THE BASICS OF SUBHARMONIC FUNCTIONS

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In this brief note, we quickly introduce the concept of a subharmonic function. In standard PDE courses, one studies harmonic functions in $\mathbb{R}^n$. This of course includes the *mean value property* and the *maximum principles* for harmonic functions. However, subharmonic functions also encode important and intersecting information. In this document, we hope to cover analogous results for subharmonic functions on $\mathbb{R}^n$.

In this note, $n \geq 1$ denotes a natural number and u is a real valued function $\Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ that is typically assumed to be twice differentiable in the interior of Ω.

Definition 1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}^n$ twice continuously differentiable in Ω. We say that u is **subharmonic** in Ω if $-\Delta u \leq 0$ on Ω. This is, of course, equivalent to the condition $\Delta u \geq 0$ in Ω.

Note that, in particular, all harmonic functions in Ω are subharmonic. The set of all subharmonic functions $\Omega \to \mathbb{R}$ is denoted by $\mathfrak{S}(\Omega)$. By definition, every subharmonic function is twice continuously differentiable on Ω. Hence, $\mathfrak{S}(\Omega) \subset C^2(\Omega)$.

1. Mean Value Property

Throughout this section, m denotes the Lebesgue measure on $\mathbb{R}^n$. If $E \subseteq \mathbb{R}^n$ is a measurable set of finite, but positive measure, we define

$$\fint_E f \,\mathrm{d}m := \frac{1}{m(E)} \int_E f \,\mathrm{d}m \tag{1.1}$$

for all functions $f \in L^1(E, m)$. We similarly define the symbol $⨍$ for a surface integral. This notation simplifies greatly the statement of the following theorem.

Theorem 1.1 (Mean Value Property). *Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty open set and $f \in C^2(\Omega)$. The following statements are equivalent:*

(1) u is subharmonic in Ω, i.e. $-\Delta u \leq 0$ in Ω;
(2) for each $x \in \Omega$ there exists $r > 0$ such that

$$u(x) \leq ⨍_{\partial B(x,\rho)} u(y)\,\mathrm{d}S(y) \tag{1.2}$$

for all $0 < \rho < r$.

The inequality in (1.2) is called the mean value formula for subharmonic functions.

Before we prove the above, let us perform a useful calculation. Suppose only the hypothesis of the theorem, i.e. let $\varnothing \neq \Omega \subseteq \mathbb{R}^n$ be open and let $u : \Omega \to \mathbb{R}$ be twice continuously differentiable. Fix a point $x \in \Omega$ and let $\varepsilon > 0$ be such that $B(x, \varepsilon) \Subset \Omega$. Here, $B(x, \varepsilon)$ denotes the open ball of radius ε about x. For $0 < \rho < \varepsilon$, we define

$$\mu(\rho) := ⨍_{\partial B(x,\rho)} u(y)\,\mathrm{d}S(y).$$

Observe that for all such ρ there holds

$$\begin{aligned}\mu(\rho) = ⨍_{\partial B(x,\rho)} u(y)\,\mathrm{d}S(y) &= \frac{1}{n\omega_n\rho^{n-1}}\int_{\partial B(x,\rho)} u(y)\,\mathrm{d}S(y)\\ &= \frac{1}{n\omega_n}\int_{\partial B(\mathbf{0},1)} u(x+\rho z)\,\mathrm{d}S(z).\end{aligned}$$

Here, ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Because $u \in C^2(\Omega)$ we may differentiate the above with respect to $\rho \in (0, \varepsilon)$. This yields;

$$\begin{aligned}\mu'(\rho) &= \frac{\partial}{\partial\rho}\left(\frac{1}{n\omega_n}\int_{\partial B(\mathbf{0},1)} u(x+\rho z)\,\mathrm{d}S(z)\right)\\ &= \frac{1}{n\omega_n}\int_{\partial B(\mathbf{0},1)} \frac{\partial}{\partial\rho}u(x+\rho z)\,\mathrm{d}S(z)\\ &= \frac{1}{n\omega_n}\int_{\partial B(\mathbf{0},1)} \nabla u(x+\rho z)\cdot z\,\mathrm{d}S(z)\\ &= \frac{1}{n\omega_n\rho^{n-1}}\int_{\partial B(x,\rho)} \nabla u(y)\cdot\nu(y)\,\mathrm{d}S(y).\end{aligned}$$

Here, $\nu(y)$ is the outwards pointing unit normal to $\partial B(x,\rho)$ at y. Thus, an application of Green's theorem grants us the following:

$$\mu'(\rho) = \frac{1}{n\omega_n \rho^{n-1}} \int_{B(x,\rho)} \Delta u(y)\, \mathrm{d}m. \tag{1.3}$$

It turns out that this identity will tell us plenty about $\mu(\rho)$. Let us now give the proof of the aforementioned theorem.

