude the following.

Corollary 3.47. *Let V be a topological vector space. The following statements are equivalent.*

(1) V is a T_1-space.

(2) V is Hausdorff.

(3) V is regular.

3.6 Exercises

Problem 3.1. Let $(X, \mathfrak{T})$ be a Hausdorff topological space and let $(x_\lambda)_{\lambda \in \Lambda}$ be a convergent net in X. Prove that the limit of $(x_\lambda)_{\lambda \in \Lambda}$ is unique.

Problem 3.2. Let X and Y be topological spaces with Y Hausdorff. Fix a subset $A \subseteq X$ and let $f : X \to Y$ be continuous. Prove **using nets** that if f can be extended to a continuous function $g : \mathrm{Cl}(A) \to Y$, then g is uniquely determined by f.

Notice how this exercise is a precisely Problem 1.15.

Problem 3.3. Let $(X, \mathfrak{T})$ be a Hausdorff space and suppose Y is a compact subspace of X. If $x_0 \notin Y$, there exist disjoint open sets U and V containing x_0 and Y, respectively.

Problem 3.4. Let $(X, \mathfrak{T})$ be a Hausdorff space and suppose A and B are compact disjoint subspaces of X. Show that there exist disjoint open sets containing A and B, respectively.

Problem 3.5. Let X and Y be non-compact locally compact Hausdorff spaces that are homeomorphic and let X^* denote a one-point compactification of the space X. If Y^* is a one-point compactification of Y, then $X^* \cong Y^*$.

Problem 3.6. Show that the one-point compactification of $\mathbb{N}$ is homeomorphic to the subspace

$$\mathcal{Z} := \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subset \mathbb{R}.$$

Problem 3.7. Let X and Y be spaces, with Y Hausdorff. Prove that if $f, g : X \to Y$ are continuous, then the slice

$$\mathscr{D} := \{x \in X : f(x) = g(x)\}$$

is closed in X.

Problem 3.8. Prove Theorem 3.45.

Problem 3.9. For this exercise, we will use the Urysohn metrization theorem, which states the following:

Every Hausdorff second countable regular space is metrizable.

Let X be a compact Hausdorff space. Show that X is metrizable if and only if it is second countable.

Part II

Algebraic Topology

Chapter 4

Fundamentals of Algebraic Topology

This chapter will serve as our first glimpse into the world of algebraic topology. The concepts we will develop here are *far* different from those we have introduced in the previous chapters and will, of course, be less analytic in nature. Nonetheless, one cannot hope to understand this chapter without having read the previous ones. The general question to keep in mind when reading this part of the book is

> *How can one determine whether any two given topological spaces are homeomorphic?*

In general, this is a very difficult problem. Sometimes, it may be possible to find a homeomorphic invariant which one space possesses, but the other does not. Or, one can sometimes give an explicit example of a homeomorphism when the context is less abstract. For the most part however, these simplistic approaches will fail. To see this, one should ask themselves how one would go about showing that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$ using the elementary methods we have listed.

4.1 Homotopy of Paths

Let $(X, \mathfrak{T})$ be a topological space. A continuous function $f : [a, b] \subset \mathbb{R} \to X$ is called a *path* in X. The point $x_0 := f(a)$ is called the *initial point* and $x_1 := f(b)$ is the *final point*. One would then call f a path from x_0 to x_1. Since one can always find a homeomorphism $[a, b]$ to $[0, 1]$ for any $a < b$, it suffices to consider paths $f : [0, 1] \to X$. For this reason, we will always use $\mathbb{I}$ to denote the compact interval $[0, 1] \subset \mathbb{R}$.

Definition 4.1 (Homotopy)**.** Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces and suppose that $f, f' : X \to Y$ are continuous functions. We say that f and f' are *homotopic*, written $f \simeq f'$, if there exists a continuous function $F : X \times \mathbb{I} \to Y$ having the property that

$$F(x, 0) \equiv f(x) \quad \text{and} \quad f(x, 1) \equiv f'(x).$$

This function F is called a *homotopy* between f and f'. We say f is *nulhomotopic* if $f \simeq f'$ and f' is constant.

Remark 4.1. Let f and f' be continuous functions $X \to Y$. Suppose also that $F : X \times \mathbb{I} \to Y$ is a homotopy. We claim that the function

$$f_a : X \to Y, \quad f_a(x) := F(x, a)$$

is continuous for each *fixed* $a \in \mathbb{I}$. First, note that the function

$$G : X \to X \times \mathbb{I}, \quad x \mapsto (x, a)$$

is continuous. Since the function f_a can be obtained via the composition

$$X \xrightarrow{G} X \times \mathbb{I} \xrightarrow{F} Y,$$

the claim is proven.

One should think of f and f' being homotopic if f can be "continuously deformed" into the function f'. A stronger (and more useful) version of this definition exists when we are considering paths instead of general maps.

Definition 4.2 (Path Homotopy)**.** Let $(X, \mathfrak{T})$ be a topological space and suppose that f and f' are paths with initial point x_0 and final point x_1. We will say that f and f' are *path homotopic*, written $f \simeq_p f'$, if there exists a continuous function $F : \mathbb{I} \times \mathbb{I} \to X$ satisfying the following:

(1) $F(s, 0) \equiv f(s)$ and $F(s, 1) \equiv f'(s)$;

(2) $F(0, t) \equiv x_0$ and $F(1, t) \equiv x_1$;

for all $s, t \in \mathbb{I}$. The map F is called a *path homotopy* between f and f'. Note that it only makes sense to write $f \simeq_p f'$ for paths f and f' sharing the same initial and final points.

As a first step, we will prove that both '$\simeq$' and '$\simeq_p$' are equivalence relations. More formally, we are claiming the following:

(1) '$\simeq$' is an equivalence relation on the set of all continuous maps $X \to Y$;

(2) given any two points x_0 and x_1 in X, '$\simeq_p$' is an equivalence relation on the set of all paths in X with initial point x_0 and final point x_1.

Let $f : X \to Y$ be continuous and consider the function

$$F : X \times \mathbb{I} \to Y, \quad F(x,t) := f(x).$$

Then F is continuous and satisfies $F(x,0) = f(x)$ and $F(x,1) = f(x)$. This shows that $f \simeq f$. If f were a path, then we would have $X = \mathbb{I}$ whence the above also shows that $f \simeq_p f$. Suppose now that f and f' are continuous functions $X \to Y$. If $f \simeq f'$, then we can choose a homotopy F. It is then easy to see that the function

$$G(x,t) := F(x, 1-t)$$

is a homotopy between f' and f. This gives $f' \simeq f$. Suppose instead that f and f' are paths with $f \simeq_p f'$. Choose a path homotopy F between f and f'. Then,

$$G : \mathbb{I} \times \mathbb{I} \to X, \quad G(s,t) := G(s, 1-t)$$

will satisfy all the criteria of a path homotopy between f' and f. We conclude that $f' \simeq_p f$. Suppose now that $f \simeq f'$ and that $f' \simeq f''$, where these are continuous functions $X \to Y$. Let F be a homotopy between f and f' and let F' be a homotopy between f' and f''. Define

$$G : X \times \mathbb{I} \to Y, \quad G(x,t) := \begin{cases} F(x, 2t), & 0 \le t \le \frac{1}{2}, \\ F'(x, 2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Then, G is continuous by the pasting lemma and is clearly well defined since $F(x,1) \equiv f'(x) \equiv F'(x,0)$. It follows that $f \simeq f''$. In the case where f, f' and f'' are path homotopic, the function G as defined above also yields $f \simeq_p f''$.

If f is a path in X, we denote by $[f]$ its equivalence class modulo $\simeq_p$.

Definition 4.3. Let $(X, \mathfrak{T})$ be a topological space and $x_{0,1,2} \in X$. Let f be a path from x_0 to x_1 and g a path from x_1 to x_2. We define the "product" $f * g$ to be the path in X prescribed by the function

$$(f * g)(s) := \begin{cases} f(2s), & 0 \le s \le \frac{1}{2}, \\ g(2s-1), & \frac{1}{2} \le s \le 1; \end{cases}$$

for $s \in \mathbb{I}$.

Again, the pasting lemma ensures that the function $f * g$ is a continuous map $\mathbb{I} \to X$. As is evident from the definition, $f * g$ is a path from x_0 to x_2. In fact, one can view $f * g$ as the path obtained by "gluing together" f and g. The important part of this definition is that the product $*$ is well behaved with respect to equivalence classes $[f]$ modulo $\simeq_p$.

Lemma 4.1. *Let f be a path from x_0 to x_1 and g a path from x_1 to x_2. Then, the product*

$$[f] * [g] := [f * g]$$

is well defined. Moreover, the following properties hold true.

Associativity *If $[f] * ([g] * [h])$ is well defined, then it is equal to $([f] * [g]) * [h]$ (which is also well defined).*

Right and Left Identities *For a fixed $x \in X$, denote by e_x the path $\mathbb{I} \to X$ equal to x on all of $\mathbb{I}$. If f is a path from x_0 to x_1, there holds*

$$[f] * [e_{x_1}] = [f], \quad [e_{x_0}] * [f] = [f]. \tag{4.1}$$

Inversion *If f is a path in X from x_0 to x_1, let $\bar{f}$ be the path $\bar{f}(s) := f(1-s)$. The path $\bar{f}$ is called the **reverse** of f and satisfies*

$$[f] * \left[\bar{f}\right] = [e_{x_0}] \quad \text{and} \quad \left[\bar{f}\right] * [f] = [e_{x_1}]. \tag{4.2}$$

I have decided to omit the proof of this result in order to reach the fundamental group "quicker". The curious reader may refer to Theorem 51.2 in [MNKS] for the proof. However, the argument used there gives rise to an important result which we will recall below:

Theorem 4.2. *Let $(X, \mathfrak{T})$ be a space and $f : \mathbb{I} \to X$ a path. Suppose that*

$$0 = a_0 < a_1 < \cdots < a_n = 1$$

*are real numbers. For each index $j \in \{1, \ldots, n\}$ let $f_i : \mathbb{I} \to X$ be the path that equals the positive linear map of $\mathbb{I}$ **onto** $[a_{i-1}, a_i]$, followed by f. Then,*

$$[f] = [f_1] * \cdots * [f_n].$$

Remark 4.2. We would like to point out that the product $[f] * [g]$ is only defined for paths $f, g : \mathbb{I} \to X$ with the added property that $f(1) = g(0)$.

4.2 The Fundamental Group

In the previous section we introduced path homotopy and constructed path homotopy equivalence classes using $\simeq_p$. We further defined an operation $*$ on for equivalence classes $[f]$ and $[g]$ such that $f(1) = g(0)$. Since this operation $*$ is not defined for *all* equivalence classes, we cannot make the set of all path homotopy classes into a group using $*$. To move past this apparent difficulty, we introduce *loops*; these will allow us to construct the fundamental group.

Henceforth, we assume the reader is very familiar with group theory. Preferably, they will have already taken several courses in abstract algebra.

Definition 4.4 (Fundamental Group)**.** Let $(X, \mathfrak{T})$ be a topological space and fix a point x_0 in X. A *loop* based at x_0 is a path in X from x_0 to x_0. The set of path homotopy classes of loops based at x_0, equipped with the operation $*$, is called the *fundamental group* of X based at x_0. We denote this group by $\pi_1(X, x_0)$.[1]

Of course, we must verify that the family of path homotopy classes of loops at x_0, equipped with $*$, is indeed a group. Luckily, this is almost immediate from Lemma 4.1. For the moment, let $\pi_1(X, x_0)$ denote the set of path homotopy classes of loops at x_0. If f, g are two loops at x_0, we see from the definition that $f * g$ is again a loop at x_0. Thus, Lemma 4.1 shows that $[f] * [g]$ belongs to $\pi_1(X, x_0)$. Associativity then follows from this same lemma. Moreover, we get that $[e_{x_0}]$ is the identity element of $\pi_1(X, x_0)$. Given $[f] \in \pi_1(X, x_0)$, Lemma 4.1 also implies that $\left[\bar{f}\right]$ is the inverse of $[f]$ in $\pi_1(X, x_0)$.[2]

Example 4.1. Let $\mathbb{R}^n$ have the standard topology. If x_0 is any point in $\mathbb{R}^n$, then $\pi_1(X, x_0)$ is isomorphic to the trivial group. More generally, any convex subset of $\mathbb{R}^n$ will have a trivial fundamental group. This includes the unit ball in $\mathbb{R}^n$.

---

[1] The notation '$\pi_1(X, x_0)$' clearly suggests the existence of groups '$\pi_n(X, x_0)$' for $n > 1$. Of course, this is no mistake at all. For $n > 1$, we can define a *higher homotopy* group based at x_0. Let $\mathfrak{P}_n(x_0)$ be the set of all continuous functions $\mathbb{I}^n \to X$ which collapse $\partial\mathbb{I}$ onto the point x_0. For any two maps $f, g \in \mathfrak{P}_n(x_0)$, we will write $f \sim g$ provided there exists a homotopy $H : \mathbb{I}^n \times \mathbb{I} \to X$ such that each $H(\cdot, t)$ throws $\partial\mathbb{I}^n$ onto x_0. The higher homotopy "group" $\pi_n(X, x_0)$ is then defined to be $\mathfrak{P}_n(x_0)$ modulo the equivalence relation $\sim$.

[2] In much the same way, one gives $\pi_n(X, x_0)$ a group structure (for $n > 1$) by defining

$$[f] * [g] := [f * g]$$

for any two equivalence classes $[f], [g]$ in $\pi_n(X, x_0)$. We will not show that this is well defined, nor will we show that $\pi_n(X, x_0)$ is a group for every $n \in \mathbb{N}$. This set $\pi_n(X, x_0)$ is then called the *higher homotopy group* of degree n. An interesting fact is that $\pi_n(X, x_0)$ is always Abelian, for each $n > 1$.

To see this, let $\mathscr{C}$ be a non-empty convex subset of $\mathbb{R}^n$ and let f and g be two loops (contained in $\mathscr{C}$) about the point $x_0 \in \mathscr{C}$. Consider the function

$$F(s,t) := (1-t)f(s) + tg(s)$$

defined on $\mathbb{I} \times \mathbb{I}$. This is called the *straight-line homotopy* between f and g. It is easy to check that F is a path homotopy between f and g whose image lives in $\mathscr{C}$. Therefore, any loop in $\mathscr{C}$ at the point x_0 will be path homotopic to the constant loop at x_0. This argument can be easily extended to convex subsets of topological vector spaces (see Problem 4.13).

Definition 4.5. Let $(X, \mathfrak{T})$ be a space and α a path from x_0 to x_1. Define

$$\widehat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1), \quad [f] \mapsto [\bar{\alpha}] * [f] * [\alpha]. \tag{4.3}$$

We call this map "α-hat".

Since f is a loop around x and $\alpha(0) = x_0$, the first product $[f] * [\alpha] = [f * \alpha]$ is well defined and yields a path from x_0 to x_1. By definition of the reverse, $\bar{\alpha}$ is a path from x_1 to x_0. Since $f * \alpha$ is a path from x_0 to x_1, the same argument shows that $\bar{\alpha} * (f * \alpha)$ is a loop at x_1. Thus, $\widehat{\alpha}$ is well defined.

Theorem 4.3. *The map described in* (4.3) *is an isomorphism of groups.*

Proof. Let us first check that $\widehat{\alpha}$ is a group homomorphism. If $[f], [g] \in \pi_1(X, x_0)$ then, by Lemma 4.1,

$$\begin{aligned}
\widehat{\alpha}([f]) * \widehat{\alpha}([g]) &= ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha]) \\
&= ([\bar{\alpha}] * [f]) * [e_{x_0}] * ([g] * [\alpha]) \\
&= [\bar{\alpha}] * [f] * [g] * [\alpha] \\
&= \widehat{\alpha}([f] * [g]) .
\end{aligned}$$

When applying the lemma, we have used the fact that $g * \alpha$ is a path in X from x_0 to x_1. It remains to show that $\widehat{\alpha}$ is a bijection. To this end, let β denote the path $\bar{\alpha}$ (from x_1 to x_0). As per the definition above, β induces a map

$$\widehat{\beta} : \pi_1(X, x_1) \to \pi_1(X, x_0), \quad [h] \mapsto \left[\bar{\beta}\right] * [h] * [\beta] = [\alpha] * [h] * [\bar{\alpha}] .$$

Thus, an easy application of Lemma 4.1 shows that

$$\left(\widehat{\beta} \circ \widehat{\alpha}\right)([f]) = [\alpha] * ([\bar{\alpha}] * [f] * [\alpha]) * [\bar{\alpha}] = [f].$$

Similarly, one can show that $\widehat{\alpha} \circ \widehat{\beta}$ is the identity on $\pi_1(X, x_1)$. It follows that $\widehat{\alpha}$ is an isomorphism of groups. ◻

As an easy corollary, we have the following.

Corollary 4.4. *Let $(X, \mathfrak{T})$ be a path connected space. For any $x_0, x_1 \in X$, the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

If $(X, \mathfrak{T})$ is not path connected, then the fundamental group at a single point does not tell us much about the structure of X. Hence, we will mostly deal with path connected spaces within this section.

Remark 4.3. We have shown that in a path connected space, the fundamental groups based at any two points are isomorphic. It therefore makes sense to speak of *the* fundamental group of a path connected space (by looking at the fundamental group at any fixed point).

Definition 4.6. Let $(X, \mathfrak{T})$ be a path connected space. We say that X is *simply connected* if there exists a point $x_0 \in X$ such that $\pi_1(X, x_0)$ is isomorphic to the trivial group.

By Corollary 4.4, if X is a simply connected space, then the fundamental group at each point will be isomorphic to the trivial group. This fact gives us the following elegant result.

Proposition 4.5. *Let $(X, \mathfrak{T})$ be a simply connected space and fix $x_0, x_1 \in X$. Any two paths from x_0 to x_1 are path homotopic.*

Proof. Let $\alpha, \beta : \mathbb{I} \to X$ be two paths from x_0 to x_1. The reverse $\bar{\beta}$ is then a loop from x_1 to x_0. It follows that $\alpha * \bar{\beta}$ is a loop based at x_0. Since X is simply connected, we have $\pi_1(X, x_0) \cong \mathbf{0}$ where $\mathbf{0}$ is the trivial group. In particular, $\alpha * \bar{\beta}$ is path homotopic to the constant loop at x_0. Then,

$$[\alpha * \bar{\beta}] * [\beta] = [\alpha] * [e_{x_1}] = [\alpha],$$
$$[\alpha * \bar{\beta}] * [\beta] = [e_{x_0}] * [\beta] = [\beta].$$

It follows that $[\alpha] = [\beta]$, whence $\alpha \simeq_p \beta$. □

4.2.1 Induced Homomorphisms

Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces. Fix two points $x_0 \in X$ and $y_0 \in Y$. We will write $h : (X, x_0) \rightharpoonup (Y, y_0)$ is say that h is a continuous function $X \to Y$ having the property that $h(x_0) = y_0$.

Definition 4.7. Let $h : (X, x_0) \rightharpoonup (Y, y_0)$ be a map. Define

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [f] \mapsto [h \circ f]. \tag{4.4}$$

We call h_* the *homomorphism induced by* h at the base point x_0.

We make sure that h_* is well defined. First, suppose that f is a loop at the point x_0. The composite $h \circ f$ is thus a continuous function $\mathbb{I} \to Y$ satisfying $(h \circ f)(0) = (h \circ f)(1) = y_0$. That is, $h \circ f$ is a loop at y_0 in Y. We get that, $h_*([f]) \in \pi_1(Y, y_0)$. Suppose that $[f] = [f']$, i.e. $f \simeq_p f'$. We must show that $h \circ f \simeq_p h \circ f'$. Suppose that

$$F : \mathbb{I} \times \mathbb{I} \to X$$

is a path homotopy between f and f'. We claim that

$$G : \mathbb{I} \times \mathbb{I} \to Y, \quad G(s, t) := h(F(s, t))$$

is a path homotopy between $h \circ f$ and $h \circ f'$. Clearly, G is continuous and satisfies

$$G(s, 0) \equiv h(F(s, 0)) \equiv h(f(s)) \equiv (h \circ f)(s)$$

and

$$G(s, 1) \equiv h(F(s, 1)) \equiv h(f'(s)) \equiv (h \circ f')(s).$$

Moreover,

$$G(0, t) \equiv h(F(0, 1)) \equiv h(x_0) \equiv y_0, \quad G(1, t) \equiv h(F(1, t)) \equiv h(x_0) \equiv y_0.$$

It follows that $[h \circ f] = [h \circ f']$ whence h_* is well defined. It remains to check that h_* is a homomorphism of groups. This fact follows at once from the identity

$$(h \circ f) * (h \circ g) \equiv h \circ (f * g).$$

One need only look back to the definition of '$*$' to see that the above holds true.

Theorem 4.6. *Let X, Y and Z be topological spaces and consider the functions*

$$h : (X, x_0) \rightharpoonup (Y, y_0), \quad k : (Y, y_0) \rightharpoonup (Z, z_0).$$

Then, $(k \circ h)_* \equiv k_* \circ h_*$. Moreover, if $i : (X, x_0) \rightharpoonup (X, x_0)$ is the identity map, then i_* is the identity homomorphism on $\pi_1(X, x_0)$.

Proof. The proof is mostly trivial. For a class $[f]$ in $\pi_1(X, x_0)$ we compute

$$(k_* \circ h_*)([f]) = k_*([h \circ f]) = [k \circ h \circ f],$$
$$(k \circ h)_*([f]) = [(k \circ h) \circ f].$$

Hence, the first part holds. The second part follows from the equality $i \circ f \equiv f$, for all functions $f : X \to X$. □

We conclude with the following "sanity check".

Corollary 4.7. *Suppose $h : (X, x_0) \rightharpoonup (Y, y_0)$ is also a homeomorphism is the topological spaces X and Y. Then,*

$$h_* : \pi_1(X, x_0) \longrightarrow \pi(Y, y_0), \quad [f] \mapsto [h \circ f]$$

is an isomorphism of groups. Especially, $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

Proof. Let k denote the inverse of h. Thus, k is a map $(Y, y_0) \rightharpoonup (X, x_0)$ satisfying

$$h_* \circ k_* \equiv (h \circ k)_* \equiv (i_Y)_*, \quad k_* \circ h_* \equiv (k \circ h)_* \equiv (i_X)_*,$$

where i_X and i_Y are the identity maps $X \to X$ and $Y \to Y$, respectively. Since $(i_X)_*$ and $(i_Y)_*$ are the identity maps of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, the claim follows. □

This corollary tells us that the fundamental group (considered up to isomorphism) is a *homeomorphic invariant.* Especially, when X and Y are path connected, the fundamental groups are isomorphic at all points.

4.3 Covering Spaces

The computation of fundamental groups is not always trivial, as was the case in Example 4.1. A useful tool for computing the fundamental groups is the notion of a *covering space*, which we introduce in this section. These are also fundamental to Riemannian geometry.

Throughout this section, E and B denote abstract non-empty sets with the implicit topologies $\mathfrak{E}$ and $\mathfrak{B}$, respectively. On the other hand, one could just as well choose to view them as informal symbols with unmentioned topologies. Also, let us recall that the notation $\bigsqcup$, rather than $\bigcup$, suggests that the elements in the union are pairwise disjoint. This notation will be particularly useful within this section.

Definition 4.8. Let $\rho : E \twoheadrightarrow B$ be a continuous surjection. Let U be an open subset of B. We say that U is *evenly covered* by ρ if $\rho^{-1}(U)$ can be written as the disjoint union of open sets V_α, for each α having the property that $p|_{V_\alpha}$ is a homeomorphism $V_\alpha \to U$. We call the V_α *slices* of $\rho^{-1}(U)$.

Definition 4.9. Let $\rho : E \twoheadrightarrow B$ be both continuous and surjective. If every $b \in B$ has a neighbourhood U that is evenly covered by ρ, then ρ is referred to as a *covering map*. We then call E a *covering space* of B.

We will now make several important remarks regarding the analytic nature of covering maps and spaces. Despite not being labeled as theorems or lemmas, these should not be glossed over and are essential things to remember.

Remark 4.4. Let $\rho : E \twoheadrightarrow B$ be a covering map and fix $b \in B$. The fiber $\rho^{-1}(\{b\})$ inherits the discrete topology from E. Indeed, let U be a neighbourhood of the point b that is *evenly covered* by ρ. Then, $U = \bigsqcup_\alpha V_\alpha$ for disjoint open sets $V_\alpha \subseteq E$ with the property that, for each index α,

$$\rho_\alpha := \rho\,|_{V_\alpha} \colon V_\alpha \longrightarrow U$$

is a homeomorphism. Then, one has

$$\rho^{-1}(\{b\}) = \bigsqcup_\alpha V_\alpha \cap \rho^{-1}(\{b\}).$$

Since every ρ_α is a homeomorphism, there is precisely one point $x_{b,\alpha}$ in each V_α with the property that $\rho(x_{b,\alpha}) = b$. In particular,

$$\rho^{-1}(\{b\}) = \bigsqcup_\alpha \{x_{b,\alpha}\}.$$

Thus, every element of $\rho^{-1}(\{b\})$ corresponds to one of these points. Finally, every singleton $\{x_{b,\alpha}\}$ is open in $\rho^{-1}(\{b\})$ since every V_α is open in E.

Remark 4.5. If $\rho : E \twoheadrightarrow B$ is a covering map, then it is also an *open map*. Fix an open set $O \subseteq E$ and consider $\rho(O) \subseteq B$. Let $b \in \rho(O)$ and choose a neighbourhood U of b that is evenly covered by ρ. Then,

$$\rho^{-1}(U) = \bigsqcup_{\alpha \in I} V_\alpha$$

for disjoint open sets $V_\alpha \subseteq E$ with the property that $\rho|_{V_\alpha} \colon V_\alpha \to U$ is a homeomorphism. Choose $e \in O$ such that $\rho(e) = b$. Since $e \in \rho^{-1}(U)$, there exists a unique index α such that $e \in V_\alpha$. Then, $O \cap V_\alpha$ is a non-empty open subset of V_α. Since ρ is a homeomorphism, $\rho(O \cap V_\alpha) \subseteq \rho(O)$ is open in B. Since $\rho(O \cap V_\alpha)$ contains b, we conclude that $b \in \operatorname{Int}(\rho(O))$. This yields the desired result.

Theorem 4.8. *The map*

$$\rho : \mathbb{R} \longrightarrow \mathbb{S}^1, \quad x \mapsto (\cos 2\pi x, \sin 2\pi x) \tag{4.5}$$

is a covering map.

