THE RIEMANN INTEGRAL FOR FUNCTIONS MAPPING TO BANACH SPACES

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Abstract. We construct a Riemann integral for functions taking values in Banach spaces over $\mathbb{R}$ or $\mathbb{C}$. We show that this integral extends the standard properties of the classical Riemann integral.

Let us get some notation out of the way. Throughout this note, $\mathbb{K}$ will denote one of the fields $\mathbb{R}$ or $\mathbb{C}$, and $\mathcal{X}$ will represent a *Banach space* over the given field $\mathbb{K}$. Let us recall that a Banach space $\mathcal{X}$ is a vector space over $\mathbb{K}$ equipped with a norm $\|\cdot\|$ such that $\mathcal{X}$ is a complete metric space with respect to the metric

$$d(x_1, x_2) := \|x_1 - x_2\|, \quad x_1, x_2 \in \mathcal{X}.$$

Henceforth, we will assume that the reader has taken courses in basic topology and functional analysis. We now move towards the development of the generalized Riemann integral.

1. Constructing the Generalized Riemann Integral

Fix a compact interval $[a, b] \subset \mathbb{R}$ and let $\mathcal{X}$ be a Banach space over $\mathbb{K}$. A function $f : [a, b] \to \mathcal{X}$ is called *bounded* if there exists $M > 0$ such that

$$\|f\|_\infty := \sup_{x \in \mathcal{X}} \|f(x)\| \leq M.$$

Here, $\|\cdot\|$ is the given norm on $\mathcal{X}$. The set of all bounded maps $f : [a, b] \to \mathcal{X}$ will be denoted by $B([a, b], \mathcal{X})$. In practice, one often takes $\mathcal{X}$ to be one of the vector spaces $\mathbb{R}$ or $\mathbb{C}$.

Definition 1. Let $a < b$ be real numbers. A partition of a compact interval $[a, b]$ is a finite family $\mathcal{P}$ consisting of closed sub-intervals of $[a, b]$ whose union is $[a, b]$. More precisely,

$$\mathcal{P} = \left\{ [x_{i-1}, x_i] : x_j \in [a, b],\, i = 1, \ldots, n \right\} \tag{1.1}$$

and $\bigcup_1^n [x_{i-1}, x_i] = [a, b]$. By convention, we always require that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

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Partitions themselves are not fundamental to the Riemann integral. However, a *tagged partition* is.

Definition 2. Let $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of a compact interval $[a,b]$. A set of *tags* for the partition $\mathcal{P}$ is a finite collection $\{t_1, \ldots, t_n\}$ of points in $[a,b]$ satisfying $t_i \in [x_{i-1}, x_i]$ for all $i = 1, \ldots, n$. A *tagged partition* is then defined to be a partition together with a set of tags, i.e.

$$\dot{\mathcal{P}} := \{\mathcal{P}, \{t_i\}_{i=1}^n\}. \tag{1.2}$$

By convention, we will often write $\dot{\mathcal{P}}$ instead of $\mathcal{P}$ to indicate that the partition $\mathcal{P}$ comes equipped with a set of tags. Given a partition $\mathcal{P}$ (possibly tagged), we define the *mesh* of $\mathcal{P}$ as

$$\|\mathcal{P}\| := \max_{1 \le i \le n} (x_i - x_{i-1}) > 0. \tag{1.3}$$

With this out of the way, we come to the definition of the Riemann integral. As in standard calculus, this begins with the introduction of a *Riemann sum*.

Definition 3. Let $f : [a,b] \to \mathcal{X}$ be a function and let $\dot{\mathcal{P}}$ be a tagged partition of $[a,b]$. The *Riemann sum* of f over the partition $\dot{\mathcal{P}}$ is defined via the equation

$$S(f; \mathcal{P}) := \sum_{i=1}^n (x_i - x_{i-1}) \cdot f(t_i) = \sum_{i=1}^n (x_i - x_{i-1}) f(t_i). \tag{1.4}$$

This brings us to another convention: for $u \in \mathcal{X}$ and $\alpha \in \mathbb{K}$, we will typically write αu instead of the scalar product $\alpha \cdot u$.

We have now come to the definition of Riemann integrability for functions taking values in a Banach space $\mathcal{X}$.