Proof of Theorem. Suppose that u is not subharmonic on Ω. Thus, there exists $x \in \Omega$ such that $\Delta u(x) < 0$. By continuity, we may choose $\varepsilon > 0$ such that $\Delta u < 0$ on $\overline{B(x,\varepsilon)} \subset \Omega$. Consider now the function $\mu(\rho)$ on $(0,\varepsilon)$. We see from (1.3) that $\mu'(\rho) < 0$ on $(0,\varepsilon)$. Hence, $\mu(\rho)$ is strictly decreasing on $(0,\varepsilon)$. By continuity,

$$u(x) = \lim_{\rho \searrow 0} \fint_{\partial B(x,\rho)} u(y)\, \mathrm{d}S(y) = \lim_{\rho \searrow 0} \mu(\rho).$$

Since μ is strictly decreasing, we have

$$\mu(\rho) = \fint_{\partial B(x,\rho)} u(y)\, \mathrm{d}S(y) < u(x)$$

for all $0 < \rho < \varepsilon$. This proves that (2) cannot hold. By contrapositive, we have thus shown that (2) implies (1).

Conversely, suppose that $\Delta u \geq 0$ on Ω. Given $x \in \Omega$, let $\varepsilon > 0$ be such that $B(x,\varepsilon) \Subset \Omega$. We now consider the function $\mu(\rho)$ on $(0,\varepsilon)$. Invoking (1.3), we see that

$$\mu'(\rho) \geq 0$$

on $(0,\varepsilon)$. Hence, $\mu(\rho)$ is increasing on this interval. By continuity we again have

$$u(x) = \lim_{\rho \searrow 0} \fint_{\partial B(x,\rho)} u(y)\, \mathrm{d}S(y) = \lim_{\rho \searrow 0} \mu(\rho).$$

Hence, $u(x) \leq \fint_{\partial B(x,\rho)} u(y)\, \mathrm{d}S(y)$ for all $\rho \in (0,\varepsilon)$. □

2. The Maximum Principles

Perhaps the most important consequence of the mean value property for harmonic functions is the infamous *strong maximum principle*. In this section, we prove an analogous result for subharmonic functions in bounded domains.

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be non-empty and open. Let $u \in \mathfrak{S}(\Omega)$ be given. For every $x \in \Omega$, there exists $r > 0$ such that*

$$u(x) \leq \fint_{B(x,\rho)} u(y)\,\mathrm{d}y$$

for all $0 < \rho < r$.

Proof. Let $x \in \Omega$ be given and fix $r > 0$ as in part (2) of the previous theorem. If $0 < \rho < r$, we calculate

$$\begin{aligned}\int_{B(x,\rho)} u(y)\,\mathrm{d}y &= \int_0^\rho \int_{\partial B(x,\delta)} u(y)\,\mathrm{d}S(y)\,\mathrm{d}\delta \\ &\geq \int_0^\rho u(x) n\omega_n \delta^{n-1}\,\mathrm{d}\delta \\ &= u(x)\omega_n \rho^n.\end{aligned}$$

Hence, $u(x) \leq \fint_{B(x,\rho)} u(y)\,\mathrm{d}y$. □

Equipped with this result, we may now give the following theorem.

Theorem 2.2 (Strong Maximum Principle)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be non-empty, open, connected, and bounded. Let $u : \Omega \to \mathbb{R}$ be twice continuously differentiable in Ω and continuous up to the boundary. Assume in addition that $u \in \mathfrak{S}(\Omega)$. If there exists a point $x_0 \in \Omega$ such that $u(x_0) = \max_{x \in \overline{\Omega}} u(x)$, then u is constant on $\overline{\Omega}$.*

Proof. Let M denote the maximum of u on the compact set $\overline{\Omega}$. Define

$$\mathfrak{X} := \{x \in \Omega : u(x) = M\}.$$

By assumption, $\mathfrak{X}$ is non-empty (it contains x_0). Since u is continuous, $\mathfrak{X}$ is closed in $\mathbb{R}^n$, and hence in Ω. We now claim that $\mathfrak{X}$ is open in $\mathbb{R}^n$, and thus in Ω. To this end, let $x \in \mathfrak{X}$ be given. Choose $r > 0$ as in the previous lemma and let $0 < \rho < r$. Clearly,

$$\begin{aligned}0 = M - u(x) &\geq M - \fint_{B(x,\rho)} u(y)\,\mathrm{d}y \\ &= \fint_{B(x,\rho)} (M - u(y))\,\mathrm{d}y \geq 0.\end{aligned}$$