Proof Sketch. Let $U \subset \mathbb{S}^1$ the the subset of $\mathbb{R}^2$ consisting of all points having a *positive* first coordinate. Then,

$$U = \mathbb{S}^1 \cap \left\{(x, y) \in \mathbb{R}^2 : x > 0\right\}$$

is an open subset of $\mathbb{S}^1$. Then,

$$\rho^{-1}(U) = \{x \in \mathbb{R} : \cos 2\pi x > 0\} = \bigsqcup_{n\in\mathbb{Z}} \left(n - \frac{1}{4}, n + \frac{1}{4}\right).$$

Fix now an index $n \in \mathbb{Z}$ and let ϱ denote the restriction of ρ to the compact interval $\left[n - \frac{1}{4}, n + \frac{1}{4}\right]$. Due to the strict monotonicity of $\sin 2\pi x$ on this interval, we see that ϱ is injective. It is also easy to see that the open interval is mapped onto U. Actually, we see that the closed interval is mapped onto $\operatorname{Cl}(U)$. Since $\mathbb{R}^2$ is Hausdorff, we conclude that ϱ is a homeomorphism. One can repeat this procedure for all the other half planes. ◻

Covering spaces and maps interact nicely with subspaces. This is demonstrated with the following theorem.

Theorem 4.9. *Let $\rho : E \twoheadrightarrow B$ be a covering map and B_0 a subspace of B. Define $E_0 := \rho^{-1}(B_0)$ and let*

$$\varrho : E_0 \longrightarrow B_0, \quad \varrho := \rho\,|_{E_0}$$

be the restriction of ρ to E_0. Then, ϱ is a covering map.

Proof. First, it is obvious that ϱ is continuous and surjective. Let $b \in B_0 \subseteq B$ be given. Choose a neighbourhood U *in* B of b that is evenly covered by ρ. Then, $\rho^{-1}(U)$ can be written as the disjoint union $\bigsqcup_\alpha V_\alpha$ for open subsets V_α of E, each mapped homeomorphically onto U by ρ. Clearly, $U \cap B_0$ is an open set in B_0 containing b. It is easy to check that

$$\rho^{-1}(B_0 \cap U) = \bigsqcup_\alpha (V_\alpha \cap E_0).$$

Since ρ is a homeomorphism $V_\alpha \to U$, we also get that $\varrho : V_\alpha \cap E_0 \to U \cap B$ is a homeomorphism. ◻

An equally elegant result holds for the product of two covering spaces.

Proposition 4.10. *Let $\rho : E \twoheadrightarrow B$ and $\rho' : E' \twoheadrightarrow B'$ be covering maps. The function*

$$\sigma : E \times E' \longrightarrow B \times B', \quad (e, e') \mapsto (\rho(e), \rho'(e'))$$

is also a covering map.

Proof. Clearly, ρ is both surjective and continuous. Fix $(b, b') \in B \times B'$. Choose evenly covered neighbourhoods $U \subseteq B$ and $U' \subseteq B'$ of b and b', respectively. Let $\{V_\alpha\}$ and $\left\{V'_\beta\right\}$ be open subsets of E and E' such that

$$\rho^{-1}(U) = \bigsqcup_\alpha V_\alpha \quad \text{and} \quad (\rho')^{-1}(U') = \bigsqcup_\beta V'_\beta$$

and $\rho : V_\alpha \to B$, $\rho' : V'_\beta \to B'$ are homeomorphisms. It is not hard to see that

$$\sigma^{-1}(U \times U') = \bigsqcup_{\alpha,\beta} (V_\alpha \times V'_\beta)$$

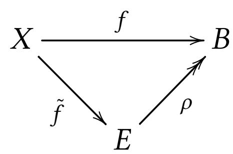
Finally, observe that σ is a homeomorphism of each $V_\alpha \times V'_\beta$ onto $U \times U'$. This completes the proof. ◻

4.4 Liftings and the Fundamental Group of $\mathbb{S}^1$

In this section, we discuss *liftings* of functions between spaces. More precisely, we begin by considering a covering map $\rho : E \twoheadrightarrow B$ and we fix a continuous function $f : X \to B$, where X is some given topological space. We then ask ourselves when the function f can be "extended" to a function $\tilde{f} : X \to E$. Informally, we would like to find a continuous function $\tilde{f}$, having a larger image than f, from which we can recover the original map f. Finally, we apply these new concepts to explicitly compute the fundamental group of the circle.

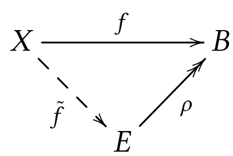
Definition 4.10 (Liftings)**.** Let $(X, \mathfrak{T})$ be a topological space and $\rho : E \twoheadrightarrow B$ a covering map. Given a continuous function $f : X \to B$, we say that a map

$\tilde{f} : X \to E$ is a *lifting* of f provided it satisfies the commutative diagram:



(4.6)

In particular, $\rho \circ \tilde{f} \equiv f$. To emphasize that the existence of a particular lifting $\tilde{f}$ has not yet been established, we might instead write



(4.7)

We now state two lemmas, without proof. Again, rigorous proofs of these results can be found in §54 of [MNKS].

Lemma 4.11 (First Lifting Lemma)**.** *Let $\rho : E \twoheadrightarrow B$ be a covering map, fix a point $b_0 \in B$ and let f be a path in B beginning at b_0. For any $e_0 \in \rho^{-1}(b_0)$, there exists a unique lifting $\tilde{f} : \mathbb{I} \to E$ of f to a path in E beginning at e_0.*

Lemma 4.12 (Second Lifting Lemma)**.** *Let $\rho : E \twoheadrightarrow B$ be a covering map, fix $b_0 \in B$, and pick $e_0 \in \rho^{-1}(b_0)$. Let $F : \mathbb{I} \times \mathbb{I} \to B$ be a continuous function satisfying $F(0,0) = b_0$. There exists a unique continuous lifting*

$$\tilde{F} : \mathbb{I} \times \mathbb{I} \to E$$

of F such that $\tilde{F}(0,0) = e_0$. If F happens to be a path homotopy, then so is $\tilde{F}$.

Theorem 4.13. *Let $\rho : E \twoheadrightarrow B$ be a covering map and suppose $\rho(e_0) = b_0$, for fixed $e_0 \in E$ and $b_0 \in B$. Let f, g be paths in B, both from b_0 to b_1. Let $\tilde{f}$ and $\tilde{g}$ denote their respective (unique) liftings to paths in E beginning at e_0. If $f \simeq_p g$, then $\tilde{f} \simeq_p \tilde{g}$. In particular, $\tilde{f}$ and $\tilde{g}$ have the same endpoint.*

Proof. Let $F : \mathbb{I} \times \mathbb{I} \to B$ be a path homotopy between f and g in B. Let $\tilde{F}$ be the continuous lifting of F guaranteed by virtue of the second lifting lemma. Then, $\tilde{F}(0,0) = e_0$. Now, consider the restriction $\tilde{F}|_{\mathbb{I}\times\{0\}}$, which is easily seen to be a lifting of f. Since $\tilde{F}(0,0) = e_0$, the uniqueness part of the first lifting lemma tells us that

$$\tilde{F}|_{\mathbb{I}\times\{0\}} \equiv \tilde{f}.$$

Similarly, one can show that $\tilde{F}|_{\mathbb{I}\times\{1\}}$ is a lifting of g. Since $\tilde{F}$ is a path homotopy (by the preceding lemma), we know that $\tilde{F}(0,1) = \tilde{F}(0,0) = e_0$. Thus, by uniqueness, $\tilde{F}|_{I\times\{1\}} \equiv \tilde{g}$. With this, the proof is complete. □

Having established this theorem, we go on to give a fundamental definition regarding liftings and covering spaces.

Definition 4.11. Let $\rho : E \twoheadrightarrow B$ be a covering map and fix a point $b_0 \in B$. Let e_0 be any point belonging to the fiber $\rho^{-1}(b_0)$. The *lifting correspondence* from ρ at e_0 is defined as the map

$$\phi = \phi_{e_0} : \pi_1(B, b_0) \longrightarrow \rho^{-1}(b_0), \quad [f] \mapsto \tilde{f}(1),$$

where $\tilde{f}$ is the *unique* lift of f to E beginning at e_0. Note that ϕ is well defined since, by the previous theorem, the lifting of any two path homotopic paths will also be path homotopic. Moreover, if f is a loop at b_0 and $\tilde{f}$ is a lifting to E, then

$$\rho \circ \tilde{f}(1) = f(1) = b_0.$$

That is, $\tilde{f}(1) \in \rho^{-1}(b_0)$. Thus, we see that ϕ is well defined. Note that ϕ *does* depend on the choice of e_0.

Theorem 4.14. *Let $\rho : E \twoheadrightarrow B$ be a covering map and suppose that e_0 belongs to the fiber $\rho^{-1}(b_0)$, for a given $b_0 \in B$. If E is path connected, then the lifting correspondence*

$$\phi : \pi_1(B, b_0) \longrightarrow \rho^{-1}(b_0)$$

is surjective. If E is simply connected, then ϕ is a bijection.

Proof. First assume that E is path connected. Let e be a point belonging to the fiber $\rho^{-1}(b_0)$ and consider a path $\gamma : \mathbb{I} \to E$ beginning at e_0 and ending at e. Then, the function

$$\delta(s) := (\rho \circ \gamma)(s)$$

describes a *loop* in B at b_0. Note that γ must be the unique lifting, beginning at e_0, of δ to E. Therefore,

$$\phi([\delta]) = \gamma(1) = e.$$

Hence, ϕ is surjective. In addition, let us suppose that E is simply connected so that $\pi_1(E, e) \cong \mathbf{0}$ at all points $e \in E$. Suppose that $\phi([f]) = \phi([g])$ for two loops f and g in B, based at b_0. It is enough to check that $f \simeq_p g$. To this end, let $\tilde{f}$ and $\tilde{g}$ be the unique liftings of f and g, respectively, starting at e_0. By the hypothesis

$\phi([f]) = \phi([g])$, $\tilde{f}$ and $\tilde{g}$ must also share the same endpoint. Since E is simply connected, this yields $\tilde{f} \simeq_p \tilde{g}$. That is, we can find a path homotopy

$$F : \mathbb{I} \times \mathbb{I} \to E$$

between $\tilde{f}$ and $\tilde{g}$. Consider the continuous map

$$G : \mathbb{I} \times I \to B, \quad (s,t) \mapsto \rho(F(s,t)).$$

Clearly, $G(s,0) = \rho(F(s,0)) = \rho(\tilde{f}(s)) = f(s)$ and, similarly, $G(s,1) = g(s)$ for all $s \in \mathbb{I}$. Finally, we observe that

$$G(0,t) = \rho(F(0,t)) = \rho(e_0) = b_0$$

and

$$G(1,t) = \rho(F(1,t)) = \rho(\tilde{f}(1)) = f(1) = b_0.$$

As was required, $f \simeq_p g$. □

Before the main result of this section, we require one final lemma.

Lemma 4.15. *Let $\rho : E \twoheadrightarrow B$ be a covering map and let $e_0 \in \rho^{-1}(b_0)$, for some $b_0 \in B$. Let $f : \mathbb{I} \to B$ be a path beginning at b_0 and g a path in B beginning at $f(1)$. Let $\tilde{f}$ be the unique lifting of f to E beginning at e_0, and let $\tilde{g}$ be the unique lifting of g to E beginning at $\tilde{f}(1)$. Then,*

$$\tilde{f} * \tilde{g} : \mathbb{I} \to E$$

is a lifting of $f * g$ beginning at e_0.

Proof. First, note that $\rho(\tilde{f}(1)) = f(1) = g(0)$. Therefore, $\tilde{f}(1) \in \rho^{-1}(g(0))$ and we can indeed choose a lifting $\tilde{g}$ beginning at $\tilde{f}(1)$. Now, we consider the product

$$\tilde{f} * \tilde{g},$$

which is a well defined path $\mathbb{I} \to E$ beginning at $\tilde{f}(0) = e_0$. All that remains is to check that

$$\rho \circ \left(\tilde{f} * \tilde{g}\right) \equiv f * g.$$

However, this is readily verified by direct calculation:

$$\begin{aligned}\rho \circ \left(\tilde{f} * \tilde{g}\right)(s) &= \begin{cases} \rho(\tilde{f}(2s)) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \rho(\tilde{g}(2s-1)) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \\ &= \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \\ &= (f * g)(s).\end{aligned}$$

□

With this, we can now compute the fundamental group of $\mathbb{S}^1$.

Theorem 4.16. *At every point, the fundamental group of $\mathbb{S}^1$ is isomorphic to the additive cyclic group $\mathbb{Z}$.*

Proof. Clearly, $\mathbb{S}^1$ is path connected and thus it suffices to compute the fundamental group at a convenient point. Let ρ be the covering map $\mathbb{R} \to \mathbb{S}^1$ discussed in the previous section and consider the point $s_0 := \rho(0)$ on $\mathbb{S}^1$. Then,

$$\rho^{-1}(s_0) = \mathbb{Z}.$$

Consider the lifting correspondence

$$\phi : \pi_1(\mathbb{S}^1, s_0) \to \rho^{-1}(s_0) = \mathbb{Z}.$$

Since E is simply connected, the previous theorem ensures that ϕ is bijective. We will therefore be done if we can show that ϕ is a homomorphism of groups. Let $[f], [g]$ be elements of $\pi_1(\mathbb{S}^1, s_0)$ and note that

$$\phi([f]) = \tilde{f}(1) =: n, \quad \phi([g]) = \tilde{g}(1) =: m,$$

where $\tilde{f}$ and $\tilde{g}$ are the unique liftings of f and g, beginning at 0. Define

$$\hat{g}(s) := n + \tilde{g}(s);$$

then $\hat{g}$ is a valid path $\mathbb{I} \to \mathbb{R}$ beginning at $n = \tilde{f}(1)$. Moreover, a direct calculation verifies that

$$\rho(\hat{g}) \equiv \rho(n + \tilde{g}) \equiv \rho(\tilde{g}) = g.$$

This means that $\hat{g}$ is the unique lifting of g beginning at $n = \tilde{f}(1)$. By the previous lemma, $\tilde{f} * \hat{g}$ will be the unique lifting of $f * g$, to E, beginning at 0. Hence,

$$\phi([f] * [g]) = \tilde{f} * \hat{g}(1) = n + \tilde{g}(1) = n + m.$$

With this, we see that ϕ is a homomorphism of groups. □

4.5 Brouwer's Fixed Point Theorem

In courses studying complete metric spaces, one often proves the *Banach fixed point theorem*, which ensures that any contractive mapping Φ of a complete metric space X has a fixed point, i.e. a point x such that $\Phi(x) = x$. In fact, this point will be unique. In this section, will prove an alternate (but still *very* useful) fixed point theorem. This result is a special case of the infamous *Brouwer fixed point theorem*:

Theorem 4.17 (Brouwer's Fixed Point Theorem for $\mathbb{B}^2$)**.** *Let $f : \mathbb{B}^2 \to \mathbb{B}^2$ be continuous. There exists a point $x \in \mathbb{B}^2$ such that $f(x) = x$.*

For the sake of clarity, we should also recall that $\mathbb{S}^{n-1}$ denotes the n-sphere

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

That is, $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ for all $n \geq 2$. When working in two dimensions it is often more convenient to work in a *field*, rather than merely a vector space. For this reason, we will assume that $\mathbb{B}^2$ is the **closed** unit disk in $\mathbb{C}$. On the other hand, for general $n \geq 2$, $\mathbb{B}^n$ will denote the closed unit disk in $\mathbb{R}^n$. Of course, there is no discrepancy here: the closed unit disk in $\mathbb{C}$ is homeomorphic to that in $\mathbb{R}^2$.

It will be a considerable effort to prove the Brouwer fixed point theorem, and before we embark upon this arduous journey we would like to point out an important consequence.

Corollary 4.18. *Let Ω be any subset of $\mathbb{R}^n$ that is homeomorphic to $\mathbb{B}^2$ and let $f : \Omega \to \Omega$ be continuous. There exists a point $x \in \Omega$ such that $f(x) = x$.*

Proof. Let $\varphi : \mathbb{B}^2 \to \Omega$ be a homeomorphism (e.g. a dilation). Then, the function

$$\psi : \mathbb{B}^2 \to \mathbb{B}^2, \quad \psi := \varphi^{-1} \circ f \circ \varphi$$

is continuous. By Brouwer's fixed point theorem, ψ has a fixed point $x \in \mathbb{B}^2$. That is,

$$\psi(x) = \varphi^{-1}(f(\varphi(x))) = x.$$

Or, rather, $f(\varphi(x)) = \varphi(x)$. It follows that $\varphi(x)$ is a fixed point for the function f. The claim then follows. □

This corollary shows that we can without harm choose to work in the closed unit disk $\mathbb{B}^2$. Let us now formalize a concept that was briefly introduced within the exercises.

Definition 4.12. Let $(X, \mathfrak{T})$ be a topological space and assume A is a subspace of X. We say that A is a retract of the space X if there exists a continuous function $r : X \to A$ with the property that $r|_A \equiv \mathbf{1}_A$. Such a function is then said to be a *retraction of X onto A.*[3]

[3]Sometimes this terminology can be hard to remember. I like to think of a retraction as a map that yields a retract. Thus, the retract should be the image under the retraction. This partially explains why A (and not X) is called the retract!

Henceforth, we will assume that we have computed the fundamental group of $\mathbb{S}^n$, for $n \geq 2$. Of course, this is not a simple process and is highly non-trivial. However, we are only currently interested in consequences of this fact.

Theorem 4.19. *For all $n \geq 2$, the sphere $\mathbb{S}^n$ is simply connected.*

We begin with the following.

Proposition 4.20. *Let $(X, \mathfrak{T})$ be a topological space and suppose A is a retract of X. At every point $a \in A$, the inclusion map $j : A \hookrightarrow X$ induces an injective homomorphism of fundamental groups:*

$$j_* : \pi_1(A, a) \hookrightarrow \pi_1(X, a), \quad [f] \mapsto [j \circ f]. \tag{4.8}$$

In particular, $\pi_1(A, a)$ is isomorphic to a subgroup of $\pi_1(X, a)$.

Proof. Let $a \in A$ be given and let $r : X \to A$ be a retraction of X onto A. Notice that r is continuous and surjective. Now, j induces a natural homomorphism j_* according to (4.8). Since r is a continuous function fixing a, r also induces a homomorphism of fundamental groups:

$$r_* : \pi_1(X, a) \to \pi_1(A, a), \quad [f] \mapsto [r \circ f].$$

Thus, the composite $r_* \circ j_*$ is a map $\pi_1(A, a) \to \pi_1(A, a)$. We claim that $r_* \circ j_*$ is the identity map $\mathbf{1}_{\pi_1(A,a)}$, whence it would follow that j_* is injective. If $[f]$ is an equivalence class in $\pi_1(A, a)$ we see that

$$(r_* \circ j_*)([f]) = r_* ([j \circ f]) = [r \circ j \circ f].$$

However, $r \circ j$ is the identity $\mathbf{1}_A$. It follows that $(r_* \circ j_*)([f]) = [f] \in \pi_1(A, a)$. This completes the proof. □

Corollary 4.21. *$\mathbb{S}^1$ is not a retract of $\mathbb{B}^2$. More generally, let $\mathscr{C}$ be a convex subset of $\mathbb{C}$ containing $\mathbb{S}^1$. Then, $\mathbb{S}^1$ is not a retract of $\mathscr{C}$.*

Proof. We argue by contradiction. Let $\mathscr{C} \subseteq \mathbb{C}$ be a convex set containing $\mathbb{S}^1$ and suppose that $r : \mathscr{C} \to \mathbb{S}^1$ is a retraction map. Denote by j the inclusion map $\mathbb{S}^1 \hookrightarrow \mathscr{C}$. Proposition 4.20 tells us that the induced homomorphism

$$j_* : \pi_1(\mathbb{S}^1, x_0) \to \pi_1(\mathscr{C}, x_0)$$

is injective at all points $x_0 \in \mathbb{S}^1$. However, this would mean that $\mathbb{Z}$ is isomorphic to a subgroup of the trivial group $\mathbf{0}$. □

We continue with some of these surprising observations. The following lemma is geometrically intuitive, but would be *very* difficult to prove without invoking the deep ideas we have developed thus far.

Lemma 4.22. *Let $(X, \mathfrak{T})$ be a topological space and let $h : \mathbb{S}^{n-1} \to X$ be continuous. If h is nulhomotopic, then*

(1) h extends to a continuous map $k : \mathbb{B}^n \to X$;

(2) at each point of $\mathbb{S}^{n-1}$, the induced map h_* is the trivial homomorphism of fundamental groups.

Conversely, if h admits a continuous extension $k : \mathbb{B}^n \to X$ then h is nulhomotopic.

Proof. Suppose that $h : \mathbb{S}^{n-1} \to X$ is continuous and nulhomotopic. Let

$$H : \mathbb{S}^{n-1} \times \mathbb{I} \longrightarrow X$$

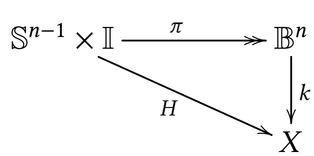
be a homotopy between h and a constant map $\mathbb{S}^{n-1} \to X$. Consider the following continuous function

$$\pi : \mathbb{S}^{n-1} \times \mathbb{I} \longrightarrow \mathbb{B}^n, \quad \pi(x, t) := (1 - t)x.$$

It is not difficult to see that π is surjective. Since $\mathbb{S}^{n-1} \times \mathbb{I}$ is compact[4] and $\mathbb{B}^n$ is Hausdorff, π is a closed map. Putting all these facts together implies that π is a *quotient map*. Note that π is injective on $\mathbb{S}^{n-1} \times [0, 1)$ and $\ker \pi = \mathbb{S}^{n-1} \times \{1\}$. Thus, for every $x \in \mathbb{B}^n$,

$$\pi^{-1}(\{x\}) = \begin{cases} \mathbb{S}^{n-1} \times \{1\}, & \text{if } x = 0, \\ \text{a singleton}, & \text{if } x \neq 0. \end{cases}$$

Since $H(\cdot, 1)$ is a constant map $\mathbb{S}^1 \to X$, we see that H is constant on every fiber $\pi^{-1}(\{x\})$. Applying Theorem 2.36 from Chapter 2 guarantees the existence of a continuous function $k : \mathbb{B}^n \to X$ with the property that $H \equiv k \circ \pi$ on $\mathbb{S}^{n-1} \times \mathbb{I}$. In the form of a diagram:



[4]Recall that finite products of compact metric spaces are compact!

It remains to check that $k \equiv h$ on $\mathbb{S}^{n-1}$. If $x \in \mathbb{S}^{n-1}$, we have $x = \pi(x, 0)$ so that $k(x) = k(\pi(x, 0)) = H(x, 0) = h(x)$.

Now, fix a point $x_0 \in \mathbb{S}^{n-1}$. We know that h induces a group homomorphism

$$h_* : \pi_1(\mathbb{S}^{n-1}, x_0) \longrightarrow \pi_1(X, h(x_0)).$$

Let $j : \mathbb{S}^{n-1} \hookrightarrow \mathbb{B}^n$ be the standard inclusion map and denote by j_* the induced homomorphism of fundamental groups $\pi_1(\mathbb{S}^{n-1}, x_0) \to \pi_1(\mathbb{B}^n, x_0)$. Since $\mathbb{B}^n$ is simply connected, j_* is necessarily trivial. Notice that $h \equiv k \circ j$ on $\mathbb{S}^{n-1}$; this implies that $h_* \equiv (k \circ j)_* \equiv k_* \circ j_*$ is trivial.

We now show the partial converse. Suppose that $h : \mathbb{S}^{n-1} \to X$ is a continuous map that admits a continuous extension $k : \mathbb{B}^n \to X$. Then, define

$$G : \mathbb{S}^{n-1} \times \mathbb{I} \longrightarrow X, \quad (x, t) \mapsto k((1-t)x).$$

Since k is continuous and defined on $\mathbb{B}^n$, this function is a homotopy between $G(x, 0) \equiv k(x) \equiv h(x)$ and $G(x, 1) \equiv k(0)$. Hence, h is nulhomotopic. □

Corollary 4.23. *The inclusion map $j : \mathbb{S}^1 \hookrightarrow \mathbb{C}^\times$ is not nulhomotopic. Similarly, the identity map $\mathbf{1}_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ is not nulhomotopic.*

Proof. First, notice that the map $r(x) := \frac{x}{|x|}$ is a retraction $\mathbb{C}^\times \to \mathbb{S}^1$. By Proposition 4.20, the inclusion map j induces an injective group homomorphism

$$j_* : \pi_1(\mathbb{S}^1, x_0) \hookrightarrow \pi_1(\mathbb{C}^\times, x_0)$$

at every point x_0 of $\mathbb{S}^1$. Since $\pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z}$ at each point, we see that j_* is never trivial. The previous lemma then tells us that j is not nulhomotopic. Similarly, $\mathbf{1}_{\mathbb{S}^1}$ induces a non-trivial homomorphism of fundamental groups at each point x_0 of $\mathbb{S}^1$. Hence, it cannot be nulhomotopic. □

Remark 4.6. We have shown above that $\mathbb{S}^1$ is a retract of $\mathbb{C}^\times$. For every $x_0 \in \mathbb{S}^1$, Proposition 4.20 implies that $\pi_1(\mathbb{C}^\times, x_0)$ contains an isomorphic copy of $\mathbb{Z}$. In particular, $\mathbb{C}^\times$ is *not* simply connected. Furthermore, since $\mathbb{C}^\times$ is homeomorphic to $\mathbb{R}^2 \setminus \{0\}$, an application of Corollary 4.7 shows that $\mathbb{R}^2 \setminus \{0\}$ is also not simply connected. In fact, we will shortly see that $\mathbb{C}^\times$ (and hence any punctured infinite plane) has fundamental group isomorphic to $\mathbb{Z}$.

We now give way to an unfortunately named theorem.

Theorem 4.24 (Hairy Ball Theorem). *Let $v : \mathbb{B}^2 \to \mathbb{C}$ be a continuous and nowhere vanishing. There exists a point on $\mathbb{S}^1$ where v points directly inwards, and another such point where v points directly outwards.*

Proof. Suppose that $v(x)$ does not point directly inwards at any point on $\mathbb{S}^1$. Let w denote the restriction of v to $\mathbb{S}^1$. Since v is a continuous extension of w, the previous lemma tells us that w is nulhomotopic. On the other hand, consider the inclusion map

$$j : \mathbb{S}^1 \hookrightarrow \mathbb{C}^\times, \quad x \mapsto x.$$

This gives rise to a continuous map (and a homotopy)

$$F : \mathbb{S}^1 \times \mathbb{I} \to \mathbb{C}^\times, \quad F(x,t) := tx + (1-t)w(x).$$

To see that the above is well defined, we need only ensure that $F(x,t)$ is never 0. If $t = 0, 1$, this is obvious. Suppose that $F(x,t) = 0$ for some $t \in (0,1)$; this means that $w(x)$ equals a negative scalar multiple of x, thereby contradicting the fact that v (and hence w) never points inwards on $\mathbb{S}^1$. It follows that F never vanishes. This means that $j \simeq w$ and, especially, that j is nulhomotopic. This contradicts the previous corollary. To see that $v(x)$ must point directly outwards at some point on $\mathbb{S}^1$, simply apply the theorem to $-v(x)$. □

We now come to the following promised result.

