ON SEMICLASSICAL DEFECT MEASURES

EDWARD CHERNYSH

Abstract. We provide a brief introduction to the concept (and applications) of semiclassical defect measures in $\mathbb{R}^n$. More precisely, given a bounded family of functions $\{u_h\}_{h\in(0,1)}$ in $L^2(\mathbb{R}^n)$, we investigate the subsequential limiting behaviour of the associated family $\big(\langle a^W(x,hD)u_h, u_h\rangle\big)_{h\in(0,1)}$ where $a^W(x,hD)$ dentotes the Weyl quantization of a given function $a \in C_c^\infty(\mathbb{R}^{2n})$. In the case where the family $\{u_h\}_{h\in(0,1)}$ are approximate solutions to a partial differential equation, we ask what more can be said about the associated defect measures.

Contents

1. Introduction

Our goal in these notes is to formally introduce the notion of a semiclassical defect measure on $\mathbb{R}^n$. More precisely, we ask what measure theoretic and functional analytic information is encoded within a bounded family $\{u_h\}_{h\in(0,1)} \subset L^2(\mathbb{R}^n)$. Specifically, we want a measure μ, interacting nicely with the topology of $\mathbb{R}^n$, that captures the limiting behaviour of the family $\{u_h\}_{h\in(0,1)}$ as $h \searrow 0$.

Following sections 5.1 and 5.2 in Zworski [7], we begin with a detailed proof that every bounded family $\{u_h\}_{h\in(0,1)}$ in $L^2(\mathbb{R}^n)$ has at least one associated defect measure. We then consider a concrete family of functions $\{u_h\}_{h\in(0,1)}$ in $L^2(\mathbb{R}^n)$ having only one associated defect measure, which we compute explicitly. This convincing example provides a clean framework in which it is clear that the associated semiclassical defect measure μ describes how the u_h concentrate as $h \searrow 0$. We also give an example

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where the functions u_h are not bounded in $L^2(\mathbb{R}^n)$, but can instead be interpreted as tempered distributions.

Finally, we briefly discuss the relationship between defect measures and PDE. That is, we ask what more can be said about defect measures associated to a family $\{u_h\}_{h\in(0,1)}$ when the $\{u_h\}_{h\in(0,1)}$ are approximate solutions to a given partial differential equation. This is a very natural question as solving a PDE is closely linked to finding critical points of an energy functional ϕ. Here, one takes a sequence (u_k) of bounded energy such that $\phi'(u_k) \to 0$ as $k \to \infty$. Up to a subsequence, this approximating sequence can encode information about both solutions to the PDE in question and how the functions u_k concentrate as $k \to \infty$. Aside from Zworski [7], for more historical context and applications of such results, a good reference is Struwe [6].

2. Existence of Defect Measures and Examples

Fix $n \in \mathbb{N}$ and denote by $\mathcal{B} := \mathcal{B}(\mathbb{R}^n)$ the Borel σ-algebra on $\mathbb{R}^n$. A *Radon* measure on $\mathbb{R}^n$ is a measure μ on the measurable space $(\mathbb{R}^n, \mathcal{B})$ that is well behaved with respect to the topological structure of $\mathbb{R}^n$. In more precise language, we have the following definition.

Definition 2.1. Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space and let $\mathcal{B}(X)$ denote the σ-algebra generated by $\mathfrak{T}$. That is, let $\mathcal{B}(X)$ be the Borel σ-algebra on X. A Radon measure μ on X is a measure on $(X, \mathcal{B}(X))$ such that each of the following hold true.

(i) For each compact set $K \subseteq X$, one has $\mu(K) < \infty$. That is, the measure μ is locally finite.

(ii) For any measurable set $E \in \mathcal{B}(X)$,
$$\mu(E) = \inf\{\mu(U) : U \supseteq E \text{ is open in } X\}.$$
Put otherwise, μ is *outer regular*.

(iii) Given any open set $O \subseteq \mathbb{R}^n$, one has
$$\mu(O) = \sup\{\mu(K) : K \subseteq O \text{ with } K \text{ compact in } X\}$$
Hence, μ is *inner regular* on open subsets of X.

A *signed* Radon measure is a Borel measure μ such that both the positive and negative variations of μ are Radon. Similarly, a *complex* Radon measure is a complex Borel measure whose real and imaginary parts are signed Radon measures.

In the case $X = \mathbb{R}^n$, a Radon measure is a Borel measure that interacts nicely with the open subsets of $\mathbb{R}^n$. That is, a Radon measure μ on $\mathbb{R}^n$ satisfies many of the same topological properties as the Lebesgue measure.

We now introduce the notion of a defect measure.

Definition 2.2. Let $\{u_h\}_{h\in(0,1)}$ be a bounded family in $L^2(\mathbb{R}^n)$. That is, assume that $\sup_{h\in(0,1)} \|u_h\|_{L^2(\mathbb{R}^n)} < \infty$. A *semiclassical defect measure* associated to the family $\{u_h\}_{h\in(0,1)}$ is a Radon measure μ on $\mathbb{R}^{2n}$ such that, for a sequence $(h_j) \subseteq (0,1)$ converging to 0 as $j \to \infty$,

$$\lim_{j\to\infty} \left\langle a^w(x, h_j D)u_{h_j}, u_{h_j}\right\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^{2n}} a(x,\xi)\mathrm{d}\mu(x,\xi)$$

for all $a \in C_c^\infty(\mathbb{R}^{2n})$. Here, $C_c^\infty(\mathbb{R}^{2n})$ denotes the complex linear space of all compactly supported smooth functions $\mathbb{R}^{2n} \to \mathbb{C}$.

It is a fundamental fact that every bounded family $\{u_h\}_{h\in(0,1)}$ admits at least one defect measure. This is Theorem 5.2 in Zworski [7], the proof of which is the main focus of this section.

