THE SPECTRAL THEOREM FOR COMPACT HERMITIAN OPERATORS

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1. Preliminaries

The goal of these notes is to prove the Spectral Theorem for *Compact and Symmetric* linear operators. Throughout this text we denote by $\mathcal{H}$ a Hilbert-space with inner-product $\langle \cdot, * \rangle$ mapping into a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Theorem 1.1 (Spectral Theorem)**.** *Let* T *be a compact, symmetric operator on a Hilbert space* $\mathcal{H}$ *over the field* $\mathbb{K}$*. There exists a countable orthonormal basis* $\{\varphi_n\}_n$ *of* $\mathcal{H}$ *such that for each index* n*:*

$$\mathrm{T}(\varphi_n) = \lambda_n \varphi_n, \quad \lambda_n \in \mathbb{R} \tag{1}$$

Moreover, in the case where there are countably infinite eigenvectors one has: $\lambda_n \xrightarrow{n\to\infty} 0$ *and every operator satisfying the above is compact and symmetric.*

We shall take a multi-step approach to this proof. First we recall that a bounded linear operator T is continuous if and only if it is bounded: i.e. if there exists $M > 0$ such that $\|\mathrm{T}(f)\| \leq M \|f\|$ for all $f \in \mathcal{H}$. We now define $\|\mathrm{T}\|$ to be the infimum over all such $M > 0$. We have the following elegant characterization of $\|\mathrm{T}\|$:

Lemma 1.2. *Let* $\mathrm{T} : \mathcal{H} \to \mathcal{H}$ *be a bounded linear-operator:*

$$\|T\| = \sup_{\|f\|=\|g\|=1} |\langle \mathrm{T}(f), g \rangle| \tag{2}$$

Proof. Label this supremum by S. Let f, g be unit vectors in $\mathcal{H}$ and $M > 0$ be admissible for $\|T\|$. We observe that

$$|\langle \mathrm{T}(f), g \rangle| \leq \|\mathrm{T}(f)\| \, \|g\| \leq M \|f\| \, \|g\| = M$$

Taking the supremum over all such f, g we find that $S \leq M$ for all admissible M, whence taking the infimum we recover $S \leq \|T\|$.

To see the reverse inequality, we shall show that S is admissible for $\|T\|$. If one of $\mathrm{T}(f)$ or $f = 0$ the result is trivial. Hence, we may suppose without harm that f and $\mathrm{T}(f)$ are non-zero. In which case we may define unit-vectors

$$\widetilde{f} := \frac{f}{\|f\|}, \quad \widetilde{g} := \frac{\mathrm{T}(f)}{\|\mathrm{T}(f)\|}$$

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We may then deduce that:

$$S \geq \left|\left\langle \mathrm{T}(\widetilde{f}), \widetilde{g} \right\rangle\right| = \frac{1}{\|f\| \, \|\mathrm{T}(f)\|} \left|\langle \mathrm{T}(f), \mathrm{T}(f)\rangle\right| = \|\mathrm{T}(f)\| \, \|f\|$$

which completes this proof.

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With this, we give a proof of the Riesz' representation theorem which is crucial in the construction of symmetric Hilbert operators:

Theorem 1.3 (Riesz). *Let $\ell : \mathcal{H} \to \mathbb{K}$ be a continuous linear functional. There exists a unique $g \in \mathcal{H}$ so that $\ell(f) = \langle f, g\rangle$ for all $f \in \mathcal{H}$.*

Proof. Let $\mathcal{S} := \{h \in \mathcal{H} : \ell(h) = 0\}$ denote the null-space of ℓ. This is a closed subspace of $\mathcal{H}$. Certainly, it is a vector subspace by linearity of ℓ and by continuity it is topologically closed. There must then exist an orthogonal decomposition of the space: $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^{\perp}$. If $\mathcal{S}^{\perp} =$ we set $g = 0$ and we are done. Otherwise we may let $h \in \mathcal{S}^{\perp}$ be a unit vector and put $g := \overline{\ell(h)}h$.