Proof of Brouwer's Fixed Point Theorem. We argue by contradiction. Suppose that the continuous function $v(x) := f(x) - x$ is non-vanishing on $\mathbb{B}^2$. By the previous theorem, there exists a point $x \in \mathbb{S}^1$ where $v(x)$ points directly outwards. That is, there exists $\xi > 0$ such that

$$v(x) = f(x) - x = \xi x.$$

Hence, we have $f(x) = (1 + \xi)x$ which lies *outside* the unit ball. □

4.6 Exercises

Unless otherwise specified, we will use X, Y and Z to denote arbitrary topological spaces.

Problem 4.1. Let $h, h' : X \to Y$ and $k, k' : Y \to Z$ be continuous functions. Suppose that $h \simeq h'$ and $k \simeq k'$. Prove that $k \circ h$ and $k' \circ h'$ are homotopic.

Problem 4.2. A topological space $(X, \mathfrak{T})$ is called contractible if the identity map $\mathbf{1}_X : X \to X$ is nulhomotopic.

(i) Prove that $\mathbb{R}$ and $\mathbb{I}$ are contractible.

(ii) Show that a contractible space is path connected.

(iii) If X is an arbitrary space and Y is contractible, show that any two continuous functions $X \to Y$ are homotopic.

(iv) If X is contractible and Y is path connected, prove that any two continuous functions $X \to Y$ are homotopic.

Problem 4.3. Let α, β be paths in X from x_0 to x_1 and from x_1 to x_2, respectively. Define $\gamma := \alpha * \beta$; prove that $\hat{\gamma} \equiv \hat{\beta} \circ \hat{\alpha}$.

Problem 4.4. Let X be path connected and fix points $x_0, x_1 \in X$. Prove that $\pi_1(X, x_0)$ is Abelian if and only if for every two paths α, β from x_0 to x_1 one has $\hat{\alpha} \equiv \hat{\beta}$.

Problem 4.5. Let A be a subspace of X and $r : X \to A$ a continuous map whose restriction to A is the identity map (i.e. a retraction map of X onto A). For every $a_0 \in A$ show that the induced homomorphism of fundamental groups,

$$r_* : \pi_1(X, a_0) \longrightarrow \pi_1(A, a_0),$$

is surjective.

Problem 4.6. Suppose that Y has the discrete topology and let $\pi : X \times Y \to X$ be the projection onto the first coordinate. Show that π is a covering map.

Problem 4.7. Let A be a subspace of $\mathbb{R}^n$ and let $h : (A, a_0) \rightharpoonup (Y, y_0)$. Suppose that h admits a continuous extension $k : \mathbb{R}^n \to Y$. Prove that h induces the trivial homomorphism $\pi_1(A, a_0) \to \pi_1(Y, y_0)$.

Problem 4.8. Let $\rho : E \to B$ be continuous and surjective. Let U be an open and evenly covered subset of B. If U is connected (as a subspace of B), show that the decomposition of $\rho^{-1}(U)$ into slices is unique.

Problem 4.9. Let B be a connected topological space and $\rho : E \twoheadrightarrow B$ a covering map. Let $k \in \mathbb{N}$ and assume that the fiber $\rho^{-1}(b_0)$ has k-elements for some $b_0 \in B$. Prove that $\rho^{-1}(b)$ has k-elements for every $b \in B$.

Problem 4.10. Let A be a retract of $\mathbb{B}^2$. If $f : A \to A$ is continuous, show that f has a fixed point in A.

Problem 4.11. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be nulhomotopic (in particular, it is continuous). Show that f has a fixed point. Prove also that f maps some point $x \in \mathbb{S}^1$ to its antipode $-x$.

Problem 4.12. The purpose of this exercise is to naïvely generalize the results proven in §4.5. We ask that the reader accept that $\mathbb{S}^{n-1}$ is **not** a retract of $\mathbb{B}^n$ for all $n \geq 2$. The exercise is then to prove the following results:

(1) The identity map $i : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is not nulhomotopic.

(2) The inclusion map $j : \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ is not nulhomotopic.

(3) Every non-vanishing continuous vector field $\mathbf{F} : \mathbb{B}^n \to \mathbb{R}^n$ points directly inwards at some point on $\mathbb{S}^{n-1}$.

(4) Every non-vanishing continuous vector field $\mathbf{F} : \mathbb{B}^n \to \mathbb{R}^n$ points directly *outwards* at some point on $\mathbb{S}^{n-1}$.

(5) Every continuous function $\mathbb{B}^n \to \mathbb{B}^n$ has a fixed point.

Problem 4.13. Let V be a topological vector space over $\mathbb{R}$ or $\mathbb{C}$. Let $\mathscr{C}$ be a convex subspace of V. Show that $\mathscr{C}$ is simply connected.

Problem 4.14. Let $\rho : E \twoheadrightarrow B$ be a covering map and assume that E is path connected. If B is simply connected, show that ρ is, in fact, a homeomorphism.

Problem 4.15. Let $\rho : E \twoheadrightarrow B$ be a covering map and fix a point $b_0 \in B$. Let $e_0 \in \rho^{-1}(b_0)$ and consider the induced homomorphism of fundamental groups

$$\rho_* : \pi_1(E, e_0) \longrightarrow \pi_1(B, b_0).$$

Prove that ρ_* is an embedding $\pi_1(E, e_0) \hookrightarrow \pi_1(B, b_0)$. *Hint*: show that $\ker \rho_*$ is trivial.

Chapter 5

Homotopy Type & Automorphism Groups of Covering Maps

5.1 Homotopy Type Theory

As mentioned previously, an important tool in the computation of fundamental groups are covering spaces. However, homotopy type theory is sometimes just as effective, if not more so. In this section, we introduce this approach. Loosely speaking, we will show that it is sometimes enough to compute the fundamental group of a different, and hopefully better understood, topological space.

Lemma 5.1. *Let X and Y be topological spaces with $h, k : (X, x_0) \rightharpoonup (Y, y_0)$. Suppose that $F(x, t)$ is a homotopy between h and k such that $F(x_0, t) = y_0$ for all $t \in \mathbb{I}$. Then, the induced homomorphisms of fundamental groups*

$$h_*, k_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

are equivalent.

Proof. Fix an element $[f]$ in the fundamental group $\pi_1(X, x_0)$; we wish to show that $h_*([f]) = k_*([f])$. For this, we need only show that $h \circ f \simeq_p k \circ f$. To this end, denote these paths in Y by γ_1 and γ_2, respectively. Since $h(x_0) = k(x_0) = y_0$, these are loops in Y based at y_0. Now, consider the continuous map

$$G : \mathbb{I} \times \mathbb{I} \to Y, \quad G(s, t) = F(f(s), t).$$

Clearly,

$$G(s, 0) \equiv F(f(s), 0) \equiv h(f(s)) \equiv \gamma_1(s)$$

and

$$G(s,1) \equiv F(f(s),1) \equiv k(f(s)) \equiv \gamma_1(s)$$

Moreover, $G(0,t) \equiv F(f(0),t) \equiv F(x_0,t) \equiv y_0$. Similarly, $G(1,t) \equiv y_0$. This shows that $\gamma_1 \simeq_p \gamma_2$ and the proof is complete. □

In the previous section we introduced the concept of a retraction, which we reiterate here. If A is a subspace of a topological space X, we say that A is a *retract* of X if there exists a continuous function $r : X \to A$ equal to the identity map on A. This function r is then called a *retraction* of X onto the set A. In this section, we will be more interested in what will be called a *deformation retract*.

Definition 5.1. Let X be a space and $A \subseteq X$ a subspace. We will call A a *deformation retract* of X if $\mathbf{1}_X$ is *homotopic* to a continuous function $r : X \to A$ equal to the identity on A via a homotopy which fixes A for each index $t \in \mathbb{I}$. Formally, we say that A is a deformation retract of X if there exists a continuous function

$$H : X \times \mathbb{I} \longrightarrow X \quad \text{with} \begin{cases} H(\cdot,0) \equiv \mathbf{1}_X(\cdot), \\ H(x,1) \in A \text{ for all } x \in X, \\ H(a,t) = a \text{ for all } a \in A \text{ and } t \in \mathbb{I}. \end{cases} \tag{5.1}$$

This map H is then dubbed a *deformation retraction* of X onto A. Note that the function $r : X \to A$ defined by $r(x) := H(x,1)$ is continuous by Remark 4.1.

Remark 5.1. Some comments on the definition of a deformation retract are in order. Let A be a deformation retract of X and suppose that $H : X \times \mathbb{I} \to X$ is an associated deformation retraction. The map $r(x) := H(x,1)$ is known to be a continuous function $X \to A$. Since it is equal to the identity on A, we see that r is a *retraction* of X onto A. Hence, A is a *retract* of X.

Second, let $j : A \hookrightarrow X$ be the inclusion map. The deformation retraction H is a homotopy between $\mathbf{1}_X$ and the composite $j \circ r : X \to X$. Indeed, this follows from the fact that $(j \circ r) \equiv r$ on X.

Theorem 5.2 (First Deformation Retraction Theorem)**.** *Let A be a deformation retract of X and fix $x_0 \in A$. The homomorphism of fundamental groups induced by the standard inclusion map*

$$j : (A, x_0) \longrightarrow (X, x_0)$$

is an isomorphism. In particular, $\pi_1(A, x_0) \cong \pi_1(X, x_0)$.

Proof. By our remark, A is in particular a retract of X. Thus, j_* is an embedding $\pi_1(A, x_0) \hookrightarrow \pi_1(X, x_0)$ of fundamental groups. We now show that j_* is surjective. Choose a deformation retraction of X onto A and denote by r the retraction $H(x, 1)$ of X onto A. By composing with the inclusion map j, we obtain a continuous map $j \circ r \equiv H(x, 1)$. In fact, H is a homotopy between $\mathbf{1}_X$ and $j \circ r$. Since $H(x_0, t) = x_0$ for all $t \in I$, Lemma 5.1 tells us that

$$j_* \circ r_* \equiv (j \circ r)_* \equiv (\mathbf{1}_X)_*.$$

Of course, this means that j_* is surjective. □

Corollary 5.3. *For each $n \geq 2$, the fundamental group of $\mathbb{R}^n \setminus \{0\}$ is isomorphic to that of $\mathbb{S}^{n-1}$. Namely, for every $x_0 \in \mathbb{S}^{n-1}$ there holds $\pi_1(\mathbb{R}^n \setminus \{0\}, x_0) \cong \pi_1(\mathbb{S}^{n-1}, x_0)$.*

Proof. We need only exhibit a deformation retraction from $\mathbb{R}^n \setminus \{0\}$ to the sphere $\mathbb{S}^{n-1}$. To achieve this, consider the continuous map

$$H : \mathbb{R}^n \setminus \{0\} \times \mathbb{I} \longrightarrow \mathbb{R}^n \setminus \{0\}$$

given by the rule

$$H(x, t) := (1 - t)x + t\frac{x}{\|x\|}.$$

Clearly, this is a well defined homotopy between $\mathbf{1}_X$ and the map $x \mapsto \frac{x}{\|x\|}$, the latter of which is easily seen to be a retraction of $\mathbb{R}^n \setminus \{0\}$ onto $\mathbb{S}^{n-1}$. In fact, for every $x \in \mathbb{S}^{n-1}$ and $t \in I$, one has $H(x, t) = x$. For $t = 1$, every non-zero point x gets taken to the sphere $\mathbb{S}^{n-1}$. Thus, H is a deformation retraction and the proof is complete. □

Typically one does not go through the pain of exhibiting a deformation retraction between spaces that are easy to visualize. In such cases, one usually argues geometrically or heuristically. This approach is what we recommend for most of the exercises.

5.1.1 Homotopic Equivalence

Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be two topological spaces. We will say that X and Y are **homotopy equivalent** if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to $\mathbf{1}_X$ and $f \circ g$ is homotopic to $\mathbf{1}_Y$. Some authors would then say that X and Y are of the same **homotopy type**. Henceforth, we will write $X \simeq Y$ to indicate (or to *assert*) that X and Y are topological spaces of the same homotopy type.

Proposition 5.4. *'$\simeq$' is an equivalence relation on the set of topological spaces.*

Clearly, $X \simeq X$. Also, it is immediate from the definitions that $X \simeq Y$ if and only if $Y \simeq X$. We leave the proof of transitivity as a straight-forward, albeit messy, exercise to the possibly masochistic reader.

Suppose that A is a deformation retract of a space X. We claim that $A \simeq X$. To see that this is so, let $r : X \to A$ be the induced retraction mapping. Denote by j the usual inclusion map $A \hookrightarrow X$. Then, $r \circ j$ is precisely the map $\mathbf{1}_A$. Conversely, the composite $j \circ r$ is known to be homotopic to the identity map $\mathbf{1}_X$ (see Remark 5.1). Hence, we see that A and X have the same homotopy type. Let us now formally summarize this discussion.

Proposition 5.5. *Let $(X, \mathfrak{T})$ be a topological space and let $A \subseteq X$ be a deformation retract of X. Then $X \simeq A$.*

We now establish the following powerhouse of a lemma.

Lemma 5.6. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces. Suppose $h, k : X \to Y$ are continuous and let*

$$y_0 := h(x_0), \quad y_1 := k(x_0).$$

If $h \simeq k$, there is a path α in Y from y_0 to y_1 satisfying $k_* \equiv \hat{\alpha} \circ h_*$. More precisely, if $H : X \times \mathbb{I} \to Y$ is a homotopy between h and k, α is the path obtained by setting $\alpha(t) := H(x_0, t)$. As a diagram,

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\ h_*\ } & \pi_1(Y, y_0) \\ & \searrow^{k_*} & \downarrow{\scriptstyle \hat{\alpha}} \\ & & \pi_1(Y, y_1). \end{array} \tag{5.2}$$

While the proof is not at all difficult, it is mechanical. We thus omit the proof for the sake of emphasizing the important ideas behind the result.

Corollary 5.7. *Let $h, k : X \to Y$ be continuous and homotopic. Put $y_0 := h(x_0)$ and $y_1 := k(x_0)$, for any choice of $x_0 \in X$. If h_* is injective, surjective, or trivial then so is k_*.*

Proof. This follows from the fact that $\hat{\alpha}$ is an isomorphism $\pi_1(Y, y_0) \cong \pi_1(Y, y_1)$. □

Corollary 5.8. *Consider a continuous map $h : X \to Y$. If h is nulhomotopic, then h_* is trivial.*

Proof. Since any constant map induces the trivial homomorphism, this follows from the previous corollary. ◻

We conclude this section by citing another result from Munkres [MNKS]. Although a very, we shall not prove said theorem. As with the previous lemma, the proof is so notation heavy and technical that we feel it best to focus on the consequences of this theorem.

Theorem 5.9. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces and f, g homotopy equivalences between X and Y. In addition, suppose that $f(x_0) = y_0$. Then, there holds the relation $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.*

5.2 *Compact Manifolds*

We now come to an important topic in geometry and algebraic topology. Our goal in this short section is to generalize (and at the same time formalize) the concept of a *surface*. For the sake of simplicity, we will only work in finite dimensions – the meaning of this will soon become clear.

Definition 5.2 (Manifold)**.** Fix $m \in \mathbb{N}$ and let $(M, \mathfrak{T})$ be a Hausdorff space with a countable basis. We say that M is a *manifold of dimension m* if for every $x \in M$, there exists a neighbourhood U of x that is homeomorphic to an open subset of $\mathbb{R}^m$. To say that M has dimension m, we shall write $\dim(M) = m$. In this case, one would call M a *finite dimensional manifold.*

Of course, one can always consider manifolds with additional properties. A manifold M is called a *compact manifold* if it is compact as a topological space. Similarly, M is said to be regular if it is...well...regular!

In this section, we will mostly be concerned with compact manifolds. To be more precise, we will be showing how finite dimensional compact manifolds can be *embedded* into $\mathbb{R}^N$, for some $N \geq 1$. For the sake of completeness, we recall the following definition from analysis:

Definition 5.3. Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ be topological spaces. We say that X is *embedded* in Y, written $X \hookrightarrow Y$, if there exists a continuous map $f : X \to Y$ such that f is a homeomorphism $X \to f(X)$. The map f is then called an *embedding of spaces.* To emphasize that f is an embedding, one usually writes $f : X \hookrightarrow Y$.

Equivalently, a space X is embedded in a space Y if X is homeomorphic to a subspace of Y. We continue to recall some analytic concepts. Let $\mathbb{F}$ be a field and $(X, \mathfrak{T})$ a topological space. If $f : X \to \mathbb{F}$ is a function, we define the support of f according to the following equation:

$$\operatorname{supp}(f) := \operatorname{Cl}\left(\left\{x \in X : f(x) \neq 0\right\}\right).$$

With this, we introduce partitions of unity.

Definition 5.4. Let $(X, \mathfrak{T})$ be a topological space and suppose $\{U_i\}_{i=1}^n$ is an open cover of the space. A collection $\{\phi_i\}_{i=1}^n$ of continuous functions $X \to \mathbb{I}$ is called a *partition of unity* dominated by the cover $\{U_i\}_{i=1}^n$ if each of the following conditions hold true:

(1) for each index i, one has $\operatorname{supp}(\phi_i) \subseteq U_i$;

(2) $\sum_{i=1}^n \phi_i \equiv 1$ on X.

Let M be a compact manifold. Since it is by definition Hausdorff, Theorem 3.23 assures us that M is normal. Hence, the following theorem applies to compact manifolds of finite dimension.

Theorem 5.10. *Let $\mathcal{U} := \{U_i\}_{i=1}^n$ be an open cover for a normal space $(X, \mathfrak{T})$. There exists a partition of unity dominated by $\mathcal{U}$.*

Proof. We chop the proof into two parts.

Step 1. We claim that we can find open sets $V_1, \ldots, V_n$, covering X, such that $\operatorname{Cl}(V_i) \subseteq U_i$ for all indices i. To prove this fact, we argue recursively. First, consider the set (possibly equal to X)

$$W := X \setminus \bigcup_{i=2}^n U_i$$

is a closed subset of X. Because $X = \bigcup_{i=1}^n U_i$, we clearly have $W \subseteq U_1$. By the normality of X, Theorem 3.15 implies the existence of an open set V_1, containing W, whose closure is contained in U_1. It is then easy to see that

$$X = V_1 \cup U_2 \cup \cdots \cup U_n.$$

For the remaining U_k, simply note that we can repeat the argument above by keeping the already chosen V_j and replacing U_1 with U_k.[1]

Step 2. Let $\mathcal{V} = \{V_i\}_{i=1}^n$ be a refinement of the cover $\mathcal{U}$ according to the first step. Now, apply the first step once more to $\mathcal{V}$ to obtain an open covering $\mathcal{W} := \{W_i\}_{i=1}^n$ of X such that $\mathrm{Cl}\,(W_i) \subseteq V_i$ for each $i = 1, \ldots, n$. Given i, the Urysohn lemma ensures the existence of a continuous function

$$\psi_i : X \longrightarrow \mathbb{I},$$

vanishing on $X \setminus V_i$, such that $\psi_i \equiv 1$ on $\mathrm{Cl}\,(W_i)$. Thus, one has $\psi_i^{-1}(\mathbb{R} \setminus \{0\}) \subseteq V_i$ for every i whence

$$\mathrm{supp}(\psi_i) \subseteq \mathrm{Cl}\,(V_i) \subseteq U_i.$$

Finally, let us put

$$\Psi : X \longrightarrow \mathbb{R}, \quad \Psi(x) := \sum_{i=1}^{n} \psi_i(x).$$

Since $X \subseteq \bigcup_{i=1}^n W_i$, the function Ψ is always positive. Hence, for every i, the map

$$\phi_i(x) := \frac{\psi_i(x)}{\Psi(x)}$$

is a well defined function $X \to \mathbb{I}$. Clearly, these $\{\phi_i\}_{i=1}^n$ are the desired partition of unity. This completes the proof. □

Here, we have made use of the infamous Urysohn lemma, which we formally recall below. Typically, such a result is proven in a course in analytic topology. Like the Hahn-Banach theorem, it is a result that any graduate student in analysis should be familiar with.

> *Let $(X, \mathfrak{T})$ be a normal space. Assume A and B are disjoint closed subsets of X. For any compact interval $[a, b] \subset \mathbb{R}$, there exists a continuous map $f : X \to [a, b]$ such that $f|_A \equiv a$ and $f|_B \equiv b$.*

We now return to the concept of a manifold. As mentioned previously, a manifold is meant to both formalize and generalize the intuitive concept of a surface. Let M be a manifold of dimension m. Sometimes, such a manifold is called a m-manifold. A perfect example of a compact 2-manifold is the sphere $\mathbb{S}^2$; 2-manifolds are often called *surfaces* whilst 1-manifolds are referred to as *curves*. An example of the latter is $\mathbb{S}^1$.

[1] We leave it as an exercise to the reader to fill in the details.

Theorem 5.11 (Embedding Theorem)**.** *Every finite dimensional compact manifold can be embedded into* $\mathbb{R}^N$*, for some* $N \geq 1$.

Proof. Let M be a compact manifold of dimension $m \in \mathbb{N}$; every point $x \in M$ has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^m$. In particular, every such neighbourhood can be embedded in $\mathbb{R}^m$. The collection of these neighbourhood forms an open cover of M. By compactness, we may choose a finite sub-cover $\{U_1, \ldots, U_n\}$ of these sets. For each index $i = 1, \ldots, n$, let

$$g_i : U_i \hookrightarrow \mathbb{R}^m$$

be an associated embedding. As mentioned previously, Theorem 3.23 tells us that M is normal. By the previous theorem, we may choose a partition of unity $\{\phi_i\}_{i=1}^n$ dominated by the cover $\{U_1, \ldots, U_n\}$. For every i, put $A_i := \operatorname{supp}(\phi_i) \subseteq U_i$ and define a function $M \to \mathbb{R}^m$ according to the rule

$$h_i(x) := \begin{cases} \phi_i(x) \cdot g_i(x), & \text{if } x \in U_i, \\ (0, \ldots, 0) & \text{if } x \in X \setminus A_i. \end{cases}$$

Since ϕ_i vanishes outside A_i, the above is well defined. Note that $\phi_i \cdot g_i$ is continuous on U_i. The same can be said for the constant map $x \mapsto 0$ on A_i. Since these are both open subsets of M whose union is X, we see that h_i is continuous.

Let us now consider the map

$$F : M \to \mathbb{R}^n \times \prod_{i=1}^n \mathbb{R}^m, \quad x \mapsto (\phi_1(x), \ldots, \phi_n(x), h_1(x), \ldots, h_n(x)) \,.$$

It is not hard to see that F is continuous. We want to show that F is an embedding. By Theorem 2.25, it is enough to check that F is injective. To this end, suppose that $F(x) = F(y)$ for some $x, y \in M$. For all $i = 1, \ldots, n$, there holds

$$\phi_i(x) = \phi_i(y) \quad \text{and} \quad h_i(x) = h_i(y).$$

Since $\sum_{i=1}^n \phi_i(x) = 1$ on X, there exists an i with the property that $\phi_i(x) > 0$. Thus, $\phi_i(y) > 0$. In particular, x and y both belong to U_i. On the other hand,

$$\phi_i(x) \cdot g_i(x) = h_i(x) = h_i(y) = \phi_i(y) \cdot g_i(y).$$

Since $\phi_i(x) = \phi_i(y)$ are non-zero, it follows that $g_i(x) = g_i(y)$. Because g_i is an embedding, it follows that $x = y$. This completes the proof. □

5.3 Covering Autohomeomorphisms

In this section, we return to the topic of covering spaces. Let us fix a topological space $(X, \mathfrak{T})$, which we will assume (for simplicity) to be path connected and locally path connected. Consider a covering space $\hat{X}$ for X (hence, $\hat{X}$ comes equipped with a covering map $\rho : \hat{X} \twoheadrightarrow X$). We are momentarily interested in *structure preserving* maps $\hat{X} \to \hat{X}$.

Definition 5.5. Let $(X, \mathfrak{T})$ be a topological space. An *autohomeomorphism* of X is a homeomorphism $\varphi : X \to X$. The set of all autohomeomorphisms $X \to X$ is denoted by $\mathrm{Auth}(X)$.

Clearly, $\mathrm{Auth}(X)$ forms a group under composition, whether or not X is (locally) path connected. In the context of covering spaces, such functions are of great importance.

Definition 5.6. Let $(X, \mathfrak{T})$ be a topological space and $\hat{X}$ a covering space of X (with covering map $\rho : \hat{X} \twoheadrightarrow X$). A *covering automorphism* of $\hat{X}$ is an autohomeomorphism φ of $\hat{X}$ such that

$$\rho \circ \varphi \equiv \rho.$$

The collection of all covering automorphisms of $\hat{X}$ is denoted by $\mathrm{Aut}(\hat{X})$. It is not hard to check that $\mathrm{Aut}(\hat{X})$ forms a group under composition. The group $\mathrm{Aut}(\hat{X})$ is called the automorphism group of $\hat{X}$.

Loosely speaking, we should interpret an automorphism of a covering space $\hat{X}$ as a homeomorphism $\hat{X} \to \hat{X}$ that respects the action of a covering map $\rho : \hat{X} \twoheadrightarrow X$. Keeping this in mind, we see that such a condition is highly geometric in nature.

Unfortunately, the theory of covering space automorphisms is one that is hard to make rigorous. As such, we will mostly give pictures that describe the covering maps involved. Using this pictures, it will be relatively easy to intuitively "conjure up" the automorphism group of a covering space $\hat{X}$. We feel that this is best summarized by way of examples.

Example 5.1. Check that the following diagram describes a covering map of a wedge of circles $\mathbb{S}^1 \vee \mathbb{S}^1$.

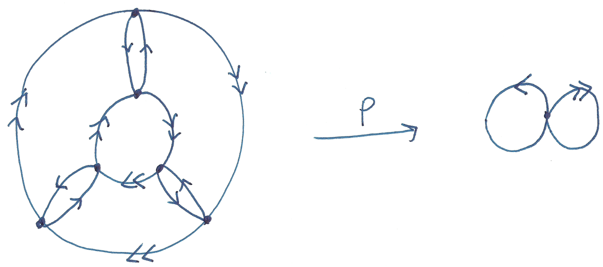


Figure 5.1: Covering Space of $\mathbb{S}^1 \vee \mathbb{S}^1$.

Also, compute its automorphism group.

Solution. Checking that the above describes a covering map of $\mathbb{S}^1 \vee \mathbb{S}^1$ is easy. To compute it's automorphism group, we first observe that we are allowed 3 clockwise rotations. After the third rotation, we obtain the identity automorphism of the covering space. Finally, the covering space diagram can be turned "inside out". Since these actions commute, we see that this covering space has automorphism group isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$. □

Example 5.2. We compute the automorphism group of the following covering space:

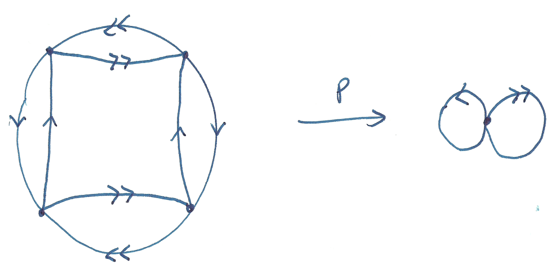


Figure 5.2: Covering Space of $\mathbb{S}^1 \vee \mathbb{S}^1$.