Theorem 2.1. *Let $\{u_h\}_{h\in(0,1)}$ be a bounded family in $L^2(\mathbb{R}^n)$. Then, there exists a sequence (h_j) in $(0,1)$ with $h_j \to 0$ (as $j \to \infty$) and a corresponding positive[1] Radon measure μ on $\mathbb{R}^{2n}$ such that*

$$\lim_{j\to\infty} \left\langle a^w(x, h_j D)u_{h_j}, u_{h_j}\right\rangle = \int_{\mathbb{R}^{2n}} a(x,\xi)\mathrm{d}\mu(x,\xi)$$

for all $a \in C_c^\infty(\mathbb{R}^{2n})$.

2.1. **The Proof of Theorem 2.1.** We follow closely the argument given in Zworski [7]. For the sake of clarity and precision, we shall divide the proof of the existence result in to several parts. Our first step is a basic, but essential, result about the separability of the space $C_0(\mathbb{R}^m)$.

Lemma 2.1. *Let $m \in \mathbb{N}$ and denote by $C_0(\mathbb{R}^m)$ the linear space of all continuous functions $\mathbb{R}^m \to \mathbb{C}$ that vanish at infinity. Endowed with the supremum norm*

$$\|\cdot\|_\infty := \sup_{x\in\mathbb{R}^m} |\cdot(x)|,$$

the space $C_0(\mathbb{R}^m)$ is separable.

Proof. We first claim that $C_c(\mathbb{R}^m)$, the subspace consisting of compactly supported functions in $C_0(\mathbb{R}^m)$, is dense in $C_0(\mathbb{R}^m)$. Fix $f \in C_0(\mathbb{R}^m)$ and let $\varepsilon > 0$ be given. By definition, there exists $R \gg 0$ such that $|f(x)| < \varepsilon$ whenever $|x| \geq R$. Choose now a bump function $\eta \in C_c^\infty(\mathbb{R}^m)$ such that

$$\begin{cases} \eta \equiv 1 & \text{on } |x| \leq R, \\ 0 \leq \eta \leq 1 & \text{in } \mathbb{R}^m, \\ \eta \equiv 0 & \text{on } |x| \geq R+1. \end{cases}$$

[1] By positive, we mean that $\int_{\mathbb{R}^{2n}} a(x,\xi)\mathrm{d}\mu \geq 0$ for all $a \in C_0(\mathbb{R}^{2m})$ with $a \geq 0$ everywhere.

Then, the function $g := \eta f$ is continuous and compactly supported. Moreover,

$$|f(x) - g(x)| = \begin{cases} 0 & \text{on } |x| \le R, \\ |f(x)|\,(1-\eta(x)) & \text{in } R < |x| < R+1, \\ |f(x)| & \text{on } |x| \ge R+1 \end{cases}$$

is clearly bounded above by ε for all $x \in \mathbb{R}^m$. We infer that $\|f-g\|_\infty \le \varepsilon$ whence our assertion follows.

In light of the above, it is now enough to check that the subspace $C_c(\mathbb{R}^m)$ is separable. For each $k \ge 1$ denote by D_k the closed ball $\overline{B(0,k)}$ in $\mathbb{R}^m$. Since D_k is compact, we know (see Royden [3]) that $C(D_k)$ must be separable. Since every compactly supported continuous function $f : \mathbb{R}^m \to \mathbb{C}$ belongs to $C_c(D_k)$ for some $k \in \mathbb{N}$, the result follows. □

More can be said about the space $C_0(\mathbb{R}^m)$. Namely, $C_0(\mathbb{R}^m)$ admits a countable dense subset consisting entirely of compactly supported smooth functions. This detail with be critical for the proof of Theorem 2.1 and as such is verified below.

Proposition 2.2. *There exists a countable subset of $C_c^\infty(\mathbb{R}^m)$ that is dense in $C_0(\mathbb{R}^m)$. Namely, there exists a countable family of compactly supported smooth functions whose closure is $C_0(\mathbb{R}^m)$ in the uniform norm.*

Proof. Let η be a standard mollifier (see Appendix C in Evans [1]). By virtue of Lemma 2.1, there exists a countable family $\{v_k\}_{k\in\mathbb{N}}$ of compactly supported continuous functions dense in $C_0(\mathbb{R}^m)$. For each $k, \ell \in \mathbb{N}$, let $u_{k,\ell}$ be the convolution

$$u_{k,l} := v_k * \eta_{1/l}$$

with $\eta_\varepsilon := \varepsilon^{-n}\eta(x/\varepsilon)$ for each $\varepsilon > 0$. Clearly, each $u_{k,l}$ is smooth and of compact support. Moreover, for every fixed k, as $l \to \infty$, we have that $u_{k,l} \to v_k$ uniformly on $\mathbb{R}^m$. Consequently, the family $\{u_{k,l}\}_{k,l\in\mathbb{N}}$ is a countable dense subset of $C_0(\mathbb{R}^m)$. □

Remark 2.1. Let $a \in C_0(\mathbb{R}^m)$ and assume that $a \ge 0$. Arguing as in Lemma 2.1, we obtain a sequence (a_k) in $C_c(\mathbb{R}^m)$ converging to a uniformly as $k \to \infty$. Furthermore, it is clear from the proof that each a_k is also non-negative. Since the convolution of non-negative functions is again non-negative, the argument from the proof of Proposition 2.2 gives us a sequence of *non-negative* functions in $C_c^\infty(\mathbb{R}^m)$ converging uniformly to a on $\mathbb{R}^m$. Note however these a_k may not belong to the countable dense subset obtained through Proposition 2.2.

Recall that S denotes the class of symbols with order function $m = 1$. That is, S is the set of all functions $f \in C^\infty(\mathbb{R}^{2n})$ such that

$$\|\partial^\alpha f(x)\|_\infty < \infty$$

for all multiindices α. For the proof of Theorem 2.1, we require one last technical result. More precisely, given a symbol a, we need to control the operator norm of $a^W(x,hD)$ on $L^2(\mathbb{R}^n)$ as $h \to 0$.

Lemma 2.3. *There exists a constant $C > 0$ such that*

$$\left\|a^W(x,hD)\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C\left\|a\right\|_\infty + O(h^{1/2}), \quad \text{as } h \searrow 0,$$

for all symbols $a \in S$.