Let $f \in \mathcal{H}$ be given and consider $u = f\ell(h) - \ell(f)h$. Clearly, it follows from the linearity of ℓ that $\ell(u) = 0$ and hence $u \in \mathcal{S}$. Therefore, $\langle u, h\rangle = 0$. Namely,

$$\begin{aligned} 0 = \langle u, h\rangle = \langle f\ell(h) - \ell(f)h, h\rangle &= \ell(h)\langle f, h\rangle - \ell(f)\langle h, h\rangle \\ &= \langle f, \overline{\ell(h)}h\rangle - \ell(f) \end{aligned}$$

yielding $\ell(f) = \langle f, g\rangle$ as was required. To see uniqueness, if g, h are two representative vectors for the inner-product in f we have

$$0 = \ell(f) - \ell(f) = \langle f, g\rangle - \langle f, h\rangle = \langle f, g - h\rangle = \langle g - h, f\rangle, \quad \forall f \in \mathcal{H}$$

Especially, this holds for all vectors in any Hilbert basis: implying that $g - h = 0$.

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Having this theorem allows us to construct a special operator, linked to each bounded operator. Indeed, let $\mathrm{T} : \mathcal{H} \to \mathcal{H}$ be a linear operator, bounded. Then we may find an operator $\mathrm{T}^* : \mathcal{H} \to \mathcal{H}$ that satisfies each of the following:

(1) $\langle \mathrm{T}(f), g\rangle = \langle f, \mathrm{T}^*(g)\rangle$ for all $f, g \in \mathcal{H}$.
(2) $\|\mathrm{T}\| = \|\mathrm{T}^*\|$.

The construction goes as follows. Fix $g \in \mathcal{H}$ and let $f \in \mathcal{H}$ vary. Define a linear functional $\ell : \mathcal{H} \to \mathbb{K}$ by $f \mapsto \langle \mathrm{T}(f), g\rangle$. By the Riesz Representation Theorem we may find a unique $h \in \mathcal{H}$ such that $\ell(f) = \langle f, h\rangle = \langle \mathrm{T}(f), g\rangle$. Therefore, we set $\mathrm{T}^* g = h$. This of course defines the map on $\mathcal{H}$.

As for uniqueness, suppose T, T' satisfy (1) in place of T^*. Fix an orthonormal basis $\{e_j\}_j$ for $\mathcal{H}$. Fix j and let $g \in \mathcal{H}$ vary

$$0 = \langle \mathrm{T}(e_j), g\rangle - \langle \mathrm{T}(e_j), g\rangle = \langle e_j - T(g) - T'(g)\rangle$$

implying that $\langle T - T', e_j\rangle$ vanishes over the basis: establishing their equality.

Note that (2) follows immediately from our first lemma.

2. Lemmata

Fix again a Hilbert space $\mathcal{H}$ over the complex numbers $\mathbb{K}$. A bounded operator $\mathrm{T} : \mathcal{H} \to \mathcal{H}$ is said to be *symmetric* (or Hermitian) provided $\mathrm{T} = \mathrm{T}^*$. T is also called *compact* whenever the closure of $\mathrm{T}(B)$ is compact. Here B denotes the ball:

$$B := \{f \in \mathcal{H} : \|f\| \leq 1\}$$

A first observation, and useful criterion, is the following:

Theorem 2.1. *Let $\mathrm{T} : \mathcal{H} \to \mathcal{H}$ be a bounded operator. The following are equivalent:*

(1) T is compact.
(2) For every bounded sequence (f_n) in $\mathcal{H}$ there is a subsequence (f_{n_k}) so that $\mathrm{T}(f_{n_k})$ is convergent.

Proof. $(1 \implies 2)$. Let (f_n) be a bounded sequence in $\mathcal{H}$. Namely, there exists $M > 0$ so that $\|f\| \leq M$ for all $n \in \mathbb{N}$. Now put $g_n := f_n/M$ and note that $g_n \in B$ for all $n \in \mathbb{N}$. Especially, $(\mathrm{T}(g_n))_n$ is a sequence in the closure of $\mathrm{T}(B)$, which is compact. There must hence be a convergent subsequence $\mathrm{T}(g_{n_k})$, with limit say, $\mathrm{T}(g)$ in $\overline{\mathrm{T}(B)}$. We now set $f = g/M$ and claim that $\mathrm{T}(f_{n_k}) \to \mathrm{T}(f)$ as $n_k \to \infty$. Certainly,

$$\|\mathrm{T}(f_{n_k}) - \mathrm{T}(f)\| = M \, \|\mathrm{T}(g_{n_k}) - \mathrm{T}(g)\|$$

by linearity of T.