As a first step, note that one can rotate by 180° exactly twice before obtaining the original identity homeomorphism. Thus, we have two permissible rotations. Furthermore, one can reflect along the horizontal middle-line. It is not hard to convince oneself that these actions commute. Thus, the automorphism group of our covering space is given by $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note that the covering space in Figure

5.2 cannot be turned "inside out" because the directions on the inner and outer segments are inverted.

Keeping in line with the notation of the previous covering map, the following examples also describe covering maps onto the copy of $\mathbb{S}^1 \vee \mathbb{S}^1$ described above.

Example 5.3. Let us compute the automorphism group of the following covering map:

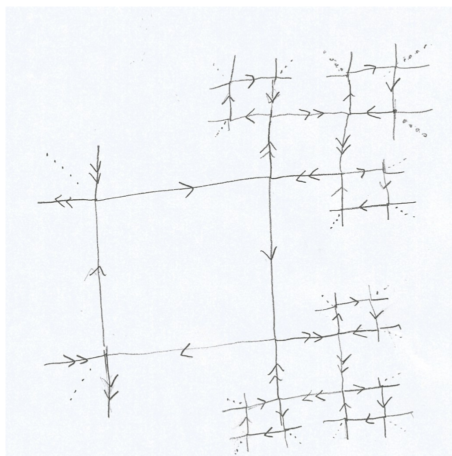


Figure 5.3: Covering Space of $\mathbb{S}^1 \vee \mathbb{S}^1$

Clearly, this covering space has "infinite degree" in the sense that every fiber $\rho^{-1}(b)$ (with $b \in \mathbb{S}^1 \vee \mathbb{S}^1$) has infinite cardinality. Right off the bat, we can see that rotations by 90° constitute *local* autohomeomorphisms of the space above. Since we can have four such rotations (with the last being the identity), the automorphism group of our covering space should contain an isomorphic copy of $\mathbb{Z}/4\mathbb{Z}$. This reasoning holds for every "square" in the graph above, and as such the automorphism group should be the free product

$$(\mathbb{Z}/4\mathbb{Z}) * (\mathbb{Z}/4\mathbb{Z}) .$$

5.3.1 *Degree of a Covering Map*

Let E and B the topological spaces, with underlying topologies $\mathfrak{E}$ and $\mathfrak{B}$. Let $\rho : E \twoheadrightarrow B$ be a covering map. Given a point $b \in B$, we define the degree of ρ *at* b as the cardinality of the fiber $\rho^{-1}(b)$. In most cases, we can extend this definition to the entire space B.

Definition 5.7 (Covering Degree)**.** Let E and B be spaces and let $\rho : E \twoheadrightarrow B$ be a covering map. We define the *degree* of ρ, denoted $\deg \rho$, to be the cardinality of any fiber $\rho^{-1}(b)$, with $b \in B$. When B is connected, this is well defined (this is a consequence of Problem 4.9).

As a series of examples, let us compute the degrees of the covering spaces for $\mathbb{S}^1 \vee \mathbb{S}^1$ given in Figures 5.1-5.2-5.3. Since each of the these covering spaces are connected, we need only compute the cardinality of the fiber $\rho^{-1}(x_0)$, where x_0 is the wedge point of the circles (i.e. the point where both circles in $\mathbb{S}^1 \vee \mathbb{S}^1$ intersect). In Figure 5.1, the degree is clearly equal to 6. In Figure 5.2, we instead have a covering space of degree equal to 4. In Figure 5.3, the covering map has infinite degree.

Theorem 5.12. *Let $\rho : E \twoheadrightarrow B$ be a covering map. Let X be a connected space and let $f : X \to B$ be a continuous function. If $\tilde{f}_1$ and $\tilde{f}_2$ are two lifts of f (to a continuous map $X \to E$) agreeing at a single point, then they are equal everywhere.*

Proof. Suppose there exists $x_* \in X$ such that $\tilde{f}_1(x_*) = \tilde{f}_2(x_*)$ and consider the non-empty set

$$T := \left\{x \in X : \tilde{f}_1(x) = \tilde{f}_2(x)\right\};$$

we will show that this is a clopen subset of X. Let $x \in X$ be given and define

$$b := (\rho \circ \tilde{f}_1)(x) = (\rho \circ \tilde{f}_2)(x) = f(x).$$

Choose a neighbourhood U of b that is evenly covered by ρ and write $\rho^{-1}(U)$ as the disjoint union $\bigsqcup_\alpha V_\alpha$, where V_α is open in E and homeomorphic to U under the action of ρ. Further, we put

$$e_1 := \tilde{f}_1(x) \quad \text{and} \quad e_2 := \tilde{f}_2(x).$$

Let V_1 and V_2 be the elements of $\{V_\alpha\}$ containing e_1 and e_2, respectively. Consider the open subset of X

$$W := \tilde{f}_1^{-1}(V_1) \cap \tilde{f}_2^{-1}(V_2),$$

which contains x. Suppose now that $x \in T$. Then, $e_1 = e_2$ and we must therefore have $V_1 = V_2$. Since ρ is injective on V_1 and

$$(\rho \circ \tilde{f}_1) \equiv f \equiv (\rho \circ \tilde{f}_2),$$

we see that $W \subseteq T$. That is, T is open. On the other hand, if $x \notin T$, we must have $e_1 \neq e_2$ and so $V_1 \neq V_2$. Let now $y \in W$ be given and assume that $y \in T$. We get that

$$\tilde{f}_1(y) = \tilde{f}_2(y) \in V_1 \quad \text{and} \quad \tilde{f}_2(y) = \tilde{f}_1(y) \in V_2.$$

Obviously, this is absurd and thus we must have $y \notin T$. More precisely, $W \subseteq X \backslash T$. Thence, T is clopen and equal to the whole of X. □

Corollary 5.13. *Let $(X, \mathfrak{T})$ be a space and assume that $\hat{X}$ is a connected covering space of X, with covering map $\rho : \hat{X} \to X$. If any two automorphisms of $\hat{X}$ agree at a single point of $\hat{X}$, then they are equal everywhere.*

Proof. Let φ, ψ be automorphisms of $\hat{X}$ and let $\hat{x} \in \hat{X}$ be a point at which they are equal. Since φ and ψ are automorphisms of $\hat{X}$, we have both

$$\rho \circ \varphi \equiv \rho \quad \text{and} \quad \rho \circ \psi \equiv \rho.$$

Hence, ϕ and ψ are lifts of the continuous map $\rho : \hat{X} \to X$ agreeing at a point. By the previous theorem, we must have $\varphi \equiv \psi$. □

Definition 5.8. Let $(X, \mathfrak{T})$ be a topological space and $\hat{X}$ a covering space of X, with covering map $\rho : \hat{X} \twoheadrightarrow X$. We say that ρ is *regular* if, for every fiber $\rho^{-1}(x)$ of $\hat{X}$ and every u_1, u_2 in this fiber, there exists $\varphi \in \mathrm{Aut}(\hat{X})$ such that $\varphi(u_1) = u_2$.

The reader can easily verify that the covering maps in Figures 5.1-5.2 are regular. However, that described in Figure 5.3 is not.

Quotients of the Fundamental Group

Let X be a path connected space and suppose that $\hat{X}$ is a path connected *regular* covering space. Denote by ρ the associated covering map $\hat{X} \twoheadrightarrow X$. Fix a point $x \in X$ and let $\hat{x} \in \hat{X}$ be such that $\rho(\hat{x}) = x$. We define a function

$$\Phi : \pi_1(X, x) \longrightarrow \mathrm{Aut}(\hat{X}), \quad [f] \mapsto \Phi[f], \tag{5.3}$$

where $\Phi[f]$ is the automorphism of $\hat{X}$ such that $\Phi[f](\hat{x}) = \tilde{f}(1)$. Here, $\tilde{f}$ denotes the *unique* lift of the loop $f : \mathbb{I} \to X$ to a path $\tilde{f} : \mathbb{I} \to X$ beginning at $\hat{x}$. Since $\hat{X}$ is a regular covering space of X, it is not hard to see that $\Phi([f])$ will always be defined. By virtue of Corollary 5.13, this map Φ is well defined.

Next, we claim that Φ is a group homomorphism. To this end, let us fix elements $[f]$ and $[g]$ of $\pi_1(X, x)$. Denoted by $\tilde{f}$ and $\tilde{g}$ the unique lifts of f and g (respectively) to paths in $\hat{X}$ beginning at $\hat{x}$. Then,

$$(\Phi[f] \circ \Phi[g])\,(\hat{x}) = \Phi[f]\,(\tilde{g}(1))$$

where, of course, $\Phi[f] \circ \Phi[g] \in \mathrm{Aut}(\hat{X})$ by closure under composition. Now, let $\tilde{\tilde{g}}$ be the unique lift of g to a path in $\hat{X}$ beginning at $\tilde{f}(1)$. This certainly exists because

$$\rho \circ \tilde{f} \equiv f$$

and $f(1) = x$. Notice that, by uniqueness, we must have

$$\tilde{\tilde{g}} = \Phi[f] \circ \tilde{g}. \tag{5.4}$$

Indeed, we first note that the right hand side is clearly a path in $\hat{X}$, beginning at

$$\Phi[f](\tilde{g}(0)) = \Phi[f](\hat{x}) = \tilde{f}(1).$$

Second,

$$\rho \circ (\Phi[f] \circ \tilde{g}) \equiv (\rho \circ \Phi[f]) \circ \tilde{g} \equiv \rho \circ \tilde{g} \equiv g.$$

Hence, $\Phi[f] \circ \tilde{g}$ a lift of g to a path in $\hat{X}$ beginning at $\tilde{f}(1)$. With this, we have established (5.4). Then, it is easy to check that $\tilde{f} * \tilde{\tilde{g}}$ is *also* a lift of $f * g$ to a path in $\hat{X}$ beginning at $\hat{x}$. Thus, by definition,

$$\Phi([f] * [g])(\hat{x}) = \Phi([f * g])(\hat{x}) = (\tilde{f} * \tilde{\tilde{g}})(1) = \tilde{\tilde{g}}(1) = \Phi[f]\,(\tilde{g}(1))$$

Therefore, $\Phi[f] \circ \Phi[g]$ and $\Phi([f] * [g])$ are two automorphisms of $\hat{X}$ agreeing at a point, and so must be equal everywhere. With this, we conclude that the map Φ defined in (5.3) is a group homomorphism.

Lemma 5.14. *Keeping in line with our earlier assumptions, the kernel of Φ is precisely equal to $\rho_*(\pi_1(\hat{X}, \hat{x}))$.*

Proof. Suppose that $[f]$ is an element of $\pi_1(X, x)$ such that $\Phi[f]$ is trivial. Or, equivalently,

$$\Phi[f](\hat{x}) = \hat{x}$$

which gives $\tilde{f}(1) = \hat{x}$; where $\tilde{f}$ is the unique lift of f to a path in $\hat{X}$ beginning at $\hat{x}$. In particular, we see that $[\tilde{f}] \in \pi_1(\hat{X}, \hat{x})$ so that

$$[f] = [\rho \circ \tilde{f}] = \rho_*([\tilde{f}]).$$

It follows that $\ker\Phi \subseteq \rho_*(\pi_1(\hat{X},\hat{x}))$. Conversely, suppose that $[f] = \rho_*([\sigma])$ for some $[\sigma] \in \pi_1(\hat{X},\hat{x})$. Then, $[f] = [\rho\circ\sigma]$. After a relabeling, we may assume that $f = \rho\circ\sigma$, for some loop σ in $\hat{X}$ based at $\hat{x}$. Then, σ is the unique lift of f to a path in $\hat{X}$ beginning at $\hat{x}$. Thus,

$$\Phi[f](\hat{x}) = \tilde{f}(1) = \sigma(1) = \hat{x}.$$

By our uniqueness result, we see that $\Phi[f]$ is trivial. This completes the proof. □

Corollary 5.15. *Let $(X,\mathfrak{T})$ be path connected and let $\hat{X}$ be a regular path connected covering space of X, with covering map ρ. For any $\hat{x} \in \rho^{-1}(x)$, there holds the following isomorphism:*

$$\mathrm{Aut}(\hat{X}) \cong \pi_1(X,x)/\rho_*(\pi_1(\hat{X},\hat{x})). \tag{5.5}$$

Proof. If we can show that Φ is an epimorphism, this will follow at once from the first isomorphism theorem. To this end, let $\varphi \in \mathrm{Aut}(\hat{X})$ and define $x^* := \varphi(\hat{x})$. Then,

$$\rho(x^*) = \rho(\varphi(\hat{x})) = \rho(\hat{x}) = x$$

whence $x^* \in \rho^{-1}(x)$. Now, let γ be a path in $\hat{X}$ from $\hat{x}$ to x^*. We see that $\rho\circ\gamma$ is a loop in X based at x. Moreover, γ is the unique lift of $\rho\circ\gamma$ to a path in $\hat{X}$ starting at $\hat{x}$. By definition,

$$\Phi[\rho\circ\gamma](\hat{x}) = \gamma(1) = x^*.$$

Thus, $\Phi[\rho\circ\gamma]$ and φ are automorphisms of $\hat{X}$ agreeing at a point, and so must be equal everywhere. □

About Quotients of $\pi_1(X,x)$

Lemma 5.16. *Let $\rho : E \twoheadrightarrow B$ be a covering map and fix $b \in B$. Let $e \in \rho^{-1}(b)$ be given and assume that E is path connected. Recall that the lifting correspondence*

$$\phi_e : \pi_1(B,b) \to \rho^{-1}(b), \quad [f] \mapsto \tilde{f}(1),$$

where $\tilde{f}$ is the unique lifting of f to a path in E beginning at e, is well defined and surjective. Then, the induced map

$$\Psi : \pi_1(B,b)/\rho_*(\pi_1(E,e)) \to \rho^{-1}(b), \quad \rho_*(\pi_1(E,e)) * [f] \mapsto \phi_e([f])$$

is well defined and bijective. Here, we are viewing $\pi_1(B,b)/\rho_*(\pi_1(E,e))$ as the collection of right cosets of $\rho_*(\pi_1(E,e))$ in $\pi_1(B,b)$.

Proof. For the sake of brevity, let H denote the subgroup $\rho_*(\pi_1(E,e))$ of $\pi_1(B,b)$. In fact, by an exercise from the previous chapter, we know that $H \hookrightarrow \pi_1(B,b)$. If B is assumed to be regular, then H will be normal in $\pi_1(B,b)$. We claim that $\phi_e([f]) = \phi_e([g])$ whenever

$$H * [f] = H * [g].$$

To this end, let $[f] = [h] * [g] = [h * g]$, for some $[h] \in H = \rho_*(\pi_1(E,e))$. Then, we must have $f \simeq_p h * g$. Let $\tilde{f}, \tilde{g}$ be the unique lifts of f and g (respectively) to paths in E beginning at e. Now, $[h] = [\rho \circ \gamma]$ for some $\gamma \in \pi_1(E,e)$. In particular, $h \simeq_p \rho \circ \gamma$. Thus, if $\tilde{h}$ denotes the unique lifting of h to a path in E beginning at e, we must have $\tilde{h} \simeq_p \gamma$. Especially, $\tilde{h}$ is a loop based at e. Now, observe that

$$\tilde{h} * \tilde{g}$$

is a lifting of $h * g$ to a path in E, beginning at e. Therefore,

$$\Psi([h * g]) = (\tilde{h} * \tilde{g})(1) = \tilde{g}(1).$$

On the other hand, $f \simeq_p h * g$ means that $\tilde{f} \simeq_p \tilde{h} * \tilde{g}$ whence

$$\tilde{f}(1) = (\tilde{h} * \tilde{g})(1) = \tilde{g}(1).$$

Consequently, we see that Ψ is well defined. Since E is path connected, the lifting correspondence $\phi_e : \pi_1(B,b) \to \rho^{-1}(b)$ is surjective. Hence, Ψ is also a surjection. All that remains is to check that Ψ is an injection. Suppose that $[f], [g] \in \pi_1(B,b)$ are such that

$$\Psi(H * [f]) = \Psi(H * [g]).$$

That is, $\tilde{f}(1) = \tilde{g}(1)$, where $\tilde{f}$ and $\tilde{g}$ are the (respective) unique lifts of f and g to paths in E beginning at e. Now, $\tilde{f} * \bar{\tilde{g}}$ will be a loop at e, and hence

$$[\tilde{f} * \bar{\tilde{g}}] \in \pi_1(E,e).$$

From this, we calculate

$$H \ni \rho_*\left([\tilde{f} * \bar{\tilde{g}}]\right) = [f * \bar{g}] = [f] * [g]^{-1}.$$

It follows that $[f] = H * [g]$ so that $H * [f] \subseteq H * [g]$. By symmetry, equality must hold. We conclude that Ψ is, in fact, a bijection. □

Corollary 5.17. *Let $(X, \mathfrak{T})$ be path connected and let $\hat{X}$ be a regular path connected covering space of X, with covering map ρ. For any $\hat{x} \in \rho^{-1}(x)$:*

$$\operatorname{Aut}(\hat{X}) \cong \pi_1(X,x)/\rho_*(\pi_1(\hat{X},\hat{x})). \tag{5.6}$$

Furthermore, $\left|\operatorname{Aut}(\hat{X})\right| = \deg \rho$.

5.4 The Seifert-Van Kampen Theorem

In this section we gloss over a useful tool for the computation of fundamental groups. Essentially, this famous result makes it easy to compute the fundamental group of a "wedge of spaces". Although we will not prove the result, we will give several examples and applications of the theorem. In fact, we will state three versions of the theorem and see examples for each. As expected, the proofs can be found in [MNKS]. Henceforth, we assume significant familiarity with free groups and amalgamation products.

Theorem 5.18 (Seifert-Van Kampen – Form 1)**.** *Let X be a space and let U, V be open subsets of X such that $X = U \cup V$ and $U \cap V \neq \emptyset$. Suppose further that U, V, and $U \cap V$ is simply connected and let $x_0 \in U \cap V$. Then,*

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0)$$

where $*$ denotes the free product operator.

This theorem makes it easy to compute the fundamental group of the figure eight $\mathbb{S}^1 \vee \mathbb{S}^1$. Indeed, let x_0 denote the point in $\mathbb{S}^1 \vee \mathbb{S}^1$ where both copies of $\mathbb{S}^1$ are joined. Let U and V consist of the respective copies of $\mathbb{S}^1$ together with a small "open portion" of the other copy of $\mathbb{S}^1$. Then, $U \cup V = \mathbb{S}^1 \vee \mathbb{S}^1$ and $U \cap V$ deformation retracts to the single point $\{x_0\}$. Hence, $U \cap V$ is simply connected and our theorem implies that

$$\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1, x_0) \cong \pi_1(\mathbb{S}^1, x_0) \vee \pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z} * \mathbb{Z}.$$

Arguing by induction, it is not hard to obtain the following.

Corollary 5.19. *For every $N \in \mathbb{N}$ there holds $\pi_1(\bigvee_{n=1}^N \mathbb{S}^1) \cong *_{n=1}^N \mathbb{Z}$.*

Example 5.4. Let X denote the wedge product of the torus with a circle, i.e. put $X := \mathbb{T}^2 \vee \mathbb{S}^1$. Let x_0 be the point in X at which $\mathbb{T}^2$ and $\mathbb{S}^1$ are identified. The Seifert-Van Kampen theorem tells us that

$$\pi_1(X, x_0) \cong \pi_1(\mathbb{T}^2, x_0) * \pi_1(\mathbb{S}^1, x_0) \cong (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}.$$

The Seifert-Van Kampen theorem continues to hold in a more general setting. However, the statement becomes significantly less elegant.

Theorem 5.20 (Seifert-Van Kampen – Form 2)**.** *Let X be a space and suppose that $X = U \cup V$ for two open subsets U, V of X such that U, V, and $U \cap V$ are non-empty and path connected. Let $x_0 \in U \cap V$ be given. Denote by i_U and i_V the group homomorphisms*

$$\pi_1(U \cap V, x_0) \xrightarrow{i_U} \pi_1(U, x_0) \quad \textit{and} \quad \pi_1(U \cap V, x_0) \xrightarrow{i_V} \pi_1(V, x_0)$$

induced by the inclusion maps $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$, respectively. Let $\{g_1, \dots, g_n\}$ be a finite generating set for $\pi_1(U \cap V, x_0)$. Then,

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{\langle\langle i_U(g_i)^{-1} i_V(g_i) : i = 1, \dots, n\rangle\rangle}. \tag{5.7}$$

In many cases, this version makes it possible to explicitly compute a presentation for the fundamental group of X. On the other hand, the statement can be significantly cleaned up if don't care about having an explicit form for the group $\pi_1(X, x_0)$. Before stating the third form of the Seifert-Van Kampen theorem, let us give a few examples.

Theorem 5.21 (Seifert-Van Kampen – Form 3)**.** *Let X be a space with $X = U \cup V$ for two open subsets U and V of X with $U \cap V \ni x_0$. If U, V and $U \cap V$ are path connected, then $\pi_1(X, x_0)$ can be expressed as the amalgamated product:*

$$\pi_1(x, x_0) \cong \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0). \tag{5.8}$$

Example 5.5. Using this form of the theorem it is easy to show that $\pi_1(\mathbb{S}^2) \cong \mathbf{0}$. Indeed, let $U \subset \mathbb{S}^2$ be the top hemisphere together with an open ring in the bottom hemisphere. Thus, U looks like an open "$\frac{3}{4}$-sphere". Similarly, let V be constructed from the bottom hemisphere. Then, U and V are path connected with $U \cup V = \mathbb{S}^2$. Furthermore, $U \cap V$ is homeomorphic to a bounded open cylinder in $\mathbb{R}^3$ which deformation retracts to $\mathbb{S}^1$. However, both U and V are homeomorphic to the open disk $\mathbb{D}^2$ in $\mathbb{R}^2$. Consequently,

$$\pi_1(\mathbb{S}^2) \cong \pi_1(\mathbb{D}^2) *_{\mathbb{Z}} \pi_1(\mathbb{D}^2) \cong \mathbf{0}.$$

5.5 Exercises

Problem 5.1. Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$, for all $n \geq 3$. You may use, without proof, that $\mathbb{S}^m$ is simply connected for all $m \geq 2$.

Problem 5.2. Calculate the fundamental group of the solid torus $\mathbb{B}^2 \times \mathbb{S}^1$. Also, compute the fundamental group of a punctured torus.

Problem 5.3. Compute the fundamental group of the infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$.

Problem 5.4. Calculate, up to isomorphic equivalence, the fundamental group of $\mathbb{R} \times [0, \infty)$.

Problem 5.5. What is the fundamental group of $\mathbb{R}^2 \setminus (0, \infty)$?

Problem 5.6. Arguing informally, explain why $\mathbb{S}^1$ is not a deformation retract of the closed unit disk $\mathbb{B}^2$ in $\mathbb{C}$.

Problem 5.7. Give an example of a covering space of $\mathbb{S}^1 \vee \mathbb{S}^1$, having infinite degree, whose automorphism group is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Provide another example whose automorphism group is $\mathbb{Z}/6\mathbb{Z}$.

Problem 5.8. Let $(X, \mathfrak{T})$ be a topological space. Show that X is contractible if and only if it has the homotopy type of a one-point space. Conclude that contractible spaces are simply connected.

Problem 5.9. Prove that a retract of a contractible space is once again contractible.

Problem 5.10. Let $(X, \mathfrak{T})$ be a space and $B \subseteq A \subseteq X$. Assume that A is a deformation retract of X, and that B is a deformation retraction of A. Show that B is a deformation retraction of X.

Problem 5.11. Compute the fundamental group of the Möbius strip. Show that there does not exist a retraction of M to its boundary[2].

Problem 5.12. Provide an example of a non-regular covering space of $\mathbb{S}^1 \vee \mathbb{S}^1$, having infinite degree, whose automorphism group is isomorphic to $\mathbb{Z}/6\mathbb{Z}$.

[2]Here, the boundary of the Möbius strip is interpreted in the sense of surfaces and is not the topological boundary. As you may check yourself with the help of a piece of paper, the boundary of a Möbius strip is homeomorphic to the circle.

Appendix A

Cell Complexes

We now touch upon a more geometric topic, that of cell complexes. We would like to warn the reader that this chapter is not meant to be taken as a rigorous treatment of the subject. In fact, we will give a rather informal discussion of complexes and we shall mostly rely upon select major theorems, of which we omit the proof. We have chosen this approach for several reasons:

- Rigorously proving these major results (e.g. the characterization of surfaces theorem) is far too involved and technical for this text.

- In order to prove everything rigorously, we would have to get our hands *really dirty*. For the most part, the proofs of the major results we present in this section are far from elegant or accessible.

- Most importantly, we wish to focus on applications of what we have accomplished thus far. By harnessing our previous work (and of course a handful of very powerful theorems), we will be able to understand some beautiful geometric phenomena.

There will, of course, be no exercises related to this material. After all, it would be largely unfair to expect students to rigorously prove statements when we have not given proofs of the major results. For the most part, this chapter should not be studied too closely. Instead, we hope the reader will be able to read this brief chapter in a way that is both fun and relaxing.

A.1 CW-Complexes

CW-complexes form a special class of topological spaces. Intuitively, they should be understood as spaces that can be *inductively* constructed by adding copies of closed n-balls to a topological space X by *identifying* the boundaries of these balls with parts of X. We formalize this notion below.

Definition A.1. Given $n \in \mathbb{N}_0$, we denote by D_n a homeomorphic copy of the closed unit ball in $\mathbb{R}^n$. For $n = 0$ this is a point, D_1 is a line, and so forth.

Definition A.2 (CW-Process)**.** We begin with a set X^0 (possibly empty), consisting of singletons. Thus, X^0 can be described as the disjoint union of 0-cells (i.e. singletons). We give X^0 the discrete topology. Taking a family $\{D_\alpha\}_{\alpha\in I_1}$ of 1-cells, i.e. $D_\alpha \cong D_1$ for all $i \in I_1$, we define a new topological space X^0_* as the disjoint union:

$$X^0_* := X^0 \amalg \coprod_{\alpha\in I_1} D_\alpha.$$

Now we choose a family $\left\{q^{(1)}_\alpha\right\}_{\alpha\in I_1}$ of maps $\partial D_\alpha \to X^0$ and define

$$X^1 := X^0_*/\left\{x \sim q^{(1)}_\alpha(x) : x \in \partial D_\alpha,\ \alpha \in I_1\right\}.$$

That is, we identify the boundaries of our adjoined 1-cells with points on X^0. Thus, we are really attaching 1-cells (i.e. lines) to the collection of points X^0. This X^1 is said to be a CW-complex of dimension 1, and it is given the induced quotient topology.