Proof. It is known from class results (see Zworski [7] equation 4.5.9, page 85) that there exists a constant $C > 0$ and $M \in \mathbb{N}$ such that

$$\left\|a^W(x,D)\right\|_{L^2\to L^2} \leq C \sum_{|\alpha|\leq Mn} \left\|\partial^\alpha a\right\|_{L^\infty}$$

for any given $a \in S$. We can therefore obtain our desired estimate by a simple re-scaling argument. To this end, we define $\tilde{x} = h^{-1/2}x$ for every $x \in \mathbb{R}^n$. Fix a function $u \in \mathscr{S}(\mathbb{R}^n)$ and consider the re-scaling

$$\tilde{u}(\tilde{x}) = h^{n/4}u\left(h^{1/2}\tilde{x}\right) = h^{n/4}u(x).$$

The reason for this choice of re-scaling becomes evident when we observe that

$$\int_{\mathbb{R}^n} \left|\tilde{u}(\tilde{x})\right|^2 \mathrm{d}\tilde{x} = \int_{\mathbb{R}^n} \left|h^{n/4}u\left(h^{1/2}\tilde{x}\right)\right|^2 \mathrm{d}\tilde{x} = \int_{\mathbb{R}^n} \left|u\left(x\right)\right|^2 \mathrm{d}x. \tag{2.1}$$

We now compute

$$\begin{aligned}
a^W(x,hD)u(x) &= \frac{h^{-n/4}}{(2\pi h)^n}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} a\left(\frac{x+y}{2},y\right)e^{\frac{i}{h}\langle x-y,\xi\rangle}h^{n/4}u(y)\mathrm{d}y\right]\mathrm{d}\xi \\
&= \frac{h^{-n/4}}{(2\pi)^n}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} a\left(h^{1/2}\frac{\tilde{x}+\tilde{y}}{2},h^{1/2}\tilde{y}\right)e^{i\left\langle\tilde{x}-\tilde{y},\tilde{\xi}\right\rangle}\tilde{u}(\tilde{y})\mathrm{d}\tilde{y}\right]\mathrm{d}\tilde{\xi} \\
&= \frac{h^{-n/4}}{(2\pi)^n}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} a_h\left(\frac{\tilde{x}+\tilde{y}}{2},\tilde{y}\right)e^{i\left\langle\tilde{x}-\tilde{y},\tilde{\xi}\right\rangle}\tilde{u}(\tilde{y})\mathrm{d}\tilde{y}\right]\mathrm{d}\tilde{\xi}
\end{aligned}$$

where $a_h(z,\eta) = a(h^{1/2}z, h^{1/2}\eta)$ for any $z,\eta \in \mathbb{R}^n$. Therefore, we have

$$a^W(x,hD)u(x) = h^{-n/4}a_h^W(\tilde{x},D)\tilde{u}(\tilde{x}) \tag{2.2}$$

for every $x \in \mathbb{R}^n$. As in the second equality of equation (2.1) we see that

$$\left\|a^W(x,hD)u(x)\right\|_{L^2} = \left\|h^{n/4}\left[a^W(x,hD)u\right]\left(h^{1/2}\tilde{x}\right)\right\|_{L^2}$$

where in the left hand side the variable of integration is x and $\tilde{x}$ on the right-hand side. Combining the above with equation (2.2) we have

$$\begin{aligned}\left\|a^W(x,hD)u(x)\right\|_{L^2} &= \left\|a_h^W(\tilde{x},D)\tilde{u}(\tilde{x})\right\|_{L^2} \\ &\leq \left\|a_h^W(\tilde{x},D)\right\|_{L^2\to L^2}\|\tilde{u}\|_{L^2} \\ &= \left\|a_h^W(\tilde{x},D)\right\|_{L^2\to L^2}\|u\|_{L^2}\end{aligned}$$

where for the last equality we have made use of equation (2.1). By the estimate mentioned at the beginning of the proof, we have

$$\begin{aligned}\left\|a^W(x,hD)u(x)\right\|_{L^2} &\leq C\left(\sum_{|\alpha|\leq Mn}\|\partial^\alpha a_h\|_{L^\infty}\right)\|u\|_{L^2} \\ &\leq \left(C\|a\|_\infty + O(h^{1/2})\right)\|u\|_{L^2}\end{aligned}$$

where we have used $\partial^\alpha a_h(z,\eta) = h^{|\alpha|/2}\partial^\alpha a(h^{1/2}z, h^{1/2}\eta)$ for any $z,\eta\in\mathbb{R}^n$. Since the space of Schwartz functions is dense in $L^2(\mathbb{R}^n)$, the desired result follows. $\square$

Armed with this lemma we begin the proof of our existence theorem.

Proof of Theorem 2.1. Citing Proposition 2.2, we may choose a countable dense subset $\{a_k\}_{k\in\mathbb{N}}$ of $C_0(\mathbb{R}^{2n})$ consisting entirely of compactly supported smooth functions.

Step 1. We first construct a functional Φ on the dense subset $\{a_k\}_{k\in\mathbb{N}}$. Fixing $k\geq 1$ and using Lemma 2.3, it is clear that

$$\begin{aligned}\left|\left\langle a_k^W(x,hD)u_h,u_h\right\rangle_{L^2(\mathbb{R}^n)}\right| &\leq \left\|a_k^W(x,hD)\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}\|u_h\|^2_{L^2(\mathbb{R}^n)} \\ &\leq \left(C\|a_k\|_\infty + O(h^{1/2})\right)\|u_h\|^2_{L^2(\mathbb{R}^n)}\end{aligned}$$

is bounded uniformly for all $h>0$ sufficiently small. In particular, we may pick a sequence $h_j^{(1)}\to 0$ in $(0,1)$ such that

$$\left\langle a_1^W(x,h_j^{(1)}D)u_{h_j^{(1)}},u_{h_j^{(1)}}\right\rangle_{L^2(\mathbb{R}^n)}\to\alpha_1,\quad\text{as } j\to\infty.$$

for some α_1. Proceeding inductively, we pick for each $k\geq 2$ a subsequence $\left(h_j^{(k)}\right)$ of $\left(h_j^{(k-1)}\right)$ such that, as $j\to\infty$,

$$h_j^{(k)}\to 0\quad\text{and}\quad\left\langle a_k^W(x,h_j^{(k)}D)u_{h_j^{(k)}},u_{h_j^{(k)}}\right\rangle_{L^2(\mathbb{R}^n)}\to\alpha_k.$$