Hence, $u \equiv M$ on $B(x,\rho)$. This implies that $B(x,\rho) \subseteq \mathfrak{X}$. It follows that $\mathfrak{X}$ is clopen. Since this set is non-empty, we conclude that $\mathfrak{X}$ is all of Ω. By continuity on $\overline{\Omega}$, we must have $u \equiv M$ on all of $\overline{\Omega}$. □

We now relax the assumption that Ω is connected. Let C be a connected component of Ω and consider its boundary $\partial C \subseteq \overline{\Omega}$. Fix a point $x \in \partial C$ but assume that $x \notin \partial\Omega$. Therefore, there exists

$\varepsilon > 0$ such that the open ball $B(x, \varepsilon)$ is contained in Ω. In particular, $x \in \Omega$. Because connected open subsets of Ω can intersect only one component, we conclude that $B(x, \varepsilon) \subseteq C$. Of course, this contradicts the assumption that $x \in \partial C$. We conclude that $\partial C \subseteq \partial\Omega$.

After breaking Ω into its connected components, we conclude the following:

Corollary 2.3. *Let Ω be a non-empty bounded open subset of $\mathbb{R}^n$. Let $u \in \mathfrak{S}(\Omega)$ be continuous up to the boundary of Ω. Then,*

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x). \tag{2.1}$$

The equation above is called the weak maximum principle *for subharmonic functions.*

Suppose that u is harmonic in Ω. Clearly, it is subharmonic as well. But, $-u$ is also subharmonic! Hence, we can also apply the weak maximum principle to $-u$. Doing so gives us the following "minimum principle":

Corollary 2.4. *Let Ω be a non-empty bounded open subset of $\mathbb{R}^n$. Let $u \in C^2(\Omega)$ be continuous up to the boundary of Ω. If u is harmonic in Ω, then*

$$\min_{x \in \overline{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x). \tag{2.2}$$

2.1. **Uniqueness to the Dirichlet Problem.** Suppose we are given an open and bounded subset Ω of $\mathbb{R}^n$. We also assume that Ω is non-empty. Given two continuous functions $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, the associated *Dirichlet problem* involves finding a function $u \in C^2(\Omega)$, continuous up to the boundary, such that

$$\begin{cases} -\Delta u \equiv f & \text{in } \Omega, \\ u \equiv g & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Whether or not there always exists a solution u to the above is a difficult question that we shall not explore here. What we *can* do is show that the problem above has *at most* a single solution.

Theorem 2.5. *There exists at most one solution u to the Dirichlet problem* (2.3).

Proof. Suppose that u and v satisfy (2.3) and put $w := u - v$. Clearly, $w \in C^2(\Omega)$ and is continuous up to the boundary. However, direct calculation gives $-\Delta w \equiv f - f \equiv 0$ in Ω. This means that w is harmonic in Ω. On $\partial\Omega$,

$$w \equiv u - v \equiv g - g \equiv 0.$$

Hence, w vanishes on $\partial\Omega$. Invoking (2.1)-(2.2) implies that

$$\max_{x\in\overline{\Omega}} w(x) = \min_{x\in\overline{\Omega}} w(x) = 0.$$

Thus, $u \equiv v$. □

2.2. **The Comparison Principle.** Again, let $\Omega \subset \mathbb{R}^n$ be a non-empty bounded open set. Let $u, v \in \mathfrak{S}(\Omega)$ both be continuous up to the boundary. What can be said about these functions? How exactly are they related? We partially answer this question with the following theorem, which often proves to be useful.

Theorem 2.6. *Let u, v, and Ω be as above. Assume that*

$$\begin{cases} -\Delta u \leq -\Delta v & \textit{in } \Omega, \\ u \leq v & \textit{on } \partial\Omega. \end{cases} \tag{2.4}$$

Then, $u \leq v$ on all of $\overline{\Omega}$.

Proof. Define $w := u - v$ and note that $w \leq 0$ in $\partial\Omega$. In Ω,

$$-\Delta w = \Delta v - \Delta u \leq 0$$

so that $w \in \mathfrak{S}(\Omega)$. By (2.1), we must have $w(x) \leq 0$ on $\overline{\Omega}$. Equivalently, $u(x) \leq v(x)$ on all of $\overline{\Omega}$. □