In general, if we have $X^{(n)}$ and a family $\{D_\alpha\}_{\alpha\in I_{n+1}}$ of $(n+1)$-cells, and a family of attachment maps $\left\{q^{(n+1)}_\alpha\right\}_{\alpha\in I_{n+1}}$, we define

$$X^{(n+1)} := \left(X^n \amalg \coprod_{\alpha\in I_{n+1}} D_\alpha\right)/\left\{x \sim q^{(n+1)}_\alpha(x) : x \in \partial D_\alpha,\ \alpha \in I_{n+1}\right\}.$$

This $X^{(n+1)}$ is called a $(n+1)$-dimensional CW-complex. Of course, we give X^{n+1} the induced quotient topology.

A topological space $(X, \mathfrak{T})$ is said to be a CW-complex if it can be constructed using the CW-process above. A finite CW-complex is one that can be built using finitely many cells (not *just* finitely many cells of every dimension n). Let us now give some examples.

Example A.1. Any n-cell is automatically a CW-complex. Indeed, it can be built up by adding only a single n-cell at the n^{th} stage.

Example A.2. The circle $\mathbb{S}^1$ is a CW-complex. Indeed, start with a single point $X^0 := \{x_0\}$. We then add a single 1-cell (i.e. a line). The boundary of this one cell will consist of exactly two points. We then choose an attaching map, which takes both these boundary points to x_0. In the quotient topology, we are wrapping both ends of the one-cell around and "gluing them together". Thus, we obtain a circle homeomorphic to $\mathbb{S}^1$.

Let $\mathbb{T}^2$ denote the torus, homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$, embedded in $\mathbb{R}^3$. This surface can also be easily constructed as a CW-complex. Certainly, we start with a 0-cell $X^0 = \{x_0\}$ and adjoin *two* 1-cells, say ℓ^1 and ℓ^2. As an attachment map, send the boundary points of both ℓ^1 and ℓ^2 to the point x_0, thereby obtaining a figure-eight consisting only of lines. After a homeomorphism, we can "twist" one of the loops by 90° so that the loops are "perpendicular" to each other in space. We continue to use the labels ℓ^1 and ℓ^2 for these loops. Let us now add a single 2-cell, in the form of a square with sides labeled *ABAB*. As an attachment map, we send the sides labeled by A to ℓ^1, and the sides labeled by B to ℓ^2. Since the endpoints of ℓ^1 and ℓ^2 coincide with x_0, the end points of A and B will also get taken to x_0. In doing so, we are rolling up the square into a tube and wrapping it around the figure-eight, with the end-circles of the tube being identified with one of the loops. The length component of this tube is identified with the other, perpendicular loop. The reader should convince themselves (with the help of a diagram) that this procedure yields a torus $\mathbb{T}^2$ (or a homeomorphic copy of it).

Theorem A.1. *Any finite CW-complex is compact.*

Proof. We argue by induction on the stage of the construction. Clearly, there can be only finitely many stages at which non-empty families of n-cells are adjoined. Moreover, at every such stage, we can only add finitely many n-cells (since the complex is finite). As a base case, note that X^0 is clearly compact as it is a finite collection of single points.

Assuming that X^n is compact, we obtain X^{n+1} from X^n by adjoining a finite number of $(n+1)$-cells. By definition of X^{n+1}, there exists a quotient map

$$\rho : X^n \amalg \coprod_{j=1}^{K_{n+1}} D_j \twoheadrightarrow X^{n+1};$$

where every D_j represents a $(n+1)$-cell. Observe that the finite disjoint union of compact spaces always yields a compact space. Since continuous maps take compacta to compacta, we see that X^{n+1} is compact whence our inductive step is complete. □

A.2 The Classification of Surfaces

A surface S is a closed manifold, i.e. a compact manifold without boundary, of dimension 2. We will pay special attention to surfaces that admit a CW-structure (i.e. surfaces that can be built through the CW-process). In fact, we will give simple methods to determine exactly which compact surface arises from a given cell structure. Although this will be far from rigorous, it is nonetheless fun and interesting. We begin with the following definition.

Definition A.3. Let X be a finite CW-complex and fix a cell structure for X. Given $n \in \mathbb{N}_0$, let $\#_n$ denote the number of n-cells present in this cell structure. The *Euler characteristic* of X is defined via the equation

$$\chi(X) = \sum_n (-1)^n \#_n.$$

We shall not prove it, but $\chi(X)$ is *independent* of the choice of cell structure.

Definition A.4. A surface S is said to be *orientable* if there exists a well defined continuous choice of normal vector on S. Equivalently, we say that S is orientable if it does not contain an embedded Möbius strip $\mathcal{M}$.

Based on the notion of orientability, we can completely characterize the closed surfaces according to their genus (take this for granted – it is very difficult to prove). Thus, if we further restrict ourselves to those orientable closed surfaces that are CW-complexes, we have a characterization in terms of the Euler characteristic. This is important, because it is much easier to directly compute the Euler characteristic of a CW-complex than its genus.

A similar thing can be said for the non-orientable closed surfaces that arise as CW-complexes. Yet again, such surfaces can be described completely in terms of their Euler characteristic, from which one can deduce the genus. We summarize this below.

Theorem A.2. *Let* S *be a closed surface that admits a CW-structure and let* g *be the genus of* S.

(1) If S *is orientable, then its Euler characteristic is given by*

$$\chi(\mathrm{S}) = 2 - 2g.$$

(2) If S *is non-orientable, then its Euler characteristic is described by the equation*

$$\chi(\mathrm{S}) = 2 - g.$$

This gives us an "easy" method for computing the genus of a closed surface that admits a CW-structure, whether or not it is orientable. To help make sense of the above, let us go over some examples for which we intuitively know the genus.

Example A.3. Consider the 2-sphere $\mathbb{S}^2$ embedded into $\mathbb{R}^3$. As a CW-complex, we can obtain $\mathbb{S}^2$ from a single 0-cell $\{x_0\}$ and one 2-cell $\mathbb{D}$, where $\mathbb{D}$ is the closed unit disk in $\mathbb{R}^2$. The attachment map would take $\partial\mathbb{D}$ onto the point x_0. Clearly, this means that

$$\chi(\mathbb{S}^2) = 1 - 0 + 1 = 2.$$

Thus, $\chi(\mathbb{S}^2) = 2 = 2 - 2g$, where g is the genus of $\mathbb{S}^2$. This forces $g = 0$, which is far from surprising.

Example A.4. Continuing in this way, the torus $\mathbb{T}^2$ should certainly have genus equal to 1. Let us verify that our theorem yields this well understood result. Thankfully, we have already given a CW-structure for $\mathbb{T}^2$ which consisted of

- one 0-cell;
- two 1-cells;
- one 2-cell.

Therefore, $\chi(\mathbb{T}^2) = 1 - 2 + 1 = 0$. Since $\mathbb{T}^2$ is orientable, we have $2 - 2g = 0$ so that $g = 1$.

As it turns out, the only closed orientable surfaces (up to homeomorphic equivalence) are the sphere and the n-hole torus (for $n \geq 1$). Thus, if we had a cell structure describing an orientable closed surface of genus 2, we could say that this surface is homeomorphic to the 2-hole torus.

A.2.1 Non-Orientable Surfaces

Next, we should discuss some non-orientable surfaces. The simplest of which is the *real projective plane*, denoted by $\mathbb{RP}^2$.

Definition A.5. Consider the two-sphere $\mathbb{S}^2$ embedded in $\mathbb{R}^3$. We can partition $\mathbb{S}^2$ into disjoint non-empty sets of the form $\{x, -x\}$, for $x \in \mathbb{S}^2$. Of course, this partition arises from an equivalence relation $\sim$. Then, we define $\mathbb{RP}^2$ to be the quotient space $\mathbb{S}^2/\sim$.

Alternatively, we could define $\mathbb{RP}^2$ according to the following CW-complex structure:

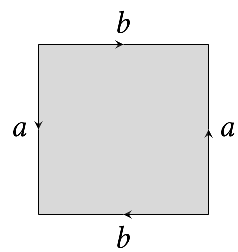


Figure A.1: Fundamental Polygon of the Real Projective Plane.

Diagrams such as that above are very useful in describing cell structures – they are called *fundamental polygons*. The vertices tell us the number of 0-cells, the lines tell us the number of 1-cells, and so forth. The direction of the arrows and labeling instead describes the attaching maps of the CW-complex. Let us work this out in an example.

Example A.5. Consider the image in Figure A.1. Since there are two different labels for the lines, this CW-complex has two 1-cells. Clearly, it has only a single 2-cell. To count the 0-cells, we see which of the four veritices end up being the "same point" after applying the attachment maps. After a moment, we see that Figure A.1 has two 0-cells. Thus,

$$\chi(\mathbb{RP}^2) = 2 - 2 + 1 = 1.$$

To see that $\mathbb{RP}^2$ contains an embedded Möbius strip, draw a strip from A to the opposite side, also labeled A. Since A reverses in orientation on the opposite side, this strip is "twisted" and reconnects with it's starting edge in an orientation reversing fashion. As a result, $\mathbb{RP}^2$ is non-orientable and therefore

$$1 = \chi(\mathbb{RP}^2) = 2 - g$$

whence $g = 1$. The genus of the real projective plane is therefore equal to 1.

As mentioned previously, the non-orientable closed surfaces can be classified (up to homeomorphic equivalence) according to their genus. Two of the most common ones are listed below:

(1) The real projective plane, $\mathbb{RP}^2$, is a non-orientable closed surface of genus equal to 1.

(2) The Klein bottle, $\mathcal{K}$, is a non-orientable closed surface of genus 2.

A.2.2 Surfaces with Boundary

What can be said about surfaces with boundary? Can we still give a homeomorphic classification of compact surfaces that *may* have a boundary? It turns out that we can, provided we again restrict ourselves to surfaces arising from cell structures whose boundary can be viewed as the disjoint union of circles. Overall, the classification of such surfaces is very hard to state concisely; we believe that the idea is best illustrated by examples and less-so by definitions and theorems.

EXAMPLE A.6. Consider the surface

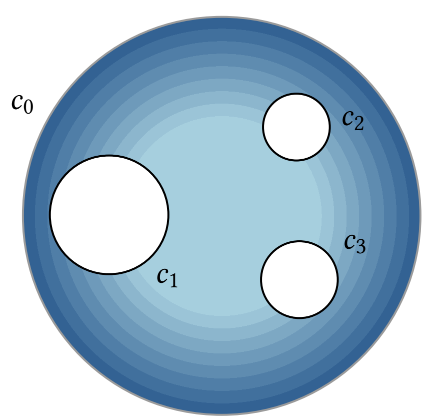


Figure A.2: Multiply Connected Domain

Clearly, this is a surface with boundary. In fact, if $\mathcal{S}$ denotes the surface in A.2, the boundary of $\mathcal{S}$ is the disjoint union of 4-circles. Thus, the surface has 4-boundary circles. On each interior boundary circle C_j, we make two cuts from C_j to C_0, thus giving rise to 12 0-cells. This places 12 points on the boundary circles, whence we have

$$12 + 6 = 18 \quad \text{1-cells.}$$

Finally, we are left with four 2-cells. Letting $\mathcal{S}^+$ denote $\mathcal{S}$ will its boundary circles filled in, we see that

$$\chi(\mathcal{S}^+) = (12 - 18 + 4) + 4 = 2.$$

Since $\mathcal{S}^+$ is orientable, the only possibility is $2 = 2 - 2g$ so that $g = 0$. Thus, $\mathcal{S}^+$ is homeomorphic to $\mathbb{S}^2$ and $\mathcal{S}$ is homeomorphic to a sphere with 4 holes.

Example A.7. We repeat the same argument for the (simpler) surface

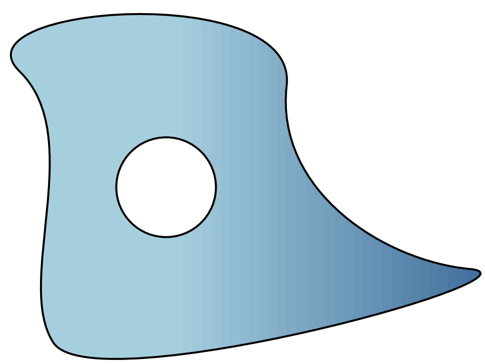


Figure A.3: Multiply Connected Domain

The trick here is to draw a line connecting the two embedded boundary circles. In doing so, we are attaching two 0-cells, one on each boundary boundary circle. Thus, we obtain

$$2 + 1 = 3$$

1-cells. After the cut, we have a single 2-cell. If $\mathcal{S}$ denotes the surface in the figure, and $\mathcal{S}^+$ denotes the same surface *after* having filled in the boundary circles, we compute:

$$\chi(\mathcal{S}^+) = (2 - 3 + 1) + \underbrace{2}_{\text{boundary circles}} = 2.$$

whence $2 = 2 - 2g$. It follows that $g = 2$ so that $\mathcal{S}^+ \cong \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$. Thus, $\mathcal{S}$ is homeomorphic to a sphere with two holes, i.e. to a bounded cylinder.

We give two more challenging examples.

EXAMPLE A.8. We will determine which compact surface the following

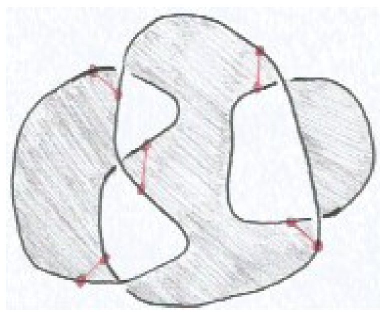


Figure A.4: Complex Structure

is homeomorphic to. Drawing a continuous normal field, it is easily seen that the surface above (denoted $\mathcal{S}$) is orientable. We make a cut at each inversion point, and in doing so we obtain 5 cuts. This gives rise to

- ten 0-cells;
- 10 + 5 = fifteen 1-cells;
- three 2-cells.

Also, $\mathcal{S}$ has two boundary circles. Let $\mathcal{S}^+$ denote $\mathcal{S}$ when we've "filled in" these boundary circles. Then, the Euler characteristic of $\mathcal{S}^+$ is given by

$$\chi(\mathcal{S}^+) = (10 - 15 + 3) + 2 = 0.$$

Thus, $0 = 2 - 2g$ whence $g = 1$. Thus, $\mathcal{S}^+$ is homeomorphic to the torus. This means that $\mathcal{S}$ is homeomorphic to a torus with two disks removed.

EXAMPLE A.9. Which surface is the following homeomorphic to?

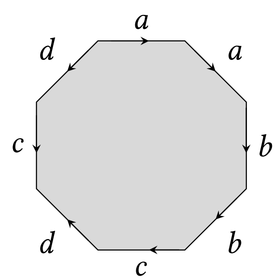


Figure A.5: Complex Structure

Let $\mathcal{S}$ be the surface described by this octagon. Then, $\mathcal{S}$ has

- 1 0-cell;
- 4 1-cells;
- 1 2-cell.

Thus, $\chi(\mathcal{S}) = 1 - 4 + 1 = -2$. Clearly, $\mathcal{S}$ has no boundary circles and is non-orientable. Thus, $-2 = 2 - g$ whence $g = 4$. We see that $\mathcal{S}$ is homeomorphic to the non-orientable closed surface of genus 4.

Example A.10. Given $\varepsilon > 0$, let B_ε denote the closed ball of radius ε in $\mathbb{C}$. Compute the one-point compactification of $\mathbb{C} - B_\varepsilon$.

Solution. First of all, we notice that for every $\varepsilon > 0$, $\mathbb{C} - B_\varepsilon$ is homeomorphic to $\mathbb{C} - B_1$. Thus, it is enough to compute the one-point compactification of the latter. However, $\mathbb{C}^\times$ is homeomorphic to $\mathbb{C} - B_1$ via the map $z \mapsto z + \frac{z}{|z|}$.

Now, $\mathbb{C}^\times$ is homeomorphic to $\mathbb{S}^2$ with the north and south poles removed. Thus, the one point compactifcation of $\mathbb{C}^\times$ must be homeomorphic to the torus of inner-radius 0.[1] □

A.3 Connected Sums of Surfaces

Let $\mathcal{M}$ and $\mathcal{N}$ be surfaces; we obtain their connected sum, denoted $\mathcal{M}\#\mathcal{N}$, by removing a disk from each and gluing them along the "circular" boundary that

[1]Such a surface is obtained by taking the two-sphere and identifying the north and south poles.

results. As it turns out, this operation is well defined! Moreover, whenever $\mathcal{M}$ and $\mathcal{N}$ can be described by CW-complexes, there holds the following:

$$\chi(\mathcal{M}\#\mathcal{N}) = \chi(\mathcal{M}) + \chi(\mathcal{N}) - 2. \tag{A.1}$$

To say even more, $\mathcal{M}\#\mathcal{N}$ is orientable *if and only if* both $\mathcal{M}$ and $\mathcal{N}$ are. It is also not too difficult to convince oneself that the 2-sphere $\mathbb{S}^2$ acts as an "identity" with respect to the connected sum. For instance, it is obvious that

$$\mathbb{S}^2\#\mathbb{S}^2 \cong \mathbb{S}^2 \quad \text{and} \quad \mathbb{T}^2\#\mathbb{S}^2 \cong \mathbb{T}^2.$$

Moreover, the operation # is commutative. Therefore, it is intuitive that the set of all closed surfaces be a monoid under the operation #, with identity element $\mathbb{S}^2$. What is very surprising, however, is that every element of this monoid is generated by the family $\{\mathbb{S}^2, \mathbb{RP}^2, \mathbb{T}^2\}$. In fact, we have the relation

$$\mathbb{T}^2\#\mathbb{RP}^2 \cong \mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2.$$

Note that, in particular, all closed orientable surfaces can be generated by $\mathbb{S}^2$ and $\mathbb{T}^2$, whilst the closed non-orientable surfaces are derived from copies of the real projective plane $\mathbb{RP}^2$.

Example A.11. We show that the Klein bottle can be described by $\mathbb{RP}^2\#\mathbb{RP}^2$. For this, we simply apply the formula

$$\chi(\mathbb{RP}^2\#\mathbb{RP}^2) = 2\chi(\mathbb{RP}^2) - 2 = 0.$$

Since $\mathbb{RP}^2\#\mathbb{RP}^2$ is non-orientable, we see that $\mathbb{RP}^2\#\mathbb{RP}^2$ has genus equal to 2, whence we are done.

A.4 Homotopy of Graphs

Let us briefly discuss the homotopy of graphs. As before, this section will not contain proofs; we instead hope to present (i.e. state) some interesting results related to the second part of this book.

Proposition A.3. *Let Γ be a connected graph. There is a bouquet of circles $\bigvee_{\alpha\in I} S_\alpha$ having the same homotopy type as Γ. In fact,*

$$\chi(\Gamma) = \chi\left(\bigvee_{\alpha\in I} S_\alpha\right). \tag{A.2}$$

In particular, $\pi_1(\Gamma) \cong \pi_1(\bigvee_{\alpha\in I} S_\alpha)$.

This property can make it relatively easy to compute the fundamental group of a graph. Indeed, given a bouquet of n-circles B, $\chi(B)$ will be given by the formula $\chi(B) = 1 - n$. Let now $\chi = \chi(\Gamma) = \chi(B)$ be the Euler characteristic of Γ. Then, $\chi = 1 - n$ so that $n = 1 - \chi$. It follows that Γ has the same fundamental group as a bouquet of $(1 - \chi)$-circles.

Theorem A.4. *Let S be a connected compact surface with boundary. Then, S has the homotopy type of a graph Γ. Especially, there is a bouquet of circles $\bigvee_{\alpha\in I} S_\alpha$ having the same homotopy type, and Euler characteristic, as S.*

We take a moment to note that this theorem implies Proposition A.3. We sketch the argument below.

Sketch of Proof. Let Γ be a connected graph. To every vertex of Γ we add a disk, and to each edge we add a compact strip strip. In doing so, we create a connected surface S with boundary that deformation retracts to Γ. In particular, $\Gamma \simeq S$. Finally, it can be shown that the Euler characteristic is homotopy type invariant. The theorem then implies the desired result. □

Appendix B

Solutions to Exercises

This chapter contains detailed solutions to the exercises given at the end of every chapter in the text. These problems are mostly taken from [MNKS] and the reader is strongly urged to attempt all problems before consulting these solutions.

B.1 Solutions to Exercises in §1.9

SOLUTION TO PROBLEM 1.1. Let $\mathfrak{T}_X$ be the topology that A inherits as a subspace of X and $\mathfrak{T}_Y$ be that from Y. We must show that $\mathfrak{T}_X = \mathfrak{T}_Y$. Recall that every open subset V of Y takes the form $V = U \cap Y$ for some open subset U of X. Therefore,

$$\begin{aligned}\mathfrak{T}_Y = \{V \cap A : V \text{ open in } Y\} &= \{U \cap Y \cap A : U \text{ open in } X\} \\ &= \{U \cap A : U \text{ open in } X\} \\ &= \mathfrak{T}_X.\end{aligned}$$

SOLUTION TO PROBLEM 1.2. Let $\mathcal{B}$ be a basis for a topology $\mathfrak{T}$ on X. We claim that $\mathfrak{T} = \bigcap_{\mathfrak{W}} \mathfrak{W}$ where the intersection is taken over all topologies $\mathfrak{W}$ containing $\mathcal{B}$. Since $\mathcal{B}$ is a basis for $\mathfrak{T}$, we know that every element of $\mathfrak{T}$ is simply the union of elements in $\mathcal{B}$. Consequently, if $\mathfrak{W}$ is a topology containing $\mathcal{B}$, we have $\mathfrak{T} \subseteq \mathfrak{W}$ since $\mathfrak{W}$ is closed with respect to unions. Especially, $\mathfrak{T} \subseteq \bigcap_{\mathfrak{W}} \mathfrak{W}$. Conversely, $\mathfrak{T}$ is a topology containing $\mathcal{B}$ and therefore appears as one of the $\mathfrak{W}$ indexing the intersection. This yields $\bigcap_{\mathfrak{W}} \mathfrak{W} \subseteq \mathfrak{T}$.

Suppose now that $\mathcal{S}$ is a subbasis for a topology $\mathfrak{T}$ on X. Let $\mathcal{B}$ denote the set of all finite intersections of elements in $\mathcal{S}$; then $\mathcal{B}$ is a basis for $\mathfrak{T}$. By the first part, $\mathfrak{T} = \bigcap_{\mathfrak{W}} \mathfrak{W}$ where the intersection is taken over all topologies $\mathfrak{W}$ containing $\mathcal{B}$. It thus suffices to show that a topology $\mathfrak{W}$ contains $\mathcal{B}$ if and only if it contains $\mathcal{S}$. Since $\mathcal{B} \supseteq \mathcal{S}$, one direction is obvious. Conversely, a topology containing $\mathcal{S}$ will contain $\mathcal{B}$ as it is closed under all finite intersections. This completes the proof.

Solution to Problem 1.3. Let X be a topological space and Y a subspace of X. Let $A \subseteq Y$ be closed in X. We claim that A is closed in Y. To this end, let $\mathrm{Cl}(A)$ denote the closure of A in Y. Then,

$$\mathrm{Cl}(A) = A \cap Y$$

since A is closed in X and hence equal to its closure (in X). Since $A \subseteq Y$, the above implies that $\mathrm{Cl}(A) = A$. This completes the proof.

Solution to Problem 1.4. We first check that $\mathcal{B}$ generates the standard topology on $\mathbb{R}$. First, notice that every element of $\mathcal{B}$ is trivially open in $\mathbb{R}$. Let now $U \subseteq \mathbb{R}$ be an open set and fix a point $x \in U$. There exists an open interval $(a, b) \subseteq U$ containing the point x. By the density of $\mathbb{Q}$ in $\mathbb{R}$, we can find rational numbers $r < q$ such that

$$x \in (r, q) \subseteq (a, b) \subseteq U.$$

Proposition 1.2 then implies that $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}$.

We now handle the second part of the problem. First, we note that every element of C is open in $\mathbb{R}_\ell$. By way of contradiction, suppose that C generates the topology on $\mathbb{R}_\ell$. Let $a, b \in \mathbb{R} \setminus \mathbb{Q}$ and consider the open set $[a, b)$ of $\mathbb{R}_\ell$. Since C is a basis for the lower limit topology,

$$[a, b) = \bigcup_{r<q} [r, q),$$

where this union is taken over some $r, q \in \mathbb{Q}$. In any case, we must have

$$a \leq r < q \leq b$$

for all r, q appearing in the union on the right hand side. Clearly, $a \in [r, q)$ for some $r, q \in \mathbb{Q}$ whence the above inequality gives $a = r$; contradicting the fact that $a \notin \mathbb{Q}$. This completes the proof.

Solution to Problem 1.5. Let $Y \subseteq X$ and let $\mathfrak{T}_Y, \mathfrak{T}'_Y$ be the topologies Y inherits from $\mathfrak{T}$ and $\mathfrak{T}'$, respectively. Then, clearly

$$\mathfrak{T}_Y = \{U \cap Y : U \in \mathfrak{T}\} \subseteq \{U \cap Y : U \in \mathfrak{T}'\} = \mathfrak{T}'_Y.$$

However, if $\mathfrak{T}'$ is strictly finer than $\mathfrak{T}$, there is no reason for $\mathfrak{T}'_Y$ to be strictly finer than $\mathfrak{T}_Y$. To prove this, we product a contrived counter example relying on $\mathbb{R}$ and $\mathbb{R}_K$. Clearly, the topology on $\mathbb{R}_K$ is strictly finer than that on $\mathbb{R}$; since

$$(-1,1) \setminus \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = (-1,0] \cup \bigcup_{n\in\mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n}\right)$$

is not open in $\mathbb{R}$ but is a basis element for the topology on $\mathbb{R}_K$. Consider the subset $[-2,-1]$ of $\mathbb{R}/\mathbb{R}_K$. The topology it inherits from $\mathbb{R}_K$ is prescribed by the basis

$$\{(a,b) \cap [-2,-1],\ [(a,b) \setminus K] \cap [-2,-1]\}$$

for $a, b \in \mathbb{R}$. It is easy to see that $[-2,-1] \subseteq K^c$ whence

$$[(a,b) \setminus K] \cap [-2,-1] = (a,b) \cap K^c \cap [-2,-1] = (a,b) \cap [-2,-1].$$

Hence, the subspace topology $[-2,-1]$ inherits from $\mathbb{R}_K$ is given by the basis

$$\{(a,b) : a < b,\ a, b \in \mathbb{R}\}$$

which also generates the topology $[-2,-1]$ inherits from $\mathbb{R}$.

Solution to Problem 1.6. Consider the complement of $A \times B$ in $X \times Y$:

$$(A \times B)^c = (X \times Y) \setminus (A \times B) = \{(x,y) : x \notin A \text{ or } y \notin B\}$$

which may be rewritten as

$$(A^c \times Y) \cup (X \times B^c).$$

Since A^c and B^c are open in X and Y, respectively, the above is the union of two basis elements for the product topology on $X \times Y$. This shows that $A \times B$ is closed in $X \times Y$.