We now consider the diagonal sequence $h_j = h_j^{(j)}$. Clearly, $h_j \to 0$ as $j \to \infty$. For each index $k \in \mathbb{N}$, we define

$$\Phi(a_k) := \lim_{j\to\infty} \left\langle a_k^W(x, h_j D) u_{h_j}, u_{h_j} \right\rangle_{L^2(\mathbb{R}^n)} = \alpha_k.$$

This gives us a functional $\Phi : \{a_k\}_{k\in\mathbb{N}} \to \mathbb{C}$. We now assert that Φ is continuous with respect to the uniform norm. Indeed, let $k, l \in \mathbb{N}$ be arbitrary. By the previous lemma there exists a constant $C > 0$ such that

$$\begin{aligned}
|\Phi(a_k) - \Phi(a_l)| &= \lim_{j\to\infty} \left| \left\langle (a_k - a_l)^W(x, h_j D) u_{h_j}, u_{h_j} \right\rangle_{L^2(\mathbb{R}^n)} \right| \\
&= \limsup_{j\to\infty} \left| \left\langle (a_k - a_l)^W(x, h_j D) u_{h_j}, u_{h_j} \right\rangle_{L^2(\mathbb{R}^n)} \right| \\
&\le \limsup_{j\to\infty} \left\| (a_k - a_l)^W(x, h_j D) \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \left\| u_{h_j} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&\le \limsup_{j\to\infty} \left(C \left\| a_k - a_l \right\|_\infty + o(1) \right) \left\| u_{h_j} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&\le C \left\| a_k - a_l \right\|_\infty \cdot \sup_{j\ge 1} \left\| u_{h_j} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&\le \tilde{C} \left\| a_k - a_l \right\|_\infty
\end{aligned}$$

for some constant $\tilde{C} > 0$. Put otherwise, we have shown that Φ is a Lipschitz continuous functional on the dense subset $\{a_k\}_{k\in\mathbb{N}}$ of $C_0(\mathbb{R}^{2n})$. In particular, Φ is uniformly continuous. Therefore, it extends uniquely to a continuous functional Φ on the whole of $C_0(\mathbb{R}^{2n})$.[2]

Step 3. We claim that

$$\Phi(a) = \lim_{j\to\infty} \left\langle a^W(x, h_j D) u_{h_j}, u_{h_j} \right\rangle_{L^2(\mathbb{R}^n)} \tag{2.3}$$

for all compactly supported symbols $a \in C_c^\infty(\mathbb{R}^{2n})$. To see this, let $\varepsilon > 0$ be given. Since the $\{a_k\}_{k\in\mathbb{N}}$ form a dense subset of $C_0(\mathbb{R}^{2n})$ and Φ is a continuous functional, there exists a function $b \in \{a_k\}_{k\in\mathbb{N}}$ such that

$$CM^2 \left\| b - a \right\|_\infty + |\Phi(b) - \Phi(a)| < \frac{\varepsilon}{2} \tag{2.4}$$

[2]Note that we cannot yet say that Φ is linear. Indeed, Φ is only a priori defined on the dense *subset* $\{a_k\}_{k\in\mathbb{N}}$ of $C_0(\mathbb{R}^{2n})$. Since this dense subset is not a linear subspace of $C_0(\mathbb{R}^{2n})$, it does not make sense to say that Φ is linear on $\{a_k\}_{k\in\mathbb{N}}$. In particular, we cannot immediately extend Φ linearly on the whole of $C_0(\mathbb{R}^{2n})$.

with $C > 0$ is as in Lemma 2.3 and $M := \sup_{h\in(0,1)} \|u_h\|_{L^2(\mathbb{R}^n)}$. Then, for any $j \in \mathbb{N}$, the triangle inequality yields

$$\begin{aligned}
&\left|\left\langle a^W(x,h_jD)u_{h_j}, u_{h_j}\right\rangle_{L^2(\mathbb{R}^n)} - \Phi(a)\right| \\
&\leq \left|\left\langle a^W(x,h_jD)u_{h_j}, u_{h_j}\right\rangle_{L^2(\mathbb{R}^n)} - \left\langle b^W(x,h_jD)u_{h_j}, u_{h_j}\right\rangle_{L^2(\mathbb{R}^n)}\right| \\
&\qquad + \left|\left\langle b^W(x,h_jD)u_{h_j}, u_{h_j}\right\rangle_{L^2(\mathbb{R}^n)} - \Phi(b)\right| + |\Phi(b)-\Phi(a)| \\
&= \left|\left\langle (b-a)^W(x,h_jD)u_{h_j}, u_{h_j}\right\rangle_{L^2(\mathbb{R}^n)}\right| + |\Phi(b)-\Phi(a)| \\
&\qquad\quad + \left|\left\langle b^W(x,h_jD)u_{h_j}, u_{h_j}\right\rangle_{L^2(\mathbb{R}^n)} - \Phi(b)\right| \\
&\leq \left\|(b-a)^W(x,h_jD)\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \left\|u_{h_j}\right\|^2_{L^2(\mathbb{R}^n)} + |\Phi(b)-\Phi(a)| \qquad (2.5) \\
&\qquad\quad + \left|\left\langle b^W(x,h_jD)u_{h_j}, u_{h_j}\right\rangle_{L^2(\mathbb{R}^n)} - \Phi(b)\right| \qquad (2.6)
\end{aligned}$$

By the definition of Φ, we have

$$\Phi(b) = \lim_{j\to\infty} \left\langle b^W(x,h_jD)u_{h_j}, u_{h_j}\right\rangle_{L^2(\mathbb{R}^n)}. \tag{2.7}$$