$(2 \implies 1)$ Pick a sequence (g_n) in $\overline{T(B)}$. Now, for each g_n there exists a sequence $g_n^{(k)} \to g_n$ as $k \to \infty$. This induces a sequence of sequences $(f_n^{(k)})$ where each $f_n^{(k)} \in B$, and is therefore bounded. Consider a diagonal subsequence $(f_{n_k}^{(k)})$ where $n_k \to \infty$ as $k \to \infty$. This is a sequence in B, and by our assumption in (2) we may assume that $\mathrm{T}(f_{n_k}^{(k)})$ converges to g as $k \to \infty$. We claim now that $\mathrm{T}(f_{n_k}^{(k)}) \to g$ as well. But this follows from the triangle inequality.

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We shall now give lemmata that will play a direct role in our proof of the spectral theorem. The first of which establishes that all eigenvalues for symmetric operators are real, and moreover shows that distinct eigenvectors are orthogonal.

Lemma 2.2. *Let $\mathrm{T} : \mathcal{H} \to \mathcal{H}$ be a bounded, symmetric linear operator. Then,*

(1) If $\lambda \in \mathbb{K}$ is an eigenvalue for T, then $\Im\lambda = 0$.
(2) For two distinct eigenvalues λ_1, λ_2 with associated eigenvectors φ_1, φ_2 one has $\varphi_1 \perp \varphi_2$.

Proof.

(1) Let $\lambda \in \mathbb{K}$ be an eigenvalue for T with associated eigenvector $\varphi \neq 0$. To see that $\Im\varphi = 0$, thereby establishing $\lambda \in \mathbb{R}$, it suffices to prove $\lambda = \overline{\lambda}$. To see this:

$$\lambda\langle\varphi, \varphi\rangle = \langle\lambda\varphi, \varphi\rangle = \langle\mathrm{T}(\varphi), \varphi\rangle = \langle\varphi, \mathrm{T}(\varphi)\rangle = \langle\varphi, \lambda\varphi\rangle = \overline{\lambda}\langle\varphi, \varphi\rangle$$

since $\|\varphi\|^2 = \langle \varphi, \varphi \rangle \neq 0$ we find $\lambda = \overline{\lambda}$.

(2) Let $\lambda_1 \neq \lambda_2$ with associated eigenvectors φ_1, φ_2: observe that (1) implies λ_1, λ_2 are real. Now let us consider:

$$\lambda_1 \langle \varphi_1, \varphi_2 \rangle = \langle \lambda_1 \varphi_1, \varphi_2 \rangle = \langle \mathrm{T}(\varphi_1), \varphi_2 \rangle = \langle \varphi_1, \mathrm{T}(\varphi_2) \rangle = \overline{\lambda_2} \langle \varphi_1, \varphi_2 \rangle$$

Since $\lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$ we find that $\langle \varphi_1, \varphi_2 \rangle = 0$ establishing $\varphi_1 \perp \varphi_2$.

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The subsequent lemma shows that there are at-most countably many eigenvalues for a compact operator $\mathrm{T} : \mathcal{H} \to \mathcal{H}$, and moreover the sequence of eigenvalues vanishes in absolute value at infinity; in the case $\dim(\mathcal{H}) = \infty$.

Lemma 2.3. *Let $\mathrm{T} : \mathcal{H} \to \mathcal{H}$ be a compact linear operator,*

(1) For each $\lambda \neq 0$ the null-space of $\mathrm{T} - \lambda I$ has finite dimension.

(2) If (λ_n) is any sequence of pairwise disjoint eigenvalues then $|\lambda_n| \to 0$ as $n \to \infty$.

(3) A compact operator has countably many eigenvalues.

Proof.