Solution to Problem 1.7. Since $\mathrm{Cl}(B)$ is a closed set containing B, and hence A, it is immediate from the definitions that $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$. Similarly, since finite unions of closed sets are closed, $\mathrm{Cl}(A) \cup \mathrm{Cl}(B)$ is a closed set containing $A \cup B$, whence $\mathrm{Cl}(A \cup B) \subseteq \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$. For the reverse inclusion, let F be a closed set containing $A \cup B$. Since F contains both A and B, we get $\mathrm{Cl}(A) \subseteq F$ and $\mathrm{Cl}(B) \subseteq F$. Thus, $\mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$.

The exact same argument shows that $\bigcup_\alpha \mathrm{Cl}(A_\alpha) \subseteq \mathrm{Cl}(\bigcup_\alpha A_\alpha)$. To show that equality does not always hold, let $\{r_n\}_{n=1}^\infty$ be an enumeration of the rationals. Then, we clearly have

$$\bigcup_{n=1}^{\infty} \mathrm{Cl}(\{r_n\}) = \bigcup_{n=1}^{\infty} \{r_n\} = \mathbb{Q} \subsetneq \mathbb{R} = \mathrm{Cl}(\mathbb{Q}) = \mathrm{Cl}\left(\bigcup_{n=1}^{\infty} \{r_n\}\right).$$

This concludes the proof.

Solution to Problem 1.8. By Problem 1.6, we have $\mathrm{Cl}(A \times B) \subseteq \mathrm{Cl}(A) \times \mathrm{Cl}(B)$. It therefore remains to prove that $\mathrm{Cl}(A) \times \mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \times B)$. Of course, this is equivalent to

$$\mathrm{Cl}(A \times B)^{\mathsf{c}} \subseteq [\mathrm{Cl}(A) \times \mathrm{Cl}(B)]^{\mathsf{c}}.$$

So, suppose that $(x, y) \notin \mathrm{Cl}(A \times B)^{\mathsf{c}}$. There exists a basis element $U \times V$ containing (x, y), with U open in X and V open in Y, such that $U \times V$ does not intersect $A \times B$. Then, either $U \cap A = \varnothing$ or $V \cap B = \varnothing$. Without loss of generality, assume that $U \cap A = \varnothing$. Then, $x \notin \mathrm{Cl}(A)$ whence $(x, y) \in [\mathrm{Cl}(A) \times \mathrm{Cl}(B)]^{\mathsf{c}}$. This completes the proof.

Solution to Problem 1.9. Suppose X and Y are Hausdorff. Let (x_1, y_1) and (x_2, y_2) be distinct points in $X \times Y$. Without loss of generality, we will assume that $x_1 \neq x_2$. Since X is Hausdorff, we may choose disjoint open sets U_1 and U_2 of X containing x_1 and x_2, respectively. Then,

$$U_1 \times Y \quad \text{and} \quad U_2 \times Y$$

are disjoint open subsets of $X \times Y$ containing (x_1, y_1) and (x_2, y_2), respectively.

Solution to Problem 1.10. Let $(X, \mathfrak{T})$ be a Hausdorff space and Y a subspace of X. If $y_1, y_2 \in Y \subseteq X$, we can find disjoint open sets U_1, U_2 in X containing y_1 and y_2, respectively. Then, $U_1 \cap Y$ and $U_2 \cap Y$ are disjoint open subsets of Y containing y_1 and y_2, respectively. We conclude that Y is Hausdorff.

Solution to Problem 1.11. Suppose that X is Hausdorff and let $(x, y) \in \Delta^c$. Then, $x \neq y$. Choose disjoint neighbourhoods U and V, of x and y respectively. Then, $U \times V \subseteq \Delta^c$. This implies that Δ^c is open in $X \times X$.

Conversely, suppose that Δ is closed in X. Fix two distinct points $x, y \in X$. Then, $(x, y) \in \Delta^c$, where this set is open in $X \times X$. We may therefore a find basis element $U \times V$ for the topology on $X \times X$ such that $(x, y) \in U \times V \subseteq \Delta^c$. By definition, U and V are open subsets of X and will contain x, y, respectively. Since $U \times V \subseteq \Delta^c$, we also get that $U \cap V = \varnothing$. It follows that X is Hausdorff.

Solution to Problem 1.12. First, notice that $X \setminus \operatorname{Int}(A) \supseteq A^c$. Since $\operatorname{Int}(A)$ is open in X, we get that $\operatorname{Cl}(A^c) \subseteq X \setminus \operatorname{Int}(A)$. For the reverse inclusion, suppose $x \in X \setminus \operatorname{Int}(A)$. Let U be a neighbourhood of the point x and assume for a contradiction that $U \cap A^c = \varnothing$. Then, $U \subseteq A$ so that $U \subseteq \operatorname{Int}(A)$. This would imply that $x \in \operatorname{Int}(A)$, which is absurd. Thus, every neighbourhood of x intersects A^c. It follows that $x \in \operatorname{Cl}(A^c)$. We conclude that $\operatorname{Cl}(A^c) = X \setminus \operatorname{Int}(A)$, as was asserted. This completes the proof.

Solution to Problem 1.13. By definition, we have $\partial A \subseteq \operatorname{Cl}(A^c)$. Invoking the previous problem implies that $\partial A \subseteq X \setminus \operatorname{Int}(A)$. In particular, ∂A and $\operatorname{Int}(A)$ are disjoint. Also, it is immediate that $\operatorname{Int}(A) \cup \partial A \subseteq \operatorname{Cl}(A)$. Conversely, let $x \in \operatorname{Cl}(A)$ and assume $x \notin \operatorname{Int}(A)$. Let U be a neighbourhood of the point x; if $U \subseteq A$ then we would have $x \in \operatorname{Int}(A)$ contradicting our assumption. Therefore, every neighbourhood of x intersects A^c whence $x \in \operatorname{Cl}(A^c)$. Since $x \in \operatorname{Cl}(A)$, we conclude that $x \in \partial A$. Thus, $\operatorname{Cl}(A) \subseteq \partial A \cup \operatorname{Int}(A)$. This proves (1).

Suppose that ∂A is empty. By the previous part, $\operatorname{Cl}(A) = \operatorname{Int}(A) \cup \partial A = \operatorname{Int}(A)$. Therefore, $A = \operatorname{Cl}(A) = \operatorname{Int}(A)$ whence A is clopen. Conversely, suppose that A is clopen. If $x \in \partial A$, then $x \in \operatorname{Cl}(A) = A = \operatorname{Int}(A)$ and $x \in \operatorname{Cl}(A^c) = A^c = \operatorname{Int}(A^c)$. Clearly, this is impossible and so ∂A must be empty.

Let now $A \subseteq X$ be open. Then, A^c is closed. Of course, this implies that $\partial A = \operatorname{Cl}(A) \cap \operatorname{Cl}(A^c) = \operatorname{Cl}(A) \cap A^c = \operatorname{Cl}(A) \setminus A$. Conversely, suppose that $\partial A = \operatorname{Cl}(A) \setminus A$. We prove that $\operatorname{Cl}(A^c) = A^c$. Suppose for a contradiction that there exists $x \in \operatorname{Cl}(A^c)$ with $x \notin A^c$. Then, $x \in A \subseteq \operatorname{Cl}(A)$ whence $x \in \partial A$. However,

$$\partial A = \operatorname{Cl}(A) \setminus A$$

implies that $x \notin A$; which is absurd. We conclude that A^c is closed, i.e. A is open.

Finally, we show that the final point fails. Let $A := \mathbb{R}^\times$ which is an open and non-closed subset of $\mathbb{R}$. Clearly, $\operatorname{Cl}(\mathbb{R}^\times) = \mathbb{R}$. However, $\operatorname{Int}(\mathbb{R}) = \mathbb{R}$ which is not equal to $\mathbb{R}^\times$.

Solution to Problem 1.14. We show that $f(x)$ need not be a limit point of $f(A)$. Consider the constant map

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 0.$$

This function is clearly continuous. Now, 0 is a limit point of $\mathbb{R}$ since the closure of $\mathbb{R} \setminus \{0\}$ is the whole space $\mathbb{R}$. However, $f(0)$ is *not* a limit point of $f(\mathbb{R}) = \{0\}$. Certainly, $f(0) = 0$ and so $f(\mathbb{R}) \setminus \{f(0)\} = \varnothing$. Since $\varnothing$ is closed, the closure of $f(\mathbb{R}) \setminus \{f(0)\}$ is empty and does not contain $f(0) = 0$.

Solution to Problem 1.15. We argue by contradiction. Suppose that g and h are distinct continuous functions $\operatorname{Cl}(A) \to Y$, both equal to f on the entirety of A. By hypothesis, there exists a point $x \in \operatorname{Cl}(A) \setminus A$ such that $g(x) \neq h(x)$. Using now that Y is Hausdorff, we choose two disjoint open sets V_1, V_2 in Y containing $g(x)$ and $h(x)$, respectively. By continuity, the sets

$$U_1 := g^{-1}(V_1) \quad \text{and} \quad U_2 := h^{-1}(V_2)$$

are open subsets of $\operatorname{Cl}(A)$. We now claim that U_1 and U_2 are disjoint. By way of contradiction, choose $\xi \in U_1 \cap U_2$. Then,

$$g(\xi) \in V_1 \quad \text{and} \quad h(\xi) \in V_2.$$

Since $g \equiv h \equiv f$ on A and $V_1 \cap V_2 = \varnothing$, it is clear that $\xi \notin A$. But then, $U_1 \cap U_2$ is a neighbourhood of the point $\xi \in \operatorname{Cl}(A) \setminus A$; it follows that $U_1 \cap U_2$ has non-empty intersection with A. Choosing $a \in U_1 \cap U_2 \cap A$, we get

$$f(a) = g(a) = h(x) \in V_1 \cap V_2$$

which is a contradiction. Therefore, $U_1 \cap U_2$ is disjoint. This is absurd since $x \in U_1 \cap U_2$ by definition.

Solution to Problem 1.18. First, we give $\mathbb{R}^\omega$ the product topology and endow $\mathbb{R}^\infty$ with the corresponding subspace topology. We claim that $\mathbb{R}^\infty$ is *dense* in $\mathbb{R}^\omega$, i.e. that $\operatorname{Cl}(\mathbb{R}^\infty) = \mathbb{R}^\omega$. To this end, let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be an element in $\mathbb{R}^\omega$. If we can show that every neighbourhood of $\mathbf{x}$ intersects $\mathbb{R}^\infty$, it will follow that $\mathbf{x} \in \operatorname{Cl}(\mathbb{R}^\infty)$. Since $\mathbf{x}$ was taken arbitrarily, we would have the desired result. In fact, it suffices to show that every *basis element* for the product topology on $\mathbb{R}^\omega$ containing $\mathbf{x}$ intersects $\mathbb{R}^\infty$ (see Theorem 1.15-(2)). To this end, let

$$\prod_{n=1}^{\infty} U_n, \quad U_n \text{ open in } \mathbb{R}$$

be a basis element for the topology on $\mathbb{R}^\omega$ that contains $\mathbf{x}$. By definition of the product topology, all but finitely many of the U_n's will be the entire space $\mathbb{R}$. Let N be the largest element of $\mathbb{N}$ such that $U_N \neq \mathbb{R}$. Then, the point

$$\mathbf{x}' := (x_1, x_2, \ldots, x_N, 0, \ldots, 0)$$

belongs to $\mathbb{R}^\infty$ *and* to $\prod_{n=1}^{\infty} U_n$. By our earlier observations, we have that $\mathbb{R}^\infty$ is dense in $\mathbb{R}^\omega$.

Let us now give $\mathbb{R}^\omega$ the box topology. We instead show that $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$. This amounts to showing that $\mathbb{R}^\omega \setminus \mathbb{R}^\infty$ is open. Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a point *not* in $\mathbb{R}^\infty$. This is to say that $x_n \neq 0$ for infinitely many $n \in \mathbb{N}$. We now define a family $\{U_n\}_{n\in\mathbb{N}}$ of open subsets of $\mathbb{R}$ according to the following rule:

> If $x_n = 0$, take U_n to be the entire space $\mathbb{R}$. If instead $x_n \neq 0$, choose an open interval centered at x_n that does not contain 0; label this interval U_n.

The product $\prod_1^\infty U_n$ is by definition a basis element for the box topology on $\mathbb{R}^\omega$. By construction, this product will certainly contain the point $\mathbf{x}$. If we can show that $\prod_1^n U_n \subseteq \mathbb{R}^\omega \setminus \mathbb{R}^\infty$ we will be done. However, inspection shows this to be the case: since we have $x_n \neq 0$ for infinitely many n, any element the product $\prod_1^\infty U_n$ will have infinitely many non-zero coordinates. This completes the proof.

Solution to Problem 1.19. We first show that Φ is continuous. Let $\varepsilon > 0$ be given; define $\delta := \varepsilon$. If $d_X(x_1, x_2) < \delta$, then we get that $d_Y(\Phi(x_1), \Phi(x_2)) = d_X(x_1, x_2) < \delta = \varepsilon$. Hence, Φ is continuous. To see that Φ is also an injection, suppose $\Phi(x_1) = \Phi(x_2)$. Then, $0 = d_Y(\Phi(x_1), \Phi(x_2)) = d_X(x_1, x_2)$. Of course, this implies that $x_1 = x_2$.

It remains to prove that Φ is a homeomorphism onto its image. One can define an inverse function to Φ, denoted Ψ, defined as

$$\Psi : \Phi(X) \twoheadrightarrow X.$$

It is easy to check that Ψ is also an isometry; the argument used above then implies that Ψ is injective and continuous.

B.2 Solutions to Exercises in §2.10

SOLUTION TO PROBLEM 2.1. The claim amounts to proving that $[a, b]$ is closed within the order topology. This is easily seen by writing

$$X \setminus [a, b] = \{x \in X : x < a \text{ or } x > b\} = (-\infty, a) \cup (b, \infty).$$

If X has a minimal or maximal element, then the above sets are (respectively) basis elements for the order topology on X. If X does not have a minimal element, then

$$(-\infty, a) = \bigcup_{\chi < a} (\chi, a)$$

is open in the order topology. Similarly, one can show that (b, ∞) is open in X provided there does not exist a maximal element. Regardless, we get that $X \setminus [a, b]$ is open. This completes the proof.

SOLUTION TO PROBLEM 2.2. First, we give a non-empty set X the discrete topology $\mathscr{P}(X)$ and prove that it is totally disconnected. Clearly, a singleton $\{x\}$ will always be a connected subspace of X as its only open sets are $\{x\}$ and $\varnothing$. We show that these are the only connected subspaces. Let Y be a connected subspace of X and assume that Y contains more than two elements. As a subspace of X, Y also inherits the discrete topology. Certainly, the subspace topology on Y is explicitly described by

$$\{U \cap Y : U \in \mathscr{P}(X)\} = \{V \subseteq Y\} = \mathscr{P}(Y).$$

Hence, if Y contains more than two points, it can be partitioned into two non-empty disjoint open subsets (all subsets are open), thereby proving that Y is disconnected.

Now, we prove that the converse need not be true. View $\mathbb{Q}$ as a subspace of $\mathbb{R}$; we first show that $\mathbb{Q}$ does **not** have the discrete topology. For this, we need only demonstrate that $\{0\}$ is not open in $\mathbb{Q}$. Recall from Lemma 1.7 that the family

$$\mathcal{B} := \{(a, b) \cap \mathbb{Q} : a, b \in \mathbb{R},\ a < b\}$$

is a basis for the topology $\mathbb{Q}$ inherits from $\mathbb{R}$. If $\{0\}$ were open in $\mathbb{Q}$, then one could find an element $(a, b) \cap \mathbb{Q}$ of $\mathcal{B}$ having the property that

$$0 \in (a, b) \cap \mathbb{Q} \subseteq \{0\},$$

which is absurd since any open interval contains infinitely many elements of $\mathbb{Q}$. We conclude that $\{0\}$ is not open in $\mathbb{Q}$, and hence that $\mathbb{Q}$ is not discrete. It remains to prove that $\mathbb{Q}$ is totally disconnected. Again, singletons are connected. Let U be a subspace of $\mathbb{Q} \subset \mathbb{R}$ and suppose that U has two distinct points x and y. Without harm, $x < y$. Choose an irrational number ξ having the property that $x < \xi < y$; we claim that

$$U = [(-\infty, \xi) \cap U] \sqcup [(\xi, \infty) \cap U],$$

where this union is disjoint. Obviously, the two sets above are open and disjoint subsets of U. Since $x < \xi < y$, they are both non-empty. To see that the '$\subseteq$' inclusion holds, simply notice that an element of $U \subseteq \mathbb{Q}$ cannot be equal to ξ, and hence belongs to one of $(-\infty, \xi)$ or (ξ, ∞). This shows that $\mathbb{Q}$ is totally disconnected and our job here is done.

Solution to Problem 2.3. We show that $\mathbb{R}_\ell$ is **not** connected. First, we notice that $\mathbb{R}_\ell$ may be partitioned in the following way[1]

$$\mathbb{R}_\ell = (-\infty, 0) \sqcup [0, \infty) = \left(\bigcup_{n \in \mathbb{Z}_{<0}} [n, 0) \right) \sqcup \left(\bigcup_{n \in \mathbb{N}} [0, n) \right).$$

Now, both $\bigcup_{n \in \mathbb{Z}_{<0}} [n, 0)$ and $\bigcup_{n \in \mathbb{N}} [0, n)$ are open subsets of $\mathbb{R}_\ell$ (they are unions of basis elements). Since both of these sets are non-empty and disjoint, we conclude that $\mathbb{R}_\ell$ is disconnected.

Solution to Problem 2.4. We argue by contradiction; let (A, B) be a separation of the space X. Then, A and B are non-empty disjoint open subsets of X. We now take a step back and show that $\rho^{-1}(\rho(A)) = A$. By symmetry, it will also follow that $\rho^{-1}(\rho(B)) = B$.

In any case, the inclusion $A \subseteq \rho^{-1}(\rho(A))$ is obvious. Conversely, suppose that $x \in \rho^{-1}(\rho(A))$. Then, $\rho(x) = \rho(y)$ for some $y \in A$. Now, this means that $x \in \rho^{-1}(\{\rho(y)\})$. Notice also that

$$\rho^{-1}(\{\rho(y)\}) = \left[\rho^{-1}(\{\rho(y)\}) \cap A\right] \sqcup \left[\rho^{-1}(\{\rho(y)\}) \cap B\right]$$

where both the sets on the right hand side are disjoint open subsets of $\rho^{-1}(\{\rho(y)\})$. Since $y \in A$, the first of these sets is non-empty. If $x \notin A$, then $x \in B$ which makes

[1] We write $\sqcup$ or $\bigsqcup$ instead of $\cup$ or $\bigcup$ to suggest that a union is disjoint.

the latter also non-empty. But then, the above is a separation of the connected subspace $\rho^{-1}(\{\rho(y)\})$ which is absurd. We get then that $x \in A$, i.e. $\rho^{-1}(\rho(A)) = A$.

Since ρ is a quotient map and A, B are open in X, we see that $\rho(A)$ and $\rho(B)$ are open in Y. More precisely, ρ is also an *open map*. In fact, they are non-empty open sets whose union is Y, since ρ is surjective. If we can show that $\rho(A)$ and $\rho(B)$ are disjoint, we will have obtained the desired contradiction. Assume that $y \in \rho(A) \cap \rho(B)$. Then $y = f(a) = f(b)$ for $a \in A$ and $b \in B$. Then, both

$$\rho^{-1}(\{y\}) \cap A \quad \text{and} \quad \rho^{-1}(\{y\}) \cap B$$

are non-empty disjoint open subsets of $\rho^{-1}(\{y\})$ whose union is the entirely of $\rho^{-1}(\{y\})$. This contradicts the fact that $\rho^{-1}(\{y\})$ is connected and we are done.

Solution to Problem 2.5. First, notice that the claim is trivial if $f(x) = f(-x)$ for all points $x \in \mathbb{S}^1$. We may thus assume there exists a point $x \in \mathbb{S}^1$ such that $f(x) \neq f(-x)$. In fact, we may as well suppose that $f(x) > f(-x)$.

Now, the function $x \mapsto -x$ is a continuous function $\mathbb{S}^1 \to \mathbb{S}^1$, and hence so is the composite $f(-x) : \mathbb{S}^1 \to \mathbb{R}$. Denote by g the function

$$g(x) := f(x) - f(-x)$$

on $\mathbb{S}^1$. Clearly, by choice of x there holds

$$g(x) = f(x) - f(-x) > 0 \quad \text{and} \quad g(-x) = f(-x) - f(x) < 0.$$

Recall that the usual topology on $\mathbb{R}$ is identical to its order topology. Theorem 2.11 then yields the existence of a point $s \in \mathbb{S}^1$ with the property that $f(s) = f(-s)$.

Solution to Problem 2.6. Define $g(x) := f(x) - x$, which is continuous $[0, 1]$. If $f(0) = 0$, we are done. Otherwise, $f(0) > 0$. Similarly, if $f(1) = 1$ we can turn off the proof. Thus, we may assume that $f(1) < 1$. Then, there exist points $x_1, x_2 \in [0, 1]$ such that

$$g(x) := f(x) - x$$

satisfies $g(x_1) < 0$ and $g(x_2) > 0$. Applying the intermediate value theorem to the continuous function g shows that there exists a point $x^* \in [0, 1]$ such that $g(x^*) = 0$. That is, $f(x^*) = x^*$.

The claim fails for a continuous function $f : (0, 1) \to (0, 1)$. Indeed, the function $f(x) = x^2$ does not have any fixed points on this interval. For the interval $[0,1)$, consider the well defined continuous map

$$f : [0,1) \to [0,1), \quad x \mapsto \frac{x}{3} + \frac{1}{2}.$$

This function is easily seen not to have any fixed points.

Solution to Problem 2.7. We argue by contradiction. Suppose there exists $n \in \mathbb{N}$ such that $\mathbb{R} \cong \mathbb{R}^n$. Let $\phi : \mathbb{R} \to \mathbb{R}^n$ be a homeomorphism. Let σ denote the restriction of ϕ to $\mathbb{R}^\times$; we thus obtain a bicontinuous function

$$\sigma : \mathbb{R}^\times \to \mathbb{R}^n \setminus \{\phi(0)\}.$$

Now, $\mathbb{R}^\times$ is disconnected as it can be written as $(-\infty, 0) \sqcup (0, \infty)$ where both these sets are open in $\mathbb{R}^\times$. On the other hand, $\mathbb{R}^n \setminus \{v\}$ is *path connected* for any $v \in \mathbb{R}^n$. In particular, $\mathbb{R}^n \setminus \{\phi(0)\}$ is connected. However, σ^{-1} is continuous and $\mathbb{R}^\times = \sigma^{-1}(\mathbb{R}^n \setminus \{\phi(0)\})$. Since continuous functions map connected spaces to connected spaces, we have a contradiction.

Solution to Problem 2.8. This is a tricky question that is useful to keep in mind. Both implications do not hold, but it will require some contrived examples to demonstrate this fact.

(1) Let $B_1(1,0)$ denote the closed ball of radius 1 centered at $(1,0)$ in $\mathbb{R}^2$. Let $B_1(-1,0)$ denote the closed ball of radius 1 centered at $(-1,0)$. These two connected sets share the point $(0,0)$, and hence their union is connected. On the other hand, the interior of this set is the union of two disjoint *open* balls, which is disconnected. In a similar vein, the connected set $(0,1)$ has $\{0,1\}$ as its boundary, which is clearly disconnected.

(2) The converse direction also need not hold true. Recall from Problem 2.2 that $\mathbb{Q}$ is a *totally disconnected* subspace of $\mathbb{R}$. Since $\mathbb{Q}^c$ is dense in $\mathbb{R}$, it is clear that $\mathrm{Int}(\mathbb{Q}) = \varnothing$. Thus, $\mathrm{Int}(\mathbb{Q})$ is connected. By definition, one has $\partial\mathbb{Q} = \mathrm{Cl}(\mathbb{Q}) \cap \mathrm{Cl}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$, which is also connected.

This completes the problem.

Solution to Problem 2.9. Clearly,

$$(X \times Y) \setminus (A \times B) = (A^c \times Y) \cup (X \times B^c).$$

Now, fix a point $(u, v) \in A^c \times B^c$. Note that we can write $(A^c \times Y) \cup (X \times B^c)$ as

$$\bigcup_{\substack{x \in X \setminus A \\ y \in Y \setminus B}} [(u \times Y) \cup (X \times y)] \cup [(x \times Y) \cup (X \times v)] .$$

Since every set indexed by the union above contains the point (u, v), the above is easily seen to be connected.

Solution to Problem 2.10. We begin by showing that

$$X = \operatorname{Int}(A) \sqcup \operatorname{Int}(A^c) \sqcup \partial A.$$

Note that the use of '$\sqcup$' is justified because ∂A never intersects $\operatorname{Int}(A)$ (this is Problem 1.13). Let now $x \in X$ but assume that $x \notin \operatorname{Int}(A) \sqcup \operatorname{Int}(A^c)$; we will show that $x \in \partial A$. Since $x \notin \operatorname{Int}(A)$, any neighbourhood of x has non-empty intersection with A^c. Similarly, $x \notin \operatorname{Int}(A^c)$ means that any neighbourhood of x intersects A. However, both of these statements mean that

$$x \in \operatorname{Cl}(A) \cap \operatorname{Cl}(X \setminus A) = \partial A.$$

Having now verified the aforementioned identity, the proof is easily within reach. If C does not intersect ∂A, then it is contained within the union $\operatorname{Int}(A) \sqcup \operatorname{Int}(A^c)$, which therefore forms a separation of C.

Solution to Problem 2.11. To be added.

Solution to Problem 2.12. Suppose that X is locally path connected and let U be a connected open subset of X. We shall show that U is path connected. To this end, let $\{\mathscr{C}_\alpha\}$ denote the path components of the subspace U. Using that X is locally path connected, Theorem 2.16 gives that every $\mathscr{C}_\alpha$ is open in X, and hence in U. On the other hand, U is the disjoint union of these $\mathscr{C}_\alpha$. Since U is connected, we see that U can only have a single path component. This implies that U is path connected.

Solution to Problem 2.13. We begin with (1). Let $y \in Y$ be given. Since $\rho \circ f \equiv \mathbf{1}_Y$, we see that $\rho(f(y)) = y$. Therefore, ρ is a surjective map. Let now $V \subseteq Y$ be such that $\rho^{-1}(V)$ is open in X. Then,

$$V = \mathbf{1}_Y^{-1}(V) = (\rho \circ f)^{-1}(V) = f^{-1}(\rho^{-1}(V)).$$

Since f is continuous and $\rho^{-1}(V)$ is open in X, the above gives that V is open in Y. Recalling that ρ is continuous, it follows that ρ is a quotient map.