Moreover, it follows from Lemma 2.3 that, as $j \to \infty$,

$$\left\|(b-a)^W(x,h_jD)\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C\,\|b-a\|_\infty + o(1).$$

Combining the above with (2.4) - (2.7) we infer that

$$\begin{aligned}
\left|\left\langle a^W(x,h_jD)u_{h_j}, u_{h_j}\right\rangle_{L^2(\mathbb{R}^n)} - \Phi(a)\right| &\leq \frac{\varepsilon}{2} + o(1) \\
&< \varepsilon
\end{aligned}$$

for all j large. This proves (2.3). In fact, this shows that Φ is linear on $C_0(\mathbb{R}^{2n})$. Indeed, (2.3) implies that Φ is a continuous linear functional on $C_c^\infty(\mathbb{R}^{2n})$. Hence, it has a continuous *linear* extension $\widetilde{\Phi}$ on the whole of $C_0(\mathbb{R}^{2n})$. However, since $\widetilde{\Phi} \equiv \Phi$ on the dense subspace $C_c^\infty(\mathbb{R}^{2n})$, we must have $\tilde{\Phi} = \Phi$ on $C_0(\mathbb{R}^{2n})$.

Finally, invoking the Riesz Representation Theorem[3] then gives us the existence of a complex Radon measure μ such that

$$\Phi(a) = \int_{\mathbb{R}^{2n}} a(x,\xi)\mathrm{d}\mu$$

[3]Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space. Given any continuous linear functional Ψ on $C_0(X)$, the Riesz Representation Theorem guarantees the existence of a complex Radon measure μ on X such that $\Psi(f) = \int_X f\mathrm{d}\mu$ for all $f \in C_0(X)$. See Theorem 7.17 in Folland [2].

for all $a \in C_0(\mathbb{R}^{2n})$. By (2.3), it follows that

$$\lim_{j\to\infty} \left\langle a^W(x, h_j D)u_{h_j}, u_{h_j}\right\rangle = \Phi(a) = \int_{\mathbb{R}^{2n}} a(x,\xi)\mathrm{d}\mu$$

for all compactly supported symbols $a \in C_c^\infty(\mathbb{R}^{2n})$.

Step 4. It remains to show that μ is a positive Radon measure. Namely, we assert that

$$\int_{\mathbb{R}^{2n}} a(x,\xi)\,\mathrm{d}\mu \geq 0 \tag{2.8}$$

for all $a \in C_0(\mathbb{R}^{2n})$ with $a \geq 0$. First, let us assume that $a \in C_c^\infty(\mathbb{R}^{2n})$. Then the Sharp Gårding inequality (see Theorem 4.32 in Zworski [7]) guarantees the existence of a constant $C > 0$ such that

$$\left\langle a^W(x, h_j D)u_{h_j}, u_{h_j}\right\rangle \geq -Ch_j \left\|u_{h_j}\right\|_{L^2(\mathbb{R}^n)}.$$

for all j large. Since our collection of functions (u_h) is uniformly bounded in $L^2(\mathbb{R}^n)$, taking the limit on either side yields (by virtue of (2.3))

$$\int_{\mathbb{R}^{2n}} a\mathrm{d}\mu \geq 0.$$

As in Remark 2.1, if a is merely assumed to be a member of $C_0(\mathbb{R}^{2m})$, we can find a non-negative sequence (v_l) in $C_c^\infty(\mathbb{R}^{2m})$ converging uniformly to a as $l \to \infty$. Then, by the above, we obtain $\Phi(a) = \lim_{l\to\infty} \Phi(v_l) \geq 0$. This completes the proof. □

Remark 2.2. Note that the Radon measure μ obtained in the proof of Theorem 2.1 is such that $C_0(\mathbb{R}^{2n}) \subseteq L^1(\mu)$. Clearly, such a measure μ cannot be a positive multiple of the Lebesgue measure. Therefore, Theorem 2.1 ensures that every bounded family $\{u_h\}_{h\in(0,1)}$ has an associated defect measure μ that is *not* merely a rescaling of the Lebesgue measure on $\mathbb{R}^{2n}$.

2.2. **Examples.** While we have only demonstrated the above result for Weyl quantizations, as noted by Zworski [7], the above result still holds (by an approximation argument) for the standard quantization. For the sake of simplicity and clarity, we use the standard quantization in the following two examples.

Example 1 (Coherent States – Zworski [7])**.** For $h \in (0,1)$ consider the family of functions $\{u_h\}_{h\in(0,1)}$ in $L^2(\mathbb{R}^n)$ given by

$$u_h(x) := \frac{1}{(\pi h)^{n/4}} e^{\frac{i}{h}\langle x-x_0,\xi_0\rangle - \frac{1}{2h}|x-x_0|^2}.$$

Note that the constant $(\pi h)^{-n/4}$ is chosen so that

$$\|u_h\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \frac{1}{(\pi h)^{n/2}} e^{-\frac{1}{h}|x-x_0|^2} \mathrm{d}x \xrightarrow{y=(x-x_0)/\sqrt{h}} \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2} \mathrm{d}y = 1.$$

Hence, the family $\{u_h\}_{h\in(0,1)}$ is bounded in $L^2(\mathbb{R}^n)$. Fix now a function $a \in C_c^\infty(\mathbb{R}^{2n})$; we claim that

$$\langle a(x, hD)u_h, u_h\rangle_{L^2(\mathbb{R}^n)} \to a(x_0, \xi_0)$$

as $h \searrow 0$. This would prove that the Dirac measure $\delta_{(x_0,\xi_0)}$ is the only defect measure (with respect to the standard quantization) associated to the family $\{u_h\}_{h\in(0,1)}$. To justify this, we write for each $h \in (0, 1)$

$$\begin{aligned}
\langle a(x, hD)u_h, u_h\rangle_{L^2(\mathbb{R}^n)} &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x, \xi) e^{\frac{i}{h}\langle x-y,\xi\rangle} u_h(y)\overline{u_h(x)} \mathrm{d}y\mathrm{d}\xi\mathrm{d}x \\
&= \frac{2^{n/2}}{(2\pi h)^{3n/2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x, \xi) e^{\frac{i(\langle x-y,\xi\rangle + \langle y-x,\xi_0\rangle)}{h}} e^{-\frac{\left(|x-x_0|^2+|y-x_0|^2\right)}{2h}} \mathrm{d}y\mathrm{d}\xi\mathrm{d}x \\
&= \frac{2^{n/2}}{(2\pi h)^{3n/2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x, \xi) e^{\frac{i\langle x-y,\xi-\xi_0\rangle}{h}} e^{-\frac{\left(|x-x_0|^2+|y-x_0|^2\right)}{2h}} \mathrm{d}y\mathrm{d}\xi\mathrm{d}x.
\end{aligned}$$