Definition 4. A function $f : [a,b] \to \mathcal{X}$ is called *Riemann integrable*[1] (on $[a,b]$) if there is a vector $\Lambda \in \mathcal{X}$ with the property that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\| S\left(f; \dot{\mathcal{P}}\right) - \Lambda \right\| < \varepsilon$$

for **all** tagged partitions $\dot{\mathcal{P}}$ of $[a,b]$ having $\left\|\dot{\mathcal{P}}\right\| < \delta$. If f is Riemann integrable, we say that Λ is the *Riemann integral* of f over $[a,b]$ and denote this quantity by

$$\int_a^b f.$$

[1]Or, sometimes *extended Riemann integrable.*

Finally, we will denote by $\mathcal{R}([a,b],\mathcal{X})$ the collection of all Riemann integrable functions $[a,b] \to \mathcal{X}$. We will also agree to the following convention:

$$\int_b^a f := -\int_a^b f.$$

Remark 1.1. Our construction technically only allows us to define the quantity $\int_a^b f$ for $a < b$ and suitable f. However, there is nothing that prevents us from using an analogous definition with $a = b$. Certainly, let $f : [a,b] \to \mathcal{X}$; there exists but a single tagged partition of $[a,a] = \{a\}$, given by

$$\dot{\mathcal{P}} = \{[a,a], a\}.$$

Then,

$$S\left(f; \dot{\mathcal{P}}\right) = f(a)(a-a) = 0.$$

Hence, it would make sense to define $\int_a^a f = 0$ for *all* functions $[a,b] \to \mathcal{X}$. In general, we will only allow this extreme case for functions that are Riemann integrable on non-trivial compact intervals $[c,d]$ containing the point a. Unless stated otherwise, one should assume that $[a,b]$ is a non-trivial interval.

1.1. **Immediate Consequences and Trivial Examples.** We would like to check that the Riemann integral is well defined whenever it exists This is accomplished by way of the following proposition.

Proposition 1.1. *Let $f : [a,b] \to \mathcal{X}$ be Riemann integrable. Then its Riemann integral is unique.*

Proof. Suppose that L and L' are two candidates for the "Riemann integral of f". Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$\left\| S\left(f; \dot{\mathcal{P}}\right) - L \right\| < \varepsilon \quad \text{and} \quad \left\| S\left(f; \dot{\mathcal{P}}\right) - L' \right\| < \varepsilon$$

for all tagged partitions $\dot{\mathcal{P}}$ of $[a,b]$ whose mesh is strictly less than δ. Clearly, such a partition of $[a,b]$ will always exist and as such we get that

$$\|L - L'\| \leq \left\| S\left(f; \dot{\mathcal{P}}\right) - L \right\| + \left\| S\left(f; \dot{\mathcal{P}}\right) - L' \right\| < 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $L = L'$. $\square$

In our development of the integral, it will be useful to have a class of "prototype functions" that we know are Riemann integrable. This begins with a quick proof that constant functions are integrable.

Proposition 1.2. *Fix a vector $u \in \mathcal{X}$ and define a function $f : [a, b] \to X$ via the equation $f(x) \equiv u$. Then, $f(x)$ is Riemann integrable on $[a, b]$. Moreover,*

$$\int_a^b f = (b - a)u.$$

Proof. First, let $\dot{\mathcal{P}}$ be any tagged partition of the interval $[a, b]$, following of course the notation in Definition 2. It is easy to see that

$$S\left(f; \dot{\mathcal{P}}\right) = \sum_{i=1}^{n} (x_i - x_{i-1}) f(t_i) = \sum_{i=1}^{n} (x_i - x_{i-1}) u = (b - a)u.$$

Thus, given $\varepsilon > 0$, take $\delta := 1$. Then, if $\dot{\mathcal{P}}$ is a tagged partition of $[a, b]$ whose mesh is strictly less than δ, the above yields

$$\left\| S\left(f; \dot{\mathcal{P}}\right) - (b - a)u \right\| < \varepsilon.$$

This concludes the proof. □

1.2. **Basic Properties of the Extended Riemann Integral.** We now turn towards some basic regularity properties that our integral enjoys. Again, these all hold true for the standard Riemann and Lebesgue integrals and are to be expected of any integral. As I like to say, an integral that does not satisfy the following theorem is no integral at all!