Let now A be a subspace of X and $\mathfrak{r} : X \to A$ a retraction. Clearly, $\mathfrak{r}$ is continuous and surjective. Using the first part, we will show that $\mathfrak{r}$ is a quotient map. Consider the inclusion map

$$\mathfrak{i} : A \hookrightarrow X, \quad a \mapsto a,$$

which we know to be continuous by Proposition 1.22. It is not difficult to see that $\mathfrak{r} \circ \mathfrak{i} \equiv \mathbf{1}_A$. Invoking (1) then yields the desired conclusion.

Solution to Problem 2.14. As the restriction of a continuous function, the function q is continuous from A to $\mathbb{R}$. Using the previous problem, we will show that q is a quotient map. Consider the function

$$f : \mathbb{R} \to A, \quad x \mapsto (x, 0).$$

This function is well defined as $\mathbb{R} \times \{0\}$ is a subspace of A. In fact, f is a homeomorphism between $\mathbb{R}$ and $\mathbb{R} \times \{0\}$. Consequently, it is an easy consequence of Proposition 1.22 that f is a continuous function $\mathbb{R} \to A$. Since $q \circ f \equiv \mathbf{1}_{\mathbb{R}}$, the previous problem implies that q is a quotient map, as was required.

It remains to show that q is neither open nor closed. First, consider the open subset of $\mathbb{R}^2$ given by $(-1, 1) \times (-1, 1)$. It is not difficult to see that

$$((-1, 1) \times (-1, 1)) \cap A = [0, 1) \times (-1, 1)$$

is an open subset of A. However, the image of this set under the action of q is simply $[0, 1)$ which is **not** open in $\mathbb{R}$. This shows that q is not an open map. The proof that q is not closed is slightly more contrived. Consider the set

$$\Gamma := \left\{ \left(x, \frac{1}{x} \right) : x > 0 \right\} \subset \mathbb{R}^2.$$

We now show that Γ is closed in $\mathbb{R}^2$. Let $\left\langle \left(x_n, \frac{1}{x_n} \right) \right\rangle_{n \in \mathbb{N}}$ be a sequence of points converging to $(x, y) \in \mathbb{R}^2$. Then, both $(x_n)_{n \in \mathbb{N}}$ and $\left(\frac{1}{x_n} \right)_{n \in \mathbb{N}}$ are convergent sequences in $\mathbb{R}$, with limits x and y respectively. Therefore, $x \notin \{0, \infty\}$. By continuity, it follows that

$$y = \lim_{n \to \infty} \frac{1}{x_n} = \frac{1}{x}.$$

We conclude that $(x, y) \in \Gamma$ and that Γ is closed in $\mathbb{R}^2$.

However, $q(\Gamma)$ is precisely the set $(0, \infty)$ which is not closed in $\mathbb{R}$. This is what we had to show.

Solution to Problem 2.15. Let us fix $x \in \mathbb{R}^2 \setminus A$. Let Λ_x be the set of all lines passing through x and observe that all such lines intersect only at $x \notin A$. Now, denote by $\mathscr{A}_x$ the family of all lines in Λ_x intersecting A at some point. Given a line $\lambda \in \mathscr{A}_x$, choose a point $a(\lambda) \in \lambda \cap A$. Since all lines in Λ_x intersect only at x, the map

$$\mathscr{A}_x \ni \lambda \longrightarrow a(\lambda) \in A$$

is injective. This proves that $\mathscr{A}_x$ is a countable subset of the uncountably family Λ_x. Thus, at every point $x \notin A$, there are uncountably many lines, passing through x, not intersecting the set A.

We now choose distinct points p, q from $\mathbb{R}^2 \setminus A$. Let ℓ be a line in Λ_p that does not intersect A. If ℓ also passes through q, we are done. Otherwise, the only line passing through q that does **not** intersect ℓ is the line $\hat{\ell}$ parallel to ℓ which passes through q. Thus, if we choose any line $\gamma \in \Lambda_q \setminus \{\hat{\ell}\}$, we obtain a line passing through q which intersects ℓ. Since both ℓ and γ never intersect points in A, we have shown that p and q are path connected.

Solution to Problem 2.16. Fix a non-empty open set $U \subseteq Y$. Let C be a connected component of U; it is enough to show that C is open in Y. To this end, we consider the pre-image $\rho^{-1}(C) \subseteq \rho^{-1}(U)$, where the latter is open in X. Decompose $\rho^{-1}(U)$ into its connected components, $\rho^{-1}(U) = \bigsqcup_t [t]$, where the components $[t]$ are open in X. Suppose we restrict ourselves to those t such that $\rho^{-1}(C) \cap [t] \neq \varnothing$. We claim that

$$\rho^{-1}(C) = \bigsqcup_{\rho^{-1}(C) \cap [t] \neq \varnothing} [t].$$

Obviously, we may assume without loss of generality that $t \in \rho^{-1}(C)$ for the indexing above. The inclusion $\rho^{-1}(C) \subseteq \bigsqcup_{\rho^{-1}(C) \cap [t] \neq \varnothing} [t]$ follows from the fact that every element of $\rho^{-1}(C)$ is contained within some connected component of $\rho^{-1}(U)$. Conversely, let $x \in [t]$ for some $t \in \rho^{-1}(C)$. Then,

$$\rho(x) \in \rho([t]).$$

Moreover, by the continuity of ρ, we know that $\rho([t])$ will be a connected subspace of U intersecting the component C. Thus, $\rho([t]) \subseteq C$ whence we see that

$x \in \rho^{-1}(C)$. Consequently,

$$\rho^{-1}(C) = \bigsqcup_{\rho^{-1}(C)\cap[t]\neq\varnothing} [t].$$

This means that $\rho^{-1}(C)$ can be expressed as the union of open subsets of X. Since ρ is a quotient map, it follows that C is open in Y.

B.3 Solutions to Exercises in §3.6

Solution to Problem 3.1. We argue by contradiction. Suppose that $(x_\lambda)_{\lambda\in\Lambda}$ is a net in X converging to distinct points x and y. Since X is Hausdorff, we may choose disjoint neighbourhoods U and V, respectively containing x and y. Since (x_λ) converges to x, there exists $\lambda_1 \in \Lambda$ with the property that $x_\lambda \in U$ for all $\lambda \gtrsim \lambda_1$. Similarly, since $x_\lambda \to y$, there exists $\lambda_2 \in \Lambda$ such that $x_\lambda \in V$ for all $\lambda \gtrsim \lambda_2$. Let $\nu \in \Lambda$ be such that $\nu \gtrsim \lambda_1$ and $\nu \gtrsim \lambda_2$. Then, $x_\nu \in U \cap V$, which is absurd. This completes the proof.

Solution to Problem 3.2. Let $x \in \mathrm{Cl}\,(A)$ be given and fix a net $(x_\lambda)_{\lambda\in\Lambda}$ of points in A which converges to x. The existence of such a net is guaranteed by Lemma 3.10. Let now g, h be two continuous functions $\mathrm{Cl}\,(A) \to Y$ agreeing with f on A. Subsequently, Theorem 3.11 implies that

$$g(x) = \lim_{\lambda\in\Lambda} g(x_\lambda) = \lim_{\lambda\in\Lambda} h(x_\lambda) = h(x).$$

By the previous problem, limits of nets in Hausdorff spaces are unique. It thus follows from the above identity that $g \equiv h$. The proof is now complete.

Solution to Problem 3.3. See Lemma 3.24.

Solution to Problem 3.4. Refer to Theorem 3.23.

Solution to Problem 3.5. This is a very nice problem. Let $\phi : X \to Y$ be a homeomorphism of spaces. Being one point compactifications, there exist formal points x^* and y^* such that $X^* \setminus X = \{x^*\}$ and $Y^* \setminus Y = \{y^*\}$. Moreover, Theorem 2.26 tells us that X^* and Y^* are themselves compact Hausdorff spaces.

Consider now the function

$$\varphi : X^* \to Y^*, \quad \varphi(x) := \begin{cases} \phi(x), & x \in X, \\ y^*, & x = x^*. \end{cases}$$

Since ϕ is a homeomorphism, it is clear that φ is a bijection between X^* and Y^*. We claim that φ is a homeomorphism. Since X and Y are both compact Hausdorff, it suffices (by Theorem 2.25) to show that φ is continuous. Let $V \subseteq Y^*$ be open; we see from (2.2) that either

(1) V is an open subset of Y;

(2) $V = Y \setminus K$ for a compact subset K of Y.

For the former, $\varphi^{-1}(V) = \phi^{-1}(V)$ is open in X, and hence X^*, by the continuity of ϕ. In the latter case, we have

$$\varphi^{-1}(V) = \varphi^{-1}(Y^*) \setminus \varphi^{-1}(K) = \varphi^{-1}(Y^*) \setminus \phi^{-1}(K) = X^* \setminus \phi^{-1}(K).$$

Clearly, ϕ^{-1} is a continuous function $Y \to X$, whence $\phi^{-1}(K)$ is compact in X. By (2.2), we get that $\varphi^{-1}(V)$ is *open* in Y^*. This is what had to be proven.

Solution to Problem 3.6. Since $\mathbb{N}$ is a subspace of $\mathbb{R}$, it is Hausdorff with the subspace topology. It is also locally compact for, given $x \in \mathbb{N}$, the set

$$\{x\} = \mathbb{N} \cap \left(x - \frac{1}{2}, x + \frac{1}{2}\right)$$

is a compact neighbourhood of x. Since $\mathbb{N}$ is clearly not compact, it has a one-point compactification. Alternatively, one could show that $\mathbb{N}$ has the discrete topology $\mathscr{P}(\mathbb{N})$.

As a first step, we will show that $\mathbb{N}$ is homeomorphic to the subspace of $\mathbb{R}$ described by

$$\mathcal{N} := \left\{\frac{1}{n} : n \in \mathbb{N}\right\}.$$

To see that this is so, consider the homeomorphism

$$f : \mathbb{R}^\times \to \mathbb{R}^\times, \quad x \mapsto \frac{1}{x}.$$

The restriction of f to $\mathbb{N}$ yields a continuous bijection $g : \mathbb{N} \to f(\mathbb{N}) = \mathcal{N}$. However, one can obtain the inverse function to g by restricting f^{-1} to a function $\mathcal{N} \to \mathbb{N}$, which is also continuous. We conclude that $\mathbb{N} \cong \mathcal{N}$.

Suppose for the moment that the one-point compactification of $\mathcal{N}$ is homeomorphic to $\mathcal{Z}$. Since $\mathbb{N} \cong \mathcal{N}$, the previous problem would imply that $\mathcal{Z}$ is homeomorphic to the one-point compactification of $\mathbb{N}$. We are therefore reduced to showing that $\mathcal{Z}$ is homeomorphic to the one-point compactification of $\mathbb{N}$. By Theorem 2.26, it is enough to show that

(1) $\mathcal{Z}$ is compact Hausdorff containing $\mathcal{N}$ as a subspace;

(2) $\mathcal{Z} \setminus \mathcal{N}$ is a singleton.

The second point is trivial. For the first, we need only check that $\mathcal{Z}$ with the subspace topology from $\mathbb{R}$ is compact Hausdorff. As a subspace of $\mathbb{R}$, $\mathcal{Z}$ is necessarily Hausdorff. Clearly, $\mathcal{Z}$ is bounded in $\mathbb{R}$. To see that it is closed, write

$$\mathbb{R} \setminus \mathcal{Z} = (-\infty, 0) \cup (1, \infty) \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n} \right)$$

which is evidently open in $\mathbb{R}$. It thus follows from the Heine-Borel theorem that $\mathcal{Z}$ is compact in $\mathbb{R}$, and hence also with the subspace topology. Since we can then view $\mathcal{N}$ as a *subspace* of $\mathcal{Z}$, we see that the one-point compactification of $\mathcal{N}$ is homeomorphic to $\mathcal{Z}$, as was required.

Solution to Problem 3.7. We will show that $\mathscr{D}^c$ is open in X. Let $x \in \mathscr{D}^c$ be given; then $f(x) \neq g(x)$. Using that Y is Hausdorff, we choose two disjoint neighbourhoods U and V of $f(x)$ and $g(x)$, respectively. Since f and g are continuous functions, the pre-images $f^{-1}(U)$ and $g^{-1}(V)$ are open subsets of X. Moreover, $f^{-1}(U) \cap g^{-1}(V)$ is open and non-empty since $f(x) \in U$ and $g(x) \in V$. We claim that $f^{-1}(U) \cap g^{-1}(V) \cap \mathscr{D} = \varnothing$. To see that this is so, suppose that $t \in f^{-1}(U) \cap g^{-1}(V)$ has the property that $f(t) = g(t)$. Then,

$$f(t) \in U \quad \text{and } f(t) = g(t) \in V;$$

contradicting the choice of U and V. We conclude that $f^{-1}(U) \cap g^{-1}(V)$ is an open subset of X containing x that is contained within $\mathscr{D}^c$.

To recap, for each $x \in \mathscr{D}^c$ we have found a neighbourhood, say W_x, of x that is contained within $\mathscr{D}^c$. Writing $\mathscr{D}^c = \bigcup_{x \in \mathscr{D}^c} W_x$ shows that $\mathscr{D}^c$ is open.

Solution to Problem 3.8. It was shown that the subtraction map $\ominus : V \times V \to V$ is continuous. Let $x \in V$ be given and fix a closed set $F \subset V$ that does not contain the point x. Thus, $U := F^{\mathsf{c}}$ is a neighbourhood of x. By continuity, $\ominus^{-1}(U)$ is a neighbourhood of the point $(x, 0)$. Hence, we can choose open sets $O_1, O_2 \subseteq V$ such that

$$(x, 0) \in O_1 \times O_2 \subseteq \ominus^{-1}(U).$$

In particular, O_1 is a neighbourhood of x. We now claim that O_1 and $F + O_2$ are disjoint. To see this, we proceed by contradiction. Assume one can find $\xi \in O_1$ belonging to $O_2 + F$. Then,

$$\xi = u_1 = u_2 + f$$

for $u_1 \in O_1$, $u_2 \in O_2$, and $f \in F$. But then, $\ominus(u_1, u_2) = u_1 - u_2 = f$ belongs to F. Since $\ominus(O_1 \times O_2) \subseteq U = F^{\mathsf{c}}$, this claim is proven. Finally, note that $F \subseteq F + O_2$. By our discussions near the start of §3.5, $O_2 + F$ is open.

Remark B.1. Let G be a group with an underlying topology $\mathfrak{T}$. We say that G is a *topological group* if the multiplication and inversion mappings

$$\otimes : G \times G \longrightarrow G, \quad (g, h) \mapsto gh,$$
$$\iota : G \longrightarrow G, \quad g \mapsto g^{-1}$$

are continuous. It is not difficult to see that every topological vector space is a topological group, with $\oplus$ being the group operation. In this case, the dilation map δ_{-1} is the inversion map. In any case, one can "easily" modify the proof in order to prove the following:

> *Let G be a topological group. For each $x \in G$ and every closed set F not containing x, there exist two disjoint open sets containing x and F, respectively.*

Thus, every T_1-topological group is necessarily regular.

Solution to Problem 3.9. One direction follows at once from the Urysohn metrization theorem. Conversely, we assume that X is a metrizable compact Hausdorff space; our goal is to prove that X has a countable basis. Given $n \in \mathbb{N}$, let us consider the family

$$\left\{B\left(x, \frac{1}{n}\right) : x \in X\right\}$$

which clearly forms an open cover of X. Since X is compact, we can find a finite subset X_n of X such that the family (of open sets)

$$\mathcal{B}_n := \left\{B\left(x, \frac{1}{n}\right) : x \in X_n\right\}$$

covers X. Define $\mathcal{B} := \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$, which is obviously a countable family of open sets in X. All that remains is to show that $\mathcal{B}$ is a basis. To this end, let $x \in X$ and fix a neighbourhood U of x. There is some $n \in \mathbb{N}$ so large that $B(x, 1/n) \subseteq U$. Consider the family $\mathcal{B}_{2n}$, which covers X. Hence, there is some $x_l \in X_{2n}$ such that $x \in B(x_l, 1/2n)$. By the triangle inequality, we have

$$B\left(x_l, \frac{1}{2n}\right) \subseteq B\left(x, \frac{1}{n}\right) \subseteq U,$$

which completes the proof.

B.4 Solutions to Exercises in §4.6

Solution to Problem 4.1. This problem is mostly notation and keeping track of symbols. Let $H : X \times \mathbb{I} \to Y$ be a continuous function such that

$$H(x, 0) \equiv h(x) \quad \text{and} \quad H(x, 1) \equiv h'(x).$$

Similarly, choose a continuous function $K : Y \times \mathbb{I} \to Z$ with

$$K(y, 0) \equiv k(y) \quad \text{and} \quad K(y, 1) \equiv k'(y).$$

Now, define a function $L : X \times \mathbb{I} \to Z$ by

$$(x, t) \xrightarrow{H(x,t)\times \mathbf{1}_{\mathbb{I}}} (H(x, t), t) \xrightarrow{K} K\left(H(x, t), t\right).$$

If $U \times J$ is an open set generating the product topology on $Y \times \mathbb{I}$ we compute

$$\left(H(x, t) \times \mathbf{1}_{\mathbb{I}}\right)^{-1} (U \times J) = H^{-1}(U) \cap (X \times J).$$

Therefore, $H(x, t) \times \mathbf{1}_{\mathbb{I}}$ is continuous and so must be the composite L. Now, we show that L is a homotopy between $k \circ h$ and $k' \circ h'$. To this end, let $x \in X$ and compute

$$L(x, 0) \equiv K(H(x, 0), 0) \equiv K(h(x), 0) \equiv k(h(x)) \equiv (k \circ h)(x).$$

Similarly, $L(x, 1) \equiv (k' \circ h')(x)$. This is precisely what was to be shown.

Solution to Problem 4.2. We handle (i) first. Let X denote $\mathbb{R}$ or $\mathbb{I}$ (the same proof applies in both cases). Consider the continuous function

$$H : X \times \mathbb{I} \to X, \quad (x, t) \mapsto (1 - t)x.$$

If $X = \mathbb{R}$, this is clearly well defined. If $X = \mathbb{I}$ instead, then $0 \leq (1 - t)x \leq 1$ for all $x, t \in \mathbb{I}$ so that H is *also* a well defined continuous function. Since H is clearly a homotopy between $\mathbf{1}_X$ and the constant function $\mathbf{0}$, the claim follows.

We now attack (ii). Let X be a contractible space and fix two points $x_0, x_1 \in X$. Since X is contractible, we can choose a continuous function

$$H : X \times \mathbb{I} \to X, \quad H(x, t) = \begin{cases} H(\cdot, 0) \equiv \mathbf{1}_X, \\ H(\cdot, 1) \equiv \xi \end{cases}$$

for some *fixed* point $\xi \in X$. Consider now the continuous function $\gamma_0 : \mathbb{I} \to X$ obtained through the following composition

$$t \longrightarrow (x_0, t) \longrightarrow H(x_0, t).$$

This is clearly a *path* starting at x_0 and ending at ξ. Similarly, one can construct a path γ_1 starting at x_1 and ending at ξ. But then, $\gamma_1(1 - t)$ is a path from ξ to x_1. This means that $\gamma_0 * \gamma_1$ is a path from x_0 to x_1. This yields the statement.

We now demonstrate (iii). Let $f : X \to Y$ be a continuous function. Denote by H a homotopy between the identity map $\mathbf{1}_Y$ and some constant function

$$\sigma : X \longrightarrow Y, \quad x \mapsto y_0.$$

It suffices to show that $f \simeq \sigma$. To this end, consider the function

$$G : X \times \mathbb{I} \longrightarrow Y, \quad (x, t) \mapsto (f(x), t) \mapsto H(f(x), t).$$

By composition, G is continuous. Also, observe that $G(x, 0) \equiv H(f(x), 0) \equiv f(x)$ and $G(x, 1) \equiv H(f(x), 1) \equiv y_0$. This proves (iii).

For the final claim, we first show that any two constant maps $X \to Y$ are homotopic. Let $\tau_1(x) \equiv y_1$ and $\tau_2(x) \equiv y_2$ be constant maps $X \to Y$. Since Y is path connected, we choose a path

$$\gamma : \mathbb{I} \to Y$$

connecting the points y_1 and y_2. Then, the composite

$$H : X \times \mathbb{I} \ni (x, t) \longmapsto t \longmapsto \gamma(t) \in Y$$

is continuous and satisfies both $H(x, 0) \equiv y_1$ and $H(x, 1) \equiv y_2$. Hence, $\tau_1 \simeq \tau_2$. Now, fix a continuous function $f : X \to Y$. In light of this previous argument, it suffices to show that f is homotopic to the constant map $x \mapsto f(x_0)$, where x_0 is chosen so that $\mathbf{1}_X$ is homotopic to the association $x \mapsto x_0$. Let $F : X \times \mathbb{I} \to X$ be a continuous function with

$$F(x, 0) \equiv \mathbf{1}_X \quad \text{and} \quad F(x, 1) \equiv x_0.$$

Then, the function $G : X \times \mathbb{I} \to Y$ obtained through the composition $f(F(x, t))$ is continuous. Moreover, $G(x, 0) \equiv f(x)$ and $G(x, 1) \equiv f(x_0)$. The claim follows from this.

Solution to Problem 4.3. The path γ can be visualized as the path from x_0 to x_2 obtained by "gluing" α and β together. Now, this means that $\bar{\gamma}$ is a path in X from x_2 to x_1. Hence, γ induces a homomorphism of groups:

$$\begin{aligned} \hat{\gamma} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_2), \quad [f] &\mapsto [\bar{\gamma}] * [f] * [\gamma] \\ &\mapsto [\bar{\gamma} * f * \gamma]. \end{aligned}$$

We are asked to show that $\hat{\gamma} \equiv \hat{\beta} \circ \hat{\alpha}$. If $[f] \in \pi_1(X, x_0)$, we get

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

so that

$$\begin{aligned} \left(\hat{\beta} \circ \hat{\alpha}\right)([f]) &= [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta] \\ &= [\bar{\beta} * \bar{\alpha}] * [f] * [\alpha * \beta]. \end{aligned}$$

Clearly, $[\alpha * \beta] \equiv [\gamma]$. If we can show that $[\bar{\beta} * \bar{\alpha}] \equiv [\bar{\gamma}]$, we will be done. By definition,

$$\left(\bar{\beta} * \bar{\alpha}\right)(s) = \begin{cases} \bar{\beta}(2s), & 0 \le s \le \frac{1}{2}, \\ \bar{\alpha}(2s - 1), & \frac{1}{2} \le s \le 1 \end{cases} = \begin{cases} \beta(1 - 2s), & 0 \le s \le \frac{1}{2}, \\ \alpha(1 - (2s - 1)), & \frac{1}{2} \le s \le 1. \end{cases}$$

Thus,

$$\left(\bar{\beta} * \bar{\alpha}\right)(s) = \begin{cases} \beta(1 - 2s), & 0 \le s \le \frac{1}{2}, \\ \alpha(2 - 2s), & \frac{1}{2} \le s \le 1. \end{cases}$$

On the other hand, for each $s \in \mathbb{I}$ one has

$$\bar{\gamma}(s) = \gamma(1 - s) = (\alpha * \beta)(1 - s) = \begin{cases} \alpha(2(1 - s)) & 0 \le (1 - s) \le \frac{1}{2}, \\ \beta(2(1 - s) - 1), & \frac{1}{2} \le 1 - s \le 1. \end{cases}$$

The above simplifies to yield

$$\bar{\gamma}(s) = \gamma(1-s) = (\alpha * \beta)(1-s) = \begin{cases} \alpha(2-2s) & \frac{1}{2} \leq s \leq 1, \\ \beta(1-2s), & 0 \leq s \leq \frac{1}{2}. \end{cases}$$

This completes the proof.

Solution to Problem 4.4. Suppose that $\pi_1(X, x_0)$ is Abelian and let α, β be paths from x_0 to x_1, in X. Since X is path connected, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$. Especially, the latter is also Abelian. For every point $[f]$ in $\pi_1(X, x_0)$ we compute

$$\begin{aligned} \hat{\alpha}([f]) &= [\bar{\alpha}] * [f] * [\alpha] \\ &= [\bar{\alpha}] * [\beta] * [\bar{\beta}] * [f] * [\beta] * [\bar{\beta}] * [\alpha] \\ &= \underbrace{([\bar{\alpha}] * [\beta])}_{\in \pi_1(X, x_1)} * \underbrace{\left([\bar{\beta}] * [f] * [\beta]\right)}_{\in \pi_1(X, x_1)} * \underbrace{\left([\bar{\beta}] * [\alpha]\right)}_{\in \pi_1(X, x_1)} \\ &= ([\bar{\alpha}] * [\beta]) * \left([\bar{\beta}] * [\alpha]\right) * \left([\bar{\beta}] * [f] * [\beta]\right) \\ &= [e_{x_1}] * \left([\bar{\beta}] * [f] * [\beta]\right) \\ &= \left([\bar{\beta}] * [f] * [\beta]\right) \\ &= \hat{\beta}([f]). \end{aligned}$$

Conversely, we argue that $\pi_1(X, x_0)$ is Abelian. Let $[f], [g]$ be two elements of $\pi_1(X, x_0)$ and fix $x_1 \in X$. Let α be a path from x_0 to x_1 and put $\beta := f * \alpha$, which means that β is a path from x_0 to x_1. By assumption, the homomorphisms

$$\hat{\alpha}, \hat{\beta} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

are equivalent. Hence, for all $[g] \in \pi_1(X, x_0)$ one has

$$\hat{\alpha}([g]) = [\bar{\alpha}] * [g] * [\alpha] = [\bar{\beta}] * [g] * [\beta] = \hat{\beta}([g]).$$

Multiplying through by $[\beta]$ yields

$$[\beta] * [\bar{\alpha}] * [g] * [\alpha] = [g] * [\beta].$$

Noticing that $[\beta] = [f * \alpha] = [f] * [\alpha]$, the above reduces to

$$[f] * [g] * [\alpha] = [g] * [f] * [\alpha].$$

Multiplying through by $[\bar{\alpha}]$ on the right shows that $[f] * [g] = [g] * [f]$.

Solution to Problem 4.5. Fix a point $a_0 \in A$ and let $j : A \hookrightarrow X$ be the standard inclusion map. Observe that the composite $r \circ j$ corresponds to the identity map $\mathbf{1}_A : A \to A$. Now, the continuous maps r and j both induce homomorphisms

$$j_* : \pi_1(A, a_0) \to \pi_1(X, a_0), \quad r_* : \pi_1(X, a_0) \to \pi_1(A, a_0).$$

Hence, $r_* \circ j_* \equiv (r \circ j)_* \equiv (\mathbf{1}_A)_*$, where $(\mathbf{1}_A)_*$ is the identity map on $\pi_1(A, a_0)$. It follows that $r_*(\cdot)$ is surjective.

Solution to Problem 4.6. Clearly, π is a continuous surjection $X \times Y \twoheadrightarrow X$. Let us fix a point $x \in X$ and consider the open set $X \ni x$. We show that X is evenly covered by π. Certainly, observe that one can write $\pi^{-1}(X) = \bigsqcup_{y \in Y} X \times \{y\}$. Since Y has the discrete topology, every product $X \times \{y\}$ is open in $X \times Y$. For every $y \in Y$, the restriction of π to $X \times \{y\}$ is clearly a homeomorphism onto X.