Now, for every fixed pair $(x, \xi) \in \mathbb{R}^{2n}$, making the change of variables $z = y - x_0$ gives

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{\frac{i\langle x-y,\xi-\xi_0\rangle}{h}} e^{-\frac{1}{2h}|y-x_0|^2} \mathrm{d}y &= \int_{\mathbb{R}^n} e^{\frac{i\langle x-x_0-z,\xi-\xi_0\rangle}{h}} e^{-\frac{|z|^2}{2h}} \mathrm{d}z \\
&= e^{\frac{i\langle x-x_0,\xi-\xi_0\rangle}{h}} \mathcal{F}\left(e^{-\frac{1}{2}\left\langle \frac{z}{h}, z\right\rangle}\right)\left(\frac{\xi-\xi_0}{h}\right) \\
&= e^{\frac{i\langle x-x_0,\xi-\xi_0\rangle}{h}} (2\pi h)^{n/2} e^{-\frac{|\xi-\xi_0|^2}{2h}}.
\end{aligned}$$

Note that this last equality follows from Theorem 3.1 in Zworski [7] with $Q = I/h$. Thus, we obtain

$$\begin{aligned}
\langle a(x, hD)u_h, u_h\rangle_{L^2(\mathbb{R}^n)} &= \frac{2^{n/2}}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x, \xi) e^{\frac{i\langle x-x_0,\xi-\xi_0\rangle}{h}} e^{-\frac{\left(|x-x_0|^2+|\xi-\xi_0|^2\right)}{2h}} \mathrm{d}\xi\mathrm{d}x \\
&= \frac{2^{n/2}}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x, \xi) e^{\frac{i\langle x_0-x,\xi_0-\xi\rangle}{h}} e^{-\frac{\left(|x_0-x|^2+|\xi_0-\xi|^2\right)}{2h}} \mathrm{d}\xi\mathrm{d}x
\end{aligned}$$

Writing $z = (x, \xi)$ and defining $z_0 := (x_0, \xi_0)$ in $\mathbb{R}^{2n}$, it is clear from the above that

$$\langle a(x, hD)u_h, u_h\rangle_{L^2(\mathbb{R}^n)} = (a * K_h)(x_0), \quad \forall h \in (0, 1), \tag{2.9}$$

where we are defining

$$K_h(z) = K_h(x, \xi) := \frac{2^{n/2}}{(2\pi h)^n} e^{\frac{i\langle x,\xi\rangle}{h}} e^{-\frac{\left(|x|^2+|\xi|^2\right)}{2h}}$$

We are now reduced to showing that $\{K_h\}_{h\in(0,1)}$ consists of "good kernels" in the sense that $(a * K_\delta)(x_0,\xi_0) \to a(x_0,\xi_0)$ as $h \searrow 0$. Indeed, by a simple change of variables, it is straightforward to check that for each $h > 0$ there holds

$$\|K_h\|_{L^1(\mathbb{R}^{2n})} = \frac{2^{n/2}}{(2\pi h)^n}\int_{\mathbb{R}^{2n}} e^{-\frac{1}{2h}|z|^2}\mathrm{d}z = \frac{2^{n/2}}{\pi^n}\int_{\mathbb{R}^{2n}} e^{-|w|^2}\mathrm{d}w < \infty$$

uniformly in h. Furthermore, using Fubini's Theorem and making the change of variables $y = -x$ shows that

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} K_h(z)\mathrm{d}z &= \frac{2^{n/2}}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} e^{-\frac{\left(|x|^2+|\xi|^2\right)}{2h}}\mathrm{d}x\mathrm{d}\xi \\
&= \frac{2^{n/2}}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{2h}}\left(\int_{\mathbb{R}^n} e^{-i\langle y,\xi/h\rangle}e^{-\frac{|y|^2}{2h}}\mathrm{d}y\right)\mathrm{d}\xi \\
&= \frac{2^{n/2}}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{2h}}\mathcal{F}\left(e^{-\frac{1}{2}\left\langle\frac{z}{h},z\right\rangle}\right)\left(\frac{\xi}{h}\right)\mathrm{d}\xi \\
&= \frac{2^{n/2}}{(2\pi h)^n}\cdot(2\pi)^{n/2}h^{n/2}\int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{2h}}e^{-\frac{|\xi|^2}{2h}}\mathrm{d}\xi \\
&= \frac{1}{\pi^{n/2}h^{n/2}}\int_{\mathbb{R}^n} e^{-\left|\xi/\sqrt{h}\right|^2}\mathrm{d}\xi \\
&= 1.
\end{aligned}$$

Finally, for any fixed $\eta > 0$,

$$\begin{aligned}
\int_{|z|\geq\eta} |K_h(z)|\,\mathrm{d}z &= \frac{2^{n/2}}{(2\pi h)^n}\int_{|z|\geq\eta} e^{-\frac{|z|^2}{2h}}\mathrm{d}z \\
&= \frac{2^{n/2}}{\pi^n}\int_{|w|\geq\frac{\eta}{\sqrt{2h}}} e^{-|w|^2}\mathrm{d}w
\end{aligned}$$

which converges to 0 as $h \to 0$ by the monotone convergence theorem. These last three properties imply that K_δ is an approximation to the identity (see Chapter 3 §2 in Stein & Shakarchi [5]). Hence, it follows from (2.9) that

$$\lim_{h\to 0}\langle a(x,hD)u_h,u_h\rangle_{L^2(\mathbb{R}^n)} = \lim_{h\to 0}(a * K_h)(z_0) = a(z_0) = a(x_0,\xi_0).$$

We conclude that the only defect measure (with respect to the standard quantization) associated to the famiy $\{u_h\}_{h\in(0,1)}$ is the Dirac measure $\delta_{(x_0,\xi_0)}$. Informally, we see that the $\{u_h\}_{h\in(0,1)}$ concentrate near the point x_0 as $h \searrow 0$.