Theorem 1.3. *Let $[a, b] \subset \mathbb{R}$ and $\mathcal{X}$ a Banach space over $\mathbb{K}$. Suppose that $f, g : [a, b] \to \mathcal{X}$ are Riemann integrable. The following hold true.*

(1) If $\alpha \in \mathbb{K}$, then αf is also Riemann integrable on $[a, b]$. Moreover,

$$\int_a^b \alpha f = \alpha \int_a^b f.$$

(2) The function $f + g$ is Riemann integrable on $[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Proof. We first prove (1). Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$\left\| S\left(f; \dot{\mathcal{P}}\right) - \int_a^b f \right\| < \varepsilon$$

whenever $\left\|\dot{\mathcal{P}}\right\| < \delta$. For any such tagged partition $\dot{\mathcal{P}}$, one also has

$$\begin{aligned}\left\|S\left(\alpha f;\dot{\mathcal{P}}\right) - \alpha\int_a^b f\right\| &= \left\|\sum_{i=1}^{n}(x_i - x_{i-1})\cdot\alpha f(t_i) - \alpha\int_a^b f\right\| \\ &= |\alpha|\left\|\sum_{i=1}^{n}(x_i - x_{i-1})f(t_i) - \int_a^b f\right\| \\ &= |\alpha|\left\|S\left(f;\dot{\mathcal{P}}\right) - \int_a^b f\right\| \\ &\le |\alpha|\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was taken arbitrarily, we see from the above that the first point holds true. For the second, we simply make use of the triangle inequality. Let $\varepsilon > 0$ and fix $\delta > 0$ such that both

$$\left\|S\left(f;\dot{\mathcal{P}}\right) - \int_a^b f\right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\|S\left(g;\dot{\mathcal{P}}\right) - \int_a^b g\right\| < \frac{\varepsilon}{2}$$

whenever $\left\|\dot{\mathcal{P}}\right\| < \delta$. Of course, this can be done since both f and g are Riemann integrable on $[a, b]$. If $\dot{\mathcal{P}}$ is a tagged partition of $[a, b]$ whose mesh is strictly less than δ, the triangle inequality gives

$$\begin{aligned}&\left\|S\left(f+g;\dot{\mathcal{P}}\right) - \left(\int_a^b f + \int_a^b g\right)\right\| \\ &= \left\|\sum_{i=1}^{n}(x_i - x_{i-1})(f(t_i) + g(t_i)) - \int_a^b f - \int_a^b g\right\| \\ &\le \left\|S\left(f;\dot{\mathcal{P}}\right) - \int_a^b f\right\| + \left\|S\left(g;\dot{\mathcal{P}}\right) - \int_a^b g\right\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This proves (2) and the proof is complete. $\square$

The following bound is intuitively obvious, but incredibly useful.

Theorem 1.4. *Let $f : [a, b] \to \mathcal{X}$ be Riemann integrable and bounded. There holds the following:*

$$\left\|\int_a^b f\right\| \le (b - a)\left\|f\right\|_\infty, \tag{1.5}$$

Proof. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$\left\| S\left(f;\dot{\mathcal{P}}\right) - \int_a^b f \right\| < \varepsilon$$

for all tagged partitions $\dot{\mathcal{P}}$ of $[a,b]$ whose mesh is strictly less than δ. Now, fix any such partition $\dot{\mathcal{P}}$ and write

$$\left\| \int_a^b f \right\| \leq \left\| \int_a^b f - S\left(f;\dot{\mathcal{P}}\right) \right\| + \left\| S\left(f;\dot{\mathcal{P}}\right) \right\| < \left\| S\left(f;\dot{\mathcal{P}}\right) \right\| + \varepsilon.$$

Following the notation used in Definition 2, we subsequently see from the above that

$$\begin{aligned} \left\| \int_a^b f \right\| &\leq \varepsilon + \sum_{i=1}^n |(x_i - x_{i-1})| \, \|f(t_j)\| \\ &\leq \varepsilon + \left(\sum_{i=1}^n (x_i - x_{i-1}) \right) \|f\|_\infty \\ &= \varepsilon + (b-a) \|f\|_\infty . \end{aligned}$$

Passing to the limit as $\varepsilon \searrow 0$ yields the desired inequality. □

Before we move on towards the next unrelated subsection, we prove a "boundedness theorem" which states that all Riemann integrable functions are necessarily bounded. In most theories of Riemann integration, one assumes a priori that the function f is bounded. In our case, however, this is something that can be deduced from integrability.