Solution to Problem 4.7. Let $h_* : \pi_1(A, a_0) \to \pi_1(Y, y_0)$ be the induced homomorphism of fundamental groups. Denote by $j : A \hookrightarrow \mathbb{R}^n$ the standard inclusion map and notice that $k \circ j \equiv h$ on A. Hence, $j_* \equiv (k \circ j)_* \equiv k_* \circ j_*$. Now, j_* is a group homomorphism $\pi_1(A, a_0) \to \pi_1(\mathbb{R}^n, a_0)$. Since $\mathbb{R}^n$ is convex, it is simply connected. This means that j_*, and hence h_*, is the trivial homomorphism.

Solution to Problem 4.8. Obviously, we may assume that U is non-empty, for otherwise the claim is trivial. Suppose that we can decompose $\rho^{-1}(U)$ in the following way:

$$\rho^{-1}(U) = \bigsqcup_{\alpha} V_\alpha \quad \text{and} \quad \rho^{-1}(U) = \bigsqcup_{\beta} W_\beta,$$

where $\rho|_{V_\alpha} : V_\alpha \to U$ and $\rho|_{W_\beta} : W_\beta \to U$ are homeomorphisms. We will show that every V_α is contained with precisely one of the W_β. Since every V_α and W_β is homeomorphic to U, these must all be connected spaces in their own right. Now, $V_\alpha \subseteq \bigsqcup_\beta W_\beta$ means we can choose an index set B such that $V_\alpha \subseteq \bigsqcup_{\beta \in B} W_\beta$ and $V_\alpha \cap W_\beta \neq \varnothing$ for all $\beta \in B$. Then,

$$V_\alpha = \bigsqcup_{\beta \in B} \left(W_\beta \cap V_\alpha \right),$$

where each $W_\beta \cap V_\alpha$ is a non-empty open subset of V_α. Since V_α is connected, we see that the indexing set B can only contain a single element. Thus, there exists some W_1 from our list such that $V_\alpha \subseteq W_1$.

The argument we have employed is symmetric, whence we can select some V_1 with the property that $W_1 \subseteq V_1$. But, this implies that $V_\alpha \subseteq V_1$. We deduce that $W_1 = V_\alpha$. Hence, every V_α is equal to precisely one of the W_β. By symmetry, we can also conclude that every W_β will be equal to precisely one V_α.

Solution to Problem 4.9. Let $\mathbb{N}^*$ denote the one-point compactification of $\mathbb{N}$ (i.e. $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$). For $n \in \mathbb{N}^*$ let us put

$$B_n := \left\{ b \in B : \left| \rho^{-1}(b) \right| = k \right\}.$$

Hence, B can be expressed as the disjoint union $\bigsqcup_{n \in \mathbb{N}^*} B_n$. Now, we know that B_k is non-empty. Since B is connected, it is enough to show that B_k is clopen in B. To achieve this, we will require the following lemma.

Lemma B.1. *For each $n \in \mathbb{N}^*$, B_n is an open subset of B.*

Proof of Lemma. Let $n \in \mathbb{N}^*$ be given and assume without harm that $B_n \neq \emptyset$. Fix a point $b \in B_n$ and choose a neighbourhood U_b of b that is evenly covered by ρ. There exists an index set I and pairwise disjoint open sets V_α in E such that

$$\rho^{-1}(U_b) = \bigsqcup_{\alpha \in I} V_\alpha.$$

We may also assume that $\rho|_{V_\alpha}$ is a homeomorphism $V_\alpha \to U_b$. Since $b \in U_b$, we see that

$$\rho^{-1}(b) = \rho^{-1}(b) \cap \rho^{-1}(U_b) = \bigsqcup_{\alpha \in I} \rho^{-1}(b) \cap V_\alpha.$$

Given that ρ is a homeomorphism $V_\alpha \to U_b$, there is precisely one element in each V_α mapping to b under ρ. In fact, this point will be given by every intersection $\rho^{-1}(b) \cap V_\alpha$. Recall that $\rho^{-1}(b)$ contains n-elements. Because the V_α are pairwise disjoint, we conclude that $|I| = n$. Now, consider a general $y \in U_b$. A similar argument yields

$$\rho^{-1}(y) = \bigsqcup_{\alpha \in I} \rho^{-1}(y) \cap V_\alpha$$

where every $\rho^{-1}(y) \cap V_\alpha$ consists of a single element. Because $|I| = n$, it follows that $\rho^{-1}(y)$ has cardinality equal to n. Especially, $y \in B_n$. Since $y \in U_b$ was arbitrary, we have $U_b \subseteq B_n$. Finally, this means that $B_n = \bigcup_{y \in A_n} U_b$ is open in B. This proves the lemma. □

As a special case of the lemma, we get that B_k (with $k \in \mathbb{N} \subset \mathbb{N}^*$) is open in B. It remains to show that $B \setminus B_k$ is open. However, this follows from the lemma as well. Certainly, we write

$$B \setminus B_k = \left(\bigsqcup_{n \in \mathbb{N}^*} B_n \right) \setminus B_k = \bigsqcup_{\substack{n \in \mathbb{N}^* \\ n \neq k}} B_n.$$

Since every B_n is open in B, the claim follows. That is, B_k is open in B. Since B is connected, we must have $B = B_k$.

Solution to Problem 4.10. Let $r : \mathbb{B}^2 \to A$ be a retraction map of $\mathbb{B}^2$ onto A. The composite $f \circ r$ is thus a continuous function $\mathbb{B}^2 \to A$. Since A is a subspace of X, Proposition 1.22 tells us that $f \circ r$ is also a continuous map $\mathbb{B}^2 \to \mathbb{B}^2$. Invoking Brouwer's fixed point theorem yields a point $x \in \mathbb{B}^2$ such that $(f \circ r)(x) = x$. But the image of lies in A, which means that, in particular, x belongs to A. Since r is a retraction map, we have $r(x) = x$. This implies that $f(x) = x$.

Solution to Problem 4.11. First, notice that Lemma 4.22 allows us to extend f to a continuous function $g : \mathbb{B}^2 \to \mathbb{S}^1$. Of course, g can also be viewed as continuous from $\mathbb{B}^2$ into $\mathbb{B}^2$. Hence, there exists a point x belonging to $\mathbb{B}^2$ satisfying $g(x) = x$. But, the image of g is contained within $\mathbb{S}^1$. Especially, $x \in \mathbb{S}^1$. Since the function g extends f, it follows that $x = g(x) = f(x)$.

For the second part, we consider the vector field

$$\mathbf{F} : \mathbb{B}^2 \longrightarrow \mathbb{R}^2, \quad x \longmapsto g(x) + x.$$

The vector field $\mathbb{F}$ is easily seen to be continuous (since $\mathbb{R}^2$ is a topological vector space whence addition is a continuous operation). By way of contradiction, suppose that $\mathbf{F}$ is non-vanishing on $\mathbb{B}^2$. By the "Hairy Ball Theorem", there exists a point x on $\mathbb{S}^1$ such that $\mathbf{F}$ points directly inwards at x. Thus, there exists $\alpha > 0$ with the property that

$$\mathbf{F}(x) = g(x) + x = -\alpha x.$$

This gives $g(x) = -(1+\alpha)x$ so that $|g(x)| > 1$, which is absurd. This contradiction shows that $\mathbf{F}$ cannot be non-vanishing on $\mathbb{B}^2$. Let $x \in \mathbb{B}^2$ be such that $\mathbf{F}(x) = 0$. This means that $g(x) = -x$. Since $|g| \equiv 1$, we must have $x \in \mathbb{S}^1$. Recalling that g extends f, we get $f(x) = -x$ as was required.

Solution to Problem 4.12. We first tackle (1). By way of contradiction, suppose that i is nulhomotopic. Lemma 4.22 tells us that there exists a continuous function $k : \mathbb{B}^n \to \mathbb{S}^{n-1}$ equal to the identity on $\mathbb{S}^{n-1}$. By definition, this means that k is a retraction of $\mathbb{B}^n$ onto $\mathbb{S}^{n-1}$. Hence, $\mathbb{S}^{n-1}$ is a retract of $\mathbb{B}^n$. This contradiction shows that i is not nulhomotopic.

(2). Here we proceed similarly. Assume that the inclusion map j is nulhomotopic. By the same lemma, we may choose a continuous map $k : \mathbb{B}^n \to \mathbb{R}^n \setminus \{0\}$ equal to j on $\mathbb{S}^{n-1}$. Now, consider the continuous function

$$r : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{S}^{n-1}, \quad x \mapsto \frac{x}{\|x\|}.$$

Clearly, r is a retraction of $\mathbb{R}^n \setminus \{0\}$ onto the sphere $\mathbb{S}^{n-1}$. What is important, however, is the fact that r is equivalent to the identity on $\mathbb{S}^{n-1}$. Consider the composite

$$\mathfrak{r} : \mathbb{B}^n \xrightarrow{\;k\;} \mathbb{R}^n \setminus \{0\} \xrightarrow{\;r\;} \mathbb{S}^{n-1}.$$

Evidently, $\mathfrak{r}$ is continuous. If $x \in \mathbb{S}^{n-1}$, then $\mathfrak{r}(x) = r(j(x)) = r(x) = x$. Thus, $\mathfrak{r}$ is in fact a *retraction* of $\mathbb{B}^n$ onto $\mathbb{S}^{n-1}$. This contradiction yields the conclusion.

(3). Let $\mathbf{F} : \mathbb{B}^n \to \mathbb{R}^n$ be a non-vanishing continuous vector field. By way of contradiction, assume that $\mathbf{F}$ does **not** point directly inwards at any point on $\mathbb{S}^{n-1}$. Let $\mathbf{w}$ be the restriction of $\mathbf{F}$ to the sphere $\mathbf{S}^{n-1}$. Observe that $\mathbf{F}$ is an extension of $\mathbf{w}$ to the entire n-ball. By Lemma 4.22, this map $\mathbf{w}$ is nulhomotopic. Now, let us consider the continuous function

$$H : \mathbb{S}^{n-1} \times \mathbb{I} \longrightarrow \mathbb{R}^n \setminus \{0\}, \quad (x, t) \mapsto tx + (1-t)\mathbf{w}(x).$$

First, let us check that the above is well defined. If $t = 0, 1$ it is obvious that $H(x, t)$ is never zero. If $H(x, t) = 0$ for some $x \in \mathbb{S}^{n-1}$ and any $t \in (0, 1)$, we have

$$\mathbf{w}(x) = \frac{t}{t-1}x$$

which means that $\mathbf{w}$, and hence $\mathbf{F}$, points directly inwards at a point on $\mathbb{S}^{n-1}$. Thus, H is continuous and maps into $\mathbb{R}^n \setminus \{0\}$. We then see that H is a homotopy between the identity map i and $\mathbf{w}$. By the transitive property, it follows that i is nulhomotopic. This contradiction shows that $\mathbf{F}$ must point directly inwards at some point on $\mathbb{S}^{n-1}$.

(4). This follows by applying the previous point to $-\mathbf{F}(x)$.

(5). We need only mimic the proof of the Brouwer fixed point theorem. Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a continuous function. Suppose for a contradiction that $f(x) \neq x$

for all $x \in \mathbb{B}^n$. Then, the function $g(x) := f(x) - x$ is a non-vanishing vector field on $\mathbb{B}^n$. By the previous part, we may choose $x \in \mathbb{S}^{n-1}$ such that $g(x)$ points directly *outwards* at x. That is, there exists $x \in \mathbb{S}^{n-1}$ and $\alpha > 0$ such that

$$g(x) = f(x) - x = \alpha x.$$

But then, $f(x) = (1 + \alpha)x$ so that $\|f(x)\| = (1 + \alpha) > 1$. This contradicts the fact that $\mathrm{Im}(f) \subseteq \mathbb{B}^n$.

Solution to Problem 4.13. Let $\mathscr{C}$ be a convex subspace of V. As a first step, we check that $\mathscr{C}$ is path connected. Fix any two points x_0, x_1 in $\mathscr{C}$ and consider the map

$$\gamma : \mathbb{I} \to V, \quad t \mapsto (1 - t)x_0 + tx_1.$$

It is not difficult to check, via composition, that γ is continuous. Thus, γ is a path in V from x_0 to x_1. However, $\mathscr{C}$ being convex implies that $\gamma(t) \in \mathscr{C}$ for all $t \in \mathbb{I}$. It follows that $\mathscr{C}$ is path connected.

Now, we prove that $\mathscr{C}$ is simply connected. This amounts to showing that any loop is path homotopic to the constant loop. In fact, it is enough to show that any two loops based at x_0 are homotopic. To this end, fix two loops f, f' based at a point $x_0 \in V$. Then, the continuous function

$$F : \mathbb{I} \times \mathbb{I} \to \mathscr{C}, \quad F(s, t) := (1 - t)f(s) + tf'(s)$$

is easily seen to be a well defined path homotopy. Since $F(s, 0) \equiv f(s)$ and $F(s, 1) \equiv f'(s)$, the claim follows.

Solution to Problem 4.14. Suppose that E is path connected and assume that B is simply connected. Since covering maps are open and continuous, it will be enough to show that ρ is bijective. In fact, it suffices to prove that ρ is injective. To this end, let $e_0, e_1 \in E$ be such that $\rho(e_0) = \rho(e_1) =: b_0$. Since E is path connected, we can choose a path Γ in E starting at e_0 and ending at e_1. Then, the composite $\gamma := \rho \circ \Gamma$ will be a loop in B based at b_0. Using that B is simply connected, we see that γ is path homotopic to the constant loop $\mathbf{b} \equiv b_0$.

Now, let ϕ and ψ be the unique liftings of γ and $\mathbf{b}$ (respectively) to E that *begin* at the point e_0. Since γ and $\mathbf{b}$ are path homotopic in B (with the same start and end-points), there must also hold $\phi \simeq_p \psi$. In particular, $\phi(1) = \psi(1)$. However, Γ is path in E, with $\Gamma(0) = e_0$, that satisfies

$$\rho \circ \Gamma \equiv \gamma.$$

Hence, by the uniqueness part of *lifting lemma 1*, we must have $\Gamma \equiv \phi$. Similarly, one can show that the constant path e_0 is equal to the lifting ψ. From this, it follows that

$$e_0 = \psi(1) = \phi(1) = \Gamma(1) = e_1.$$

We conclude that ρ is injective, whence the proof is complete.

Solution to Problem 4.15. Since ρ_* is a group homomorphism, the injectivity of ρ_* is equivalent to ρ_* having a trivial kernel. Now, let $[f]$ be an element of the kernel of ρ_*. This means that f is a loop in E, based at e_0, such that

$$\rho_*([f]) = [\rho \circ f] = [e_{b_0}].$$

Especially, $\rho \circ f \simeq_p e_{b_0}$. Let now $F : \mathbb{I} \times \mathbb{I} \to B$ be a path homotopy between $\rho \circ f$ and e_{b_0}. We will be done provided we can show that f is path homotopic to the constant loop at e_0. Now, denote by $\tilde{f}$ the constant loop at e_0. Clearly, $\tilde{f}$ is the unique lifting of e_{b_0} beginning at e_0. Moreover, f is the unique lifting of $\rho \circ f$ beginning at e_0. By Theorem 4.13, we see that f will be path homotopic to the constant loop at e_0. Thus, the kernel of ρ_* is trivial.

B.5 Solutions to Exercises in §5.5

Solution to Problem 5.1. Arguing by contradiction, suppose that $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^n$, for some $n \geq 3$. Then, the punctured plane $\mathbb{R}^n_*$ will be homeomorphic to the punctured space $\mathbb{R}^n_*$. However, $\mathbb{R}^n_*$ deformation retracts to the sphere $\mathbb{S}^{n-1}$ while the punctured plane $\mathbb{R}^2_*$ deformation retracts to the circle $\mathbb{S}^1$. Since $\mathbb{R}^2_* \cong \mathbb{R}^n_*$, it follows that $\mathbb{Z} \cong \mathbf{0}$, which is a contradiction.

Solution to Problem 5.2. The solid torus $\mathbb{B}^2 \times \mathbb{S}^1$ deformation retracts to the circle $\mathbb{S}^1$. Therefore, it has fundamental group isomorphic to $\mathbb{Z}$. On the other hand, the punctured torus deformation retracts to the wedge of circles $\mathbb{S}^1 \vee \mathbb{S}^1$ which, although not proven in this text, has fundamental group isomorphic to the free product $\mathbb{Z} * \mathbb{Z}$.

Solution to Problem 5.3. Clearly, the infinite cylinder is a deformation retract of $\mathbb{S}^1$. Hence, it has fundamental group isomorphic to $\mathbb{Z}$.

Solution to Problem 5.4. This set is clearly convex, and thus must be simply connected.

Solution to Problem 5.5. The set $\mathbb{R}^2 \smallsetminus (0,\infty)$ deformation retracts to the left half space

$$\mathbb{H}^- := \{(x,y) : x \leq 0\}.$$

Since $\mathbb{H}^-$ is convex, we see that $\mathbb{R}^2 \smallsetminus (0,\infty)$ has the trivial fundamental group.

Solution to Problem 5.6. If we define $\mathbb{B}^2_*$ to be the closed unit disk in $\mathbb{C}$ without the origin, then it is easy to see that this deformation retracts to $\mathbb{S}^1$. However, this argument fails for the entire unit disk $\mathbb{B}^2$. To exhibit a deformation retract of $\mathbb{B}^2$ onto the circle $\mathbb{S}^1$, one would have to "tear" the disk $\mathbb{B}^2$ at some point which is *not* a continuous operation.

Solution to Problem 5.8. First assume that X is contractible and let $x_0 \in X$ be such that $\mathbf{1}_X$ is homotopic to the constant map $x \mapsto x_0$. Label the point space $\{x_0\}$ by X_0 (and note that this set admits a unique topology, which it also inherits as a subspace of X). Now, define maps

$$f : X \to X_0, \quad x \mapsto x_0$$

and

$$g : X_0 \to X, \quad x_0 \mapsto x_0.$$

Clearly, $f \circ g$ is the identity map $X_0 \to X_0$. Similarly, $g \circ f$ is the constant map $x \mapsto x_0$, which we assumed to be homotopic with $\mathbf{1}_X$. This shows that f and g are homotopy equivalences. Since singleton spaces have the trivial fundamental group, it follows that X is simply connected.

Conversely, let $A = \{a\}$ is a singleton space and assume that X has the homotopy type of A. Let $f : X \to A$ and $g : A \to X$ be the respective homotopy equivalences. Denote by x_0 the point in X mapped to by $g(a)$. By assumption, $g \circ f$ is a map $X \to X$ homotopic to the identity map $\mathbf{1}_X$. However, $g \circ f$ is easily seen to be the constant map $x \mapsto a \mapsto x_0$, whence X is contractible.

Solution to Problem 5.9. Let X be a contractible space and $A \subseteq X$ a retraction of X. Since X is contractible, we can find a homotopy $H : X \times \mathbb{I} \to X$ between the identity map $\mathbf{1}_X$ and some constant function $x \mapsto x_0$. Now, we consider the composite

$$\tilde{H} : A \times \mathbb{I} \to A, \quad (x,t) \mapsto r(H(x,t)).$$

Evidently, this function is continuous. In fact, a direct computation shows that for all $x \in A$ one has

$$\tilde{H}(x,0) \equiv r(H(x,0)) = r(x) = x$$

and

$$\tilde{H}(x,1) = r(H(x,1)) = r(x_0).$$

With this, we see that A is contractible.

Solution to Problem 5.10. Let $H : X \times \mathbb{I} \to X$ be a deformation retraction of X onto A. Then, H will satisfy all of the following:

$$\begin{cases} H(x,0) = x, & x \in X, \\ H(a,t) = a, & a \in A,\ t \in \mathbb{I}, \\ H(x,1) \in A, & x \in X. \end{cases}$$

Note that the function $r(x) := H(x,1)$ is a retraction of X onto A. Similarly, let $G : A \times \mathbb{I} \to A$ be a deformation retraction of A onto B. As above,

$$\begin{cases} G(a,0) = a, & a \in A, \\ G(b,t) = b, & b \in B,\ t \in \mathbb{I}, \\ G(a,1) \in B, & a \in B. \end{cases}$$

Now, consider the following function:

$$F : X \times \mathbb{I} \to X, \quad (x,t) \mapsto \begin{cases} H(x,2t), & t \in [0,\frac{1}{2}], \\ G(r(x),2t-1), & t \in [\frac{1}{2},1]. \end{cases}$$

Since $H(\cdot,1) \equiv r(\cdot) \equiv G(r(\cdot),0)$, the pasting lemma ensures that F will be a well defined continuous function. All that remains is to check that F is a deformation retraction of X onto B. Given $x \in X$, there holds

$$F(x,0) = H(x,0) = x.$$

Given $t \in \mathbb{I}$ and $b \in B \subseteq A$ we see that

$$F(b,t) = \begin{cases} H(b,2t), & t \in [0,\frac{1}{2}], \\ G(r(b),2t-1), & t \in [\frac{1}{2},1] \end{cases}$$

which is always equal to b. Finally, note that $F(x, 1) = G(r(x), 1) \in B$ for all $x \in X$. This completes the proof.

Solution to Problem 5.11. Denote by $\mathbb{M}$ the Möbius strip, which is a compact orientable surface whose boundary (in the sense of manifolds) is homeomorphic to $\mathbb{S}^1$. Clearly, $\mathbb{M}$ is path connected. Moreover, $\mathbb{M} \setminus \partial\mathbb{M}$ has an embedded homeomorphic copy of $\mathbb{S}^1$ (to see this, follow the line in the very middle of the Möbius strip until you return to the starting point). Hence, $\mathbb{M}$ deformation retracts onto this copy of $\mathbb{S}^1$ (picture the width of the band getting thinner and thinner). In particular, the Möbius strip has fundamental group isomorphic to $\mathbb{Z}$.

Arguing by contradiction, suppose that there is a retraction of $r : \mathbb{M} \to \partial\mathbb{M}$. Fixing a point $x_0 \in \partial\mathbb{M}$, we know that the inclusion map $j : \partial\mathbb{M} \hookrightarrow \mathbb{M}$ induces an embedding of fundamental groups

$$\mathbb{Z} \cong \pi_1(\partial\mathbb{M}, x_0) \hookrightarrow \pi_1(\mathbb{M}, x_0) \cong \mathbb{Z}.$$

For the moment let us view $\partial\mathbb{M}$ as the circle $\mathbb{S}^1$. Then, $\pi_1(\partial\mathbb{M}, x_0)$ has $[f]$ as a generator, where f is a loop based at x_0 wrapping around $\partial\mathbb{M}$ exactly once. If we look at this loop as a loop in the whole of $\mathbb{M}$, this path wraps around the strip twice. Hence, $j_*([f]) = [j \circ f] = [g] * [g]$, where g is a loop in $\mathbb{M}$ (based at x_0) wrapping around the strip exactly once. In particular, $[g]$ is a generator of $\pi_1(\mathbb{M}, x_0)$. On the other hand,

$$[f] = (r_* \circ j_*)([f]) = r_*([g] * [g]) = r_*([g])^2.$$

This means that a generator of the infinite cyclic group $\pi_1(\partial\mathbb{M}, x_0)$ is the square of some element, which cannot be the case. Hence, no retraction $\mathbb{M} \to \partial\mathbb{M}$ exists.

Nomenclature

Functional Notation

$\rightarrow$ A standard function arrow; $f : X \rightarrow Y$ means that f associates to each point in X a point in Y.

$\dashrightarrow$ Indicates that a function has not yet been shown to exist.

$\hookrightarrow$ Indicates that a function is an embedding of topological spaces (or groups).

$\rightharpoonup$ If $X \ni x_0$ and $Y \ni y_0$ are spaces, we write $h : (X, x_0) \rightharpoonup (Y, y_0)$ to indicate that $h : X \rightarrow Y$ is continuous with $h(x_0) = y_0$.

$\rightarrowtail$ Indicates that a function is injective.

$\twoheadrightarrow$ Indicates that a function is surjective.

h_* Given $h : (X, x_0) \rightharpoonup (Y, y_0)$, h_* denotes the induced homomorphism of fundamental groups: $\pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$.

Sets

$\mathbb{C}$ The set of all complex numbers, i.e. the complex plane.

$\mathfrak{T}$ A dummy topology.

$\mathfrak{W}$ A dummy topology.

$\mathbb{N}$ The set of natural numbers, i.e. $\{1, 2, \ldots, \}$.

$\mathbb{N}_0$ The set of all non-negative integers, i.e. $\mathbb{N} \cup \{0\}$.

Aut($\hat{\mathrm{X}}$) Group of automorphisms of a covering space $\hat{X}$.

Auth(X) Group of autohomeomorphisms of a space X.

$\pi_1(X, x_0)$ The fundamental group of a space X based at x_0.

$\pi_n(X, x_0)$ The higher homotopy groups of a space X based at a point x_0.

$\mathbb{Q}$ The set of all rational numbers.

$\mathbb{Z}$ The set of all integers.

Symbolic Notation

$[x]$ The connected (or path) component of the point x relative to a space. In Part II, this often denotes a hopotopy equivalence class.

$\bigotimes_\alpha \mathfrak{T}_\alpha$ The product of an indexed families of topologies. *Warning*: this notation is not standard and rarely used throughout this text.

$\bigsqcup$ A disjoint union.

$\bigvee_{\alpha \in I} X_\alpha$ The wedge product of a family of topological spaces.

$\mathrm{Cl}\,(A)$ The closure of the set A relative to some space.

$\coprod_{\alpha \in I} X_\alpha$ The topological sum of an indexed family of topological spaces.

$\mathrm{Int}\,(A)$ The interior of the set A relative to some space.

$\prod_\alpha X_\alpha$ The cartesian product of an indexed family of sets.

X/A The collapsing of a subspace A of X to a single point.

Common Topological Spaces

$\mathbb{B}^n$ The closed unit ball in $\mathbb{R}^n$. For $n = 2$, we take the closed unit disk in $\mathbb{C}$.

$\mathbb{S}^{n-1}$ The set of all points in $\mathbb{R}^n$ (or $\mathbb{C}$ in the case $n = 2$) having norm equal to 1. Equivalently, the boundary of the n-Sphere.

$\mathbb{R}^n$ n-dimensional Euclidean space.

$\mathbb{R}_\ell$ $\mathbb{R}$ equipped with the lower-limit topology.

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Bibliography

[MNKS] Munkres, James R. *Topology*. Pearson, 2018.

[DBMN] Berman, Dana. *The Topology of Metric Spaces*. Accessed May 2018, https://www.cs.mcgill.ca/~dberma3/notes/Topo.pdf.

[FLND] Folland, Gerald. *Real Analysis: Modern Techniques and Their Applications*. Second edition. Wiley.

[SEMS] Semmes, Stephen. *Notes on Topological Vector Spaces*. https://arxiv.org/abs/math/0304032v4