Let us consider one last explicit example in which the family $\{u_h\}_{h\in(0,1)}$ does not consist of $L^2(\mathbb{R}^n)$ functions, but rather of tempered distributions. Note that Theorem 2.1 will not apply in this case, however.

Example 2. Fix $\ell \in \mathbb{Z}^n$, $a \in C_c^\infty(\mathbb{R}^{2n})$, and consider for any positive h the function $u_h(x) = e^{\frac{i}{h}\langle \ell, x\rangle}$. Clearly, for each $h > 0$, $u_h \notin L^2(\mathbb{R}^n)$. In particular, $\{u_h\}_{h\in(0,1)}$ is not a bounded family in $L^2(\mathbb{R}^n)$. However, we *can* interpret these u_h as tempered distributions. On the other hand, every u_h is bounded. Therefore, since $a(x, hD)u_h$ is Schwartz, we can still make sense of the inner product

$$\langle a(x, hD)u_h, u_h\rangle_{L^2(\mathbb{R}^n)}.$$

Now, for each $h > 0$ a simple calculation yields

$$\begin{aligned}
a(x, hD)u_h(x) &= \int_{\mathbb{R}^n} u_h(y) \left(\frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x, \xi)\mathrm{d}\xi \right) \mathrm{d}y \\
&= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} e^{\frac{i}{h}\langle \ell,y\rangle} a(x, \xi)\mathrm{d}\xi\mathrm{d}y \\
&= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} e^{\frac{i}{h}\langle \ell,y\rangle} a(x, \xi)\mathrm{d}\xi\mathrm{d}y.
\end{aligned}$$

Making the change of variables $z = y - x$, the above becomes

$$\begin{aligned}
a(x, hD)u_h(x) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle z,\xi\rangle} e^{\frac{i}{h}\langle \ell,x+z\rangle} a(x, \xi)\mathrm{d}\xi\mathrm{d}z \\
&= \frac{e^{\frac{i}{h}\langle \ell,x\rangle}}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle \ell,z\rangle} \left(\int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle z,\xi\rangle} a(x, \xi)\mathrm{d}\xi \right) \mathrm{d}z
\end{aligned}$$

Using the notation $a_x(\xi) = a(x, \xi)$ the above is precisely

$$\begin{aligned}
a(x, hD)u_h(x) &= \frac{e^{\frac{i}{h}\langle \ell,x\rangle}}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle \ell,z\rangle} \left[\mathcal{F}_h a_x\right](z)\mathrm{d}z \\
&= e^{\frac{i}{h}\langle \ell,x\rangle} \left(\mathcal{F}_h^{-1}\left[\mathcal{F}_h a_x\right]\right)(\ell) \\
&= e^{\frac{i}{h}\langle \ell,x\rangle} a(x, \ell)
\end{aligned}$$

It follows that

$$\langle a(x, hD)u_h, u_h\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} a(x, \ell) e^{\frac{i}{h}\langle \ell,x\rangle} e^{-\frac{i}{h}\langle \ell,x\rangle}\mathrm{d}x = \int_{\mathbb{R}^n} a(x, \ell)\mathrm{d}x$$

A standard application of the Tonelli-Fubini theorem shows that

$$\int_{\mathbb{R}^n} a(x, \ell)\mathrm{d}x = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} a(x, \xi)\mathrm{d}\delta_\ell(\xi) \right) dx = \int_{\mathbb{R}^{2n}} a(x, \xi)[\mathrm{d}\delta_\ell(\xi) \times \mathrm{d}x].$$

Thus, even though the family $\{u_h\}_{h\in(0,1)}$ does not belong to $L^2(\mathbb{R}^n)$, one can still make sense (in the context of defect measures) of the behaviour of $\langle a(x, hD)u_h, u_h\rangle_{L^2(\mathbb{R}^n)}$ as $h \to 0$.

3. Applications to PDE

As previously stated, we would like to conclude more about defect measures μ associated to a bounded family $\{u_h\}_{h\in(0,1)}$ in $L^2(\mathbb{R}^n)$ when our collection $\{u_h\}_{h\in(0,1)}$ are approximate solutions to a PDE. Consider once again the family of functions in Example 2. There, we had worked with a family of eigenfunctions for the Laplacian. More precisely, each function u_h in Example 2 satisfies the Helmholtz equation

$$-h^2\Delta u_h = |\ell|^2\, u_h$$

or, equivalently, $P_h u_h = -h^2\Delta u_h - |\ell|^2\, u_h = 0$ where P_h is the quantization (Weyl and standard) of $p(x,\xi) = |\xi|^2 - |\ell|^2$. In particular, $\|P_h u_h\|_{L^2(\mathbb{R}^n)} = o(h)$. Before moving forward, we re-iterate an important definition.

Definition 3.1. We define the Poisson bracket, denoted $\{\cdot,*\}$, by the rule

$$\{p,q\} = \partial_\xi p\cdot\partial_x q - \partial_x p\cdot\partial_\xi q.$$

where (p,q) are functions on $\mathbb{R}^{2n}$. The above is defined at all points of $\mathbb{R}^{2n}$ where p,q are differentiable.

Returning to Example 2, we can make the following observation:

$$\begin{aligned}\int_{\mathbb{R}^{2n}}\{p,a\}\mathrm{d}\mu &= \int_{\mathbb{R}^{2n}}(\partial_\xi p\cdot\partial_x a - \partial_x p\cdot\partial_\xi a)\,\mathrm{d}\mu\\ &= \int_{\mathbb{R}^{2n}}(\partial_\xi p\cdot\partial_x a)\,\mathrm{d}\mu\\ &= 2\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\xi\cdot\partial_x a(x,\xi)\mathrm{d}\delta_\ell(\xi)\right)\mathrm{d}x\\ &= 2\int_{\mathbb{R}^n}\ell\cdot\partial_x a(x,\ell)\mathrm{d}x = 0\end{aligned}$$

where for the last equality we have used the assumption $a\in C_c^\infty(\mathbb{R}^{2n})$.