Theorem 1.5. *Let $f : [a,b] \to \mathcal{X}$ be Riemann integrable. Then, f is bounded on $[a,b]$.*

Proof. We proceed by way of contradiction. Suppose there exists a Riemann integrable function f on $[a,b]$ that is unbounded. Let L denote the value $\int_a^b f$. Taking $\varepsilon := 1$, there exists $\delta > 0$ such that

$$\left\| S\left(f;\dot{\mathcal{P}}\right) - L \right\| < 1$$

for all tagged partitions $\dot{\mathcal{P}}$ of $[a,b]$ having mesh strictly less than δ. The reverse triangle inequality then yields

$$\left\| S\left(f;\dot{\mathcal{P}}\right) \right\| < 1 + \|L\| \tag{1.6}$$

for all such $\dot{\mathcal{P}}$. We now construct a special tagged partition $\dot{\mathcal{Q}}$ which will contradict the above. To this end, let $\mathcal{Q} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be any partition of $[a,b]$ whose mesh is strictly less than δ. Since f is unbounded on $[a,b]$, it is unbounded one at least one interval in $\mathcal{Q}$. Hence, there is an index $k = 1, \ldots, n$ and $t_k \in [x_{k-1}, x_k]$ such that

$$\|(x_k - x_{k-1}) f(t_k)\| > 1 + \|L\| + \left\| \sum_{j \neq k} (x_j - x_{j-1}) f(t_k) \right\|.$$

Here, we have chosen $t_j := x_j$ for all $j \neq k$. But then, if we give $\mathcal{Q}$ these tags,

$$\begin{aligned} \left\| S\left(f; \dot{\mathcal{Q}}\right) \right\| &= \left\| (x_k - x_{k-1}) f(t_k) + \sum\nolimits_{j \neq k} (x_j - x_{j-1}) f(t_k) \right\| \\ &\geq \|(x_k - x_{k-1}) f(t_k)\| - \left\| \sum_{j \neq k} (x_j - x_{j-1}) f(t_k) \right\| \\ &> 1 + \|L\|. \end{aligned}$$

This of course contradicts (1.6). Our proof is now complete. □

Remark 1.2. We conclude from the theorem above that the estimate in Theorem 1.4 holds for all Riemann integrable functions $[a,b] \to \mathcal{X}$.

2. Classes of Extended Riemann Integrable Functions

The ultimate goal of this section is to prove that continuous functions $[a,b] \to \mathcal{X}$ are Riemann integrable in the extended sense. This result is to be expected, but it nonetheless fundamental to the calculus we are developing. Certainly, we will mostly be dealing with functions that are *at least* continuous.

We begin with a useful criterion for Riemann integrability.

Theorem 2.1. *A function $f : [a,b] \to \mathcal{X}$ is Riemann integrable if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left\| S\left(f; \dot{\mathcal{P}}\right) - S\left(f; \dot{\mathcal{Q}}\right) \right\| < \varepsilon$$

for all tagged partitions $\dot{\mathcal{P}}$, $\dot{\mathcal{Q}}$ of $[a,b]$ whose meshes are strictly less than δ.

Proof. First, suppose that $f \in \mathcal{R}([a,b], \mathcal{X})$. Given $\varepsilon > 0$, we choose $\delta > 0$ with the property that

$$\left\| S\left(f; \dot{\mathcal{P}}\right) - \int_a^b f \right\| < \frac{\varepsilon}{2}$$

for all tagged partitions $\dot{\mathcal{P}}$ of $[a,b]$ whose mesh is strictly less than δ. Then, if both $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ have messes strictly less than δ, the triangle inequality gives

$$\left\| S\left(f; \dot{\mathcal{P}}\right) - S\left(f; \dot{\mathcal{Q}}\right) \right\| \leq \left\| S\left(f; \dot{\mathcal{P}}\right) - \int_a^b f \right\| + \left\| \int_a^b f - S\left(f; \dot{\mathcal{Q}}\right) \right\| < \varepsilon,$$

as was required. Conversely, we must show that f is Riemann integrable. Given $n \in \mathbb{N}$, let $\delta_n > 0$ be such that

$$\left\| S\left(f; \dot{\mathcal{P}}\right) - S\left(f; \dot{\mathcal{Q}}\right) \right\| < \frac{1}{n}$$

for all tagged partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ having meshes strictly less than δ_n. Of course, we may assume that $\delta_{n+1} \leq \delta_n$ for all $n \in \mathbb{N}$. For each index $n \in \mathbb{N}$, let $\dot{\mathcal{P}}_n$ be a tagged partition of $[a, b]$ whose mesh is strictly less than δ_n. For any two $n \geq m$ we then have

$$\left\| S\left(f; \dot{\mathcal{P}}_n\right) - S\left(f; \dot{\mathcal{P}}_m\right) \right\| < \frac{1}{m}.$$