(1) We first consider the operator $\Lambda := \mathrm{T} - \lambda I$. We set $V_\lambda := \ker \Lambda$. The claim is that $\dim(V_\lambda) < \infty$. We assume now for a contradiction that $\dim(V_\lambda) = \infty$. Observe that Λ is continuous (it is bounded) and therefore V_λ is a closed subspace of $\mathcal{H}$, as the null-space of a continuous operator. In this case, pick an orthonormal subset $\{\varphi_n\}_{n \in \mathbb{N}}$ for V_λ. This is obviously bounded since the collection is orthonormal. By compactness of the operator T there must exist a subsequence (φ_{n_k}) so that $\mathrm{T}(\varphi_{n_k})$ is convergent. Especially, this sequence is Cauchy in $\mathcal{H}$. However, for all $\ell \neq k$ one has

$$\|\mathrm{T}(\varphi_{n_k}) - \mathrm{T}(\varphi_{n_\ell})\|^2 = \lambda \|\varphi_{n_k} - \varphi_{n_\ell}\|^2 = 2\lambda \neq 0$$

which is absurd.

(2) Here we claim that for all $\varepsilon > 0$ there exist at-most finitely many λ_n so that $|\lambda_n| \geq \varepsilon$. Clearly, this implies (2). Otherwise, for some $\varepsilon_0 > 0$ we may select a subsequence $(\lambda_{n_k})_{n_k \in \mathbb{N}}$ with $|\lambda_{n_k}| \geq \varepsilon_0$ for all n_k. Now, by the previous lemma all the corresponding eigenvectors are orthogonal and hence by the linearity of T we may presume without harm that this sequence is orthonormal. Using the compactness of T there must be a subsequence with a convergent image sequence: after a relabeling we may assume that $\mathrm{T}(\varphi_{n_k})$ converges. That is, it is Cauchy in $\mathcal{H}$. On the other-hand,

$$\|\mathrm{T}(\varphi_{n_k}) - \mathrm{T}(\varphi_{n_\ell})\|^2 = \|\lambda_{n_k}\varphi_{n_k} - \lambda_{n_\ell}\varphi_{n_\ell}\|^2 = \lambda_{n_\ell}^2 + \lambda_{n_k}^2 \geq 2\varepsilon_0^2$$

for all $k \neq \ell$.

(3) Observe that our argument used in (2) implies (3).

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The following lemma shows that there is always an eigenvalue:

Lemma 2.4. *Let $\mathrm{T} : \mathcal{H} \to \mathcal{H}$ with $\mathrm{T} \neq 0$ be a linear operator that is compact and symmetric. Then, $\|\mathrm{T}\| = \lambda$ is an eigenvalue.*

Proof. Given our previous lemmata and propositions we know already that for a symmetric operator T one has

$$\|\mathrm{T}\| = \sup_{f:\|f\|=1} |\langle \mathrm{T}(f), f\rangle|$$

Hence, we must have one of the subsequent cases:

$$\|\mathrm{T}\| = \sup_{\|f\|=1} \langle \mathrm{T}(f), f\rangle \tag{3}$$

$$\|\mathrm{T}\| = -\inf_{\|f\|=1} \langle \mathrm{T}(f), f\rangle \tag{4}$$

We shall consider only the first possibility, the second-case differs by a sign-change and a verbatim argument applies. I t is possible to select a sequence of unit vectors (f_n) so that $\langle \mathrm{T}(f_n), f_n\rangle$ converges to $\|T\| = \lambda \in \mathbb{R}$. Now, this is most certainly a bounded sequence in $\mathcal{H}$ and therefore we find by compactness of T some subsequence $(\mathrm{T}(f_{n_k}))$ that converges to $g \in \mathcal{H}$. We now claim that this $g \in \mathcal{H}$ is an eigenvector associated to λ:

$$\mathrm{T}(g) = \lambda g \tag{5}$$

We first claim that $\mathrm{T}(f_{n_k}) - \lambda f_{n_k} \to 0$ as $n_k \to \infty$. Certainly, we write