As it turns out, the above result continues to hold for a class of symbols. However, keeping in line with the context of Theorem 2.1, we restrict ourselves to the case where $\{u_h\}_{h\in(0,1)}$ are normalized in $L^2(\mathbb{R}^n)$. Formally, we have the following:

Proposition 3.1. *Consider a real-valued function $p\in S$ and denote by P_h the Weyl quantization $p^W(x,hD)$. Let $\{u_h\}_{h\in(0,1)}$ be a family in $L^2(\mathbb{R}^n)$ satisfying*

$$\begin{cases}\|P_h u_h\|_{L^2(\mathbb{R}^n)} = o(h) & \text{as } h\to 0,\\ \|u_h\|_{L^2(\mathbb{R}^n)} = 1 & \forall h\in(0,1).\end{cases}\tag{3.1}$$

Then for any defect measure μ of the family $\{u_h\}_{h\in(0,1)}$ there holds

$$\int_{\mathbb{R}^{2n}} \{p,a\}\mathrm{d}\mu = 0$$

for all compactly supported symbols $a \in C_c^\infty(\mathbb{R}^{2n})$.

Remark 3.1. This statement carries a significant physical interpretation. As pointed out in Zworski [7], this means that μ is invariant under the Hamiltonian flow arising from the function p.

Proof. Fix a function $a \in C_c^\infty(\mathbb{R}^{2n})$ and notice that by separating between the real and imaginary parts, we may suppose without loss of generality that a is real-valued. Define $A_h = a^W(x, hD)$ and notice that since Weyl quantizations of real symbols are self adjoint (see Zworski [7] page 58), both A_h and P_h are self-adjoint operators on $L^2(\mathbb{R}^n)$. We consider the operator

$$[P_h, A_h] := P_hA_h - A_hP_h.$$

For each $h \in (0,1)$, we compute

$$\begin{aligned}|\langle [P_h, A_h]u_h, u_h\rangle| &= |\langle P_hA_hu_h, u_h\rangle - \langle A_hP_hu_h, u_h\rangle| \\ &\le |\langle A_hu_h, P_hu_h\rangle| + |\langle P_hu_h, A_hu_h\rangle| \\ &\le 2\,\|P_hu_h\|_{L^2(\mathbb{R}^n)}\,\|A_hu_h\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

where we have used that A_h, P_h are self-adjoint. Appealing to Lemma 2.3, there exists a constant $C > 0$ depending only on a such that

$$\begin{aligned}|\langle [P_h, A_h]u_h, u_h\rangle| &\le 2\,\|P_hu_h\|_{L^2(\mathbb{R}^n)}\,\|A_h\|_{L^2\to L^2}\,\|u_h\|_{L^2(\mathbb{R}^n)} \\ &= 2\,\|P_hu_h\|_{L^2(\mathbb{R}^n)}\,\|A_h\|_{L^2\to L^2} \\ &\le C\,\|P_hu_h\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

for all $h > 0$ sufficiently small. Finally, assumption (3.1) implies that

$$\langle [P_h, A_h]u_h, u_h\rangle = o(h) \quad \text{as } h \to 0 \tag{3.2}$$

On the other hand, by Theorem 4.18 in Zworski [7] we have

$$[P_h, A_h] = \frac{h}{i}\{p,a\}^W(x, hD) + f^W(x, D) \tag{3.3}$$

for some smooth function f in $O_S(h^2)$. That is, for all multi-indices α there exists $C_\alpha > 0$ such that $|\partial^\alpha f| \le C_\alpha h^2$ on $\mathbb{R}^{2n}$. In particular, since we are only considering

$h \in (0,1)$, we see that $f \in S$. By Theorem 4.23 in Zworski [7], there exist constants $C, M > 0$ such that

$$\begin{aligned}\left\|f^W(x,hD)\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} &\leq C \sum_{|\alpha|\leq Mn} h^{|\alpha|/2} \sup_{\mathbb{R}^{2n}} |\partial^\alpha f| \\ &\leq Ch^2 \sum_{|\alpha|\leq Mn} C_\alpha h^{|\alpha|/2} = O(h^2).\end{aligned}$$

Since our functions (u_h) are normalized in $L^2(\mathbb{R}^n)$, we see that

$$\left|\frac{1}{h}\left\langle f^W(x,D)u_h, u_h\right\rangle\right| \leq \frac{1}{h}\left\|f^W(x,hD)\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = O(h). \tag{3.4}$$

Combining (3.2) - (3.4), we see that

$$\left\langle\{p,a\}^W(x,hD)u_h, u_h\right\rangle = \frac{1}{h}\left\langle[P_h, A_h]u_h, u_h\right\rangle - \frac{1}{h}\left\langle f^W(x,D)u_h, u_h\right\rangle = o(1) \tag{3.5}$$

as $h \to 0$. Finally, let μ be a defect measure associated to $\{u_h\}_{h\in(0,1)}$ and let $h_j \to 0$ be the sequence such that

$$\lim_{j\to\infty}\left\langle b^W(x,h_jD)u_{h_j}, u_{h_j}\right\rangle = \int_{\mathbb{R}^{2n}} b\,\mathrm{d}\mu$$

for each $b \in C_c^\infty(\mathbb{R}^{2n})$. Since p is smooth and a is smooth and of compact support, we notice that $\{p,a\}$ is also a smooth function of compact support on $\mathbb{R}^{2n}$. Hence,

$$\int_{\mathbb{R}^{2n}}\{p,a\}\mathrm{d}\mu = \lim_{j\to\infty}\left\langle\{p,a\}^W(x,h_jD)u_{h_j}, u_{h_j}\right\rangle = 0$$

where the last equality follows from (3.5). □

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