$$\begin{aligned}
\|\mathrm{T}(f_{n_k}) - \lambda f_{n_k}\|^2 &= \langle \mathrm{T}(f_{n_k}) - \lambda f_{n_k}, \mathrm{T}(f_{n_k}) - \lambda f_{n_k}\rangle \\
&= \|\mathrm{T}(f_{n_k})\|^2 - 2\lambda\langle \mathrm{T}(f_{n_k}), f_{n_k}\rangle + \lambda^2 \|f_{n_k}\|^2 \\
&\leq \|\mathrm{T}\|^2 + \lambda^2 - 2\lambda\langle \mathrm{T}(f_{n_k}), f_{n_k}\rangle \\
&= 2\lambda^2 - 2\lambda\langle \mathrm{T}(f_{n_k}), f_{n_k}\rangle
\end{aligned}$$

where this last equation clearly converges to 0 as $n_k \to \infty$. This yields that $\lambda f_{n_k} \to g$. Taking limits:

$$\mathrm{T}(g) = \lim_{n_k\to\infty} \lambda \mathrm{T}(f_{n_k}) = \lambda g$$

We now only show that $g \neq 0$. Assume for a contradiction that $g = 0$. Then, $\|\mathrm{T}(f_{n_k})\| \to 0$ as $n_k \to \infty$. Applying Cauchy-Schwarz gives:

$$\langle \mathrm{T}(f_{n_k}), f_{n_k}\rangle \leq \|\mathrm{T} f_{n_k}\| \to 0$$

which is absurd since $\lambda \neq 0$.

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3. Proof of the Spectral Theorem for Compact Operators

We are now equipped to prove the Spectral Theorem.

Proof of The Spectral Theorem. Let $\mathcal{S}$ denote the closure of the subspace of $\mathcal{H}$ spanned by all eigenvectors of T. This is most certainly a closed subspace of $\mathcal{H}$ and therefore inherits a Hilbert space structure itself. Moreover, there exists an orthogonal decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$. Of course, $\mathcal{S} \cap \mathcal{S}^\perp = \{0\}$. The claim here is that $\mathcal{S}^\perp = \{0\}$.

We first must show that $\mathrm{T}(\mathcal{S}) \subseteq \mathcal{S}$ and $\mathrm{T}(\mathcal{S}^{\perp}) \subseteq \mathcal{S}^{\perp}$. Certainly, we recall now that any vector $f \in \mathcal{S}$ may be written as $\sum_{n=1}^{\infty} \zeta_n \varphi_n$, where the φ_n are the eigenvectors of T. Therefore, by continuity

$$\mathrm{T}(f) = \lim_{N\to\infty} \mathrm{T}\left(\sum_{n=1}^{N} \zeta_n \varphi_n\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \zeta_n \mathrm{T}(\varphi_n) = \sum_{n\in\mathbb{N}} \zeta_n \mathrm{T}(\varphi_n)$$

Consequently, for any $f \in \mathcal{S}$ and each $g \in \mathcal{S}^{\perp}$ we note that by continuity of the inner-product

$$\langle \mathrm{T}(f), g\rangle = \lim_{N\to\infty} \left\langle \sum\nolimits_{n=1}^{N} \zeta_n \mathrm{T}(\varphi_n), g \right\rangle = \sum_{n\in\mathbb{N}} \zeta_n \langle \mathrm{T}(\varphi_n), g\rangle = 0$$

We shall now use symmetry to show that $\mathrm{T}(\mathcal{S}^{\perp}) \subseteq \mathcal{S}^{\perp}$. Fix $g \in \mathcal{S}$ and consider any $f \in \mathcal{S}$. Then we have:

$$\langle \mathrm{T}(g), f\rangle = \langle g, \mathrm{T}(f)\rangle = \overline{\langle \mathrm{T}(f), g\rangle} = 0$$

By way of contradiction suppose that $\mathcal{S}^{\perp} \supsetneq \{0\}$. Then, let $\mathfrak{T}$ denote the restriction of T to $\mathcal{S}^{\perp}$, which is a Hilbert space in its own right. Of course, by hypothesis we know that $\|\mathfrak{T}\| = 0$ then T must vanish on $\mathcal{S}^{\perp}$. In which case take φ_* to be any non-zero vector in $\mathcal{S}^{\perp}$ and $\lambda_* = 0$. Otherwise, we have $\|\mathfrak{T}\| > 0$ and by the previous lemma we have a non-zero eigenvector in $\mathcal{S}^{\perp}$. Any eigenvector of $\mathfrak{T}$ is an eigenvector of T and we have reached a contradiction.

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