The above shows that the sequence $\left(S\left(f; \dot{\mathcal{P}}_n\right)\right)_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{X}$. Since $\mathcal{X}$ is a Banach space, there exists a point $\Lambda \in \mathcal{X}$ such that

$$\lim_{n \to \infty} S\left(f; \dot{\mathcal{P}}_n\right) = \Lambda.$$

We now prove that f is Riemann integrable on $[a, b]$, and that $\int_a^b f$ is equal to Λ. Let $\varepsilon > 0$ be given and select $N_1 \in \mathbb{N}$ such that

$$\left\| S\left(f; \dot{\mathcal{P}}_n\right) - \Lambda \right\| < \frac{\varepsilon}{2}$$

for all $n \geq N_1$. Let $N_2 \in \mathbb{N}$ be such that

$$\frac{1}{N_2} < \frac{\varepsilon}{2}.$$

Thus, if $n, m \geq N_2$ there holds

$$\left\| S\left(f; \dot{\mathcal{P}}_n\right) - S\left(f; \dot{\mathcal{P}}_m\right) \right\| < \frac{1}{N_2} < \frac{\varepsilon}{2}.$$

Set now $N := \max(N_1, N_2)$ and let $\dot{\mathcal{P}}$ be any tagged partition of $[a, b]$ whose mesh is strictly less than δ_N. The triangle inequality subsequently yields

$$\begin{aligned} \left\| S\left(f; \dot{\mathcal{P}}\right) - \Lambda \right\| &\leq \left\| S\left(f; \dot{\mathcal{P}}\right) - S\left(f; \dot{\mathcal{P}}_N\right) \right\| + \left\| S\left(f; \dot{\mathcal{P}}_N\right) - \Lambda \right\| \\ &< \frac{1}{N} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This yields the desired conclusion. □

Using the above criterion, we will show that all continuous functions from $[a, b]$ into $\mathcal{X}$ are Riemann integrable. But first, we must quickly introduce some terminology. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are partitions of an interval $[a, b]$. Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

denote the endpoints of those intervals in $\mathcal{P}$; let

$$a = y_0 < y_1 < \cdots < y_m = b$$

be those of intervals in $\mathcal{Q}$. The partition $\mathcal{Q}$ of $[a, b]$ is called a **refinement** of $\mathcal{P}$ if every x_i is equal to some y_j (in the lists above). In this case, we will agree to write $\mathcal{P} \subseteq \mathcal{Q}$.

The **common refinement** of the partitions $\mathcal{P}$ and $\mathcal{Q}$ consists of all those intervals formed by rearranging the x_i's and y_j's into a strictly increasing list of real numbers. This common refinement is often denoted by $\mathcal{P} \vee \mathcal{Q}$.

Lemma 2.2. *Let $f : [a, b] \to \mathcal{X}$ be continuous. For each $\varepsilon > 0$ there exists $\delta > 0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a, b]$ whose mesh is strictly less than δ, and any refinement $\dot{\mathcal{Q}}$ of $\dot{\mathcal{P}}$, there holds*

$$\left\| S\left(f; \dot{\mathcal{P}}\right) - S\left(f; \dot{\mathcal{Q}}\right) \right\| < \varepsilon.$$

Proof. Since f is continuous and $[a, b]$ is compact, the function f is uniformly continuous. Therefore, given $\varepsilon > 0$, we may find $\delta > 0$ such that

$$\|f(x) - f(y)\| < \frac{\varepsilon}{b - a}$$

for all $x, y \in [a, b]$ satisfying $|x - y| < \delta$. Let $\dot{\mathcal{P}}$ be a tagged partition of $[a, b]$ with $\left\|\dot{\mathcal{P}}\right\| < \delta$. Suppose that $\dot{\mathcal{Q}}$ is any refinement of $\dot{\mathcal{P}}$ with its own set of tags. Any subinterval $[x_{i-1}, x_i]$ of $\dot{\mathcal{P}}$ may then be decomposed into a finite union of subintervals from $\dot{\mathcal{Q}}$

$$[x_{i-1}, x_i] = \bigcup_{j=1}^{n_i} [y_{i,j-1}, y_{i,j}].$$

Therefore,

$$\begin{aligned}
\left\| S\left(f; \dot{\mathcal{P}}\right) - S\left(f; \dot{\mathcal{Q}}\right) \right\| &= \left\| \sum_{i=1}^{n} (x_i - x_{i-1}) f(x_i^*) - \sum_{i=1}^{n} \sum_{j=1}^{n_i} (y_{i,j} - y_{i,j-1}) f(y_{i,j}^*) \right\| \\
&= \left\| \sum_{i=1}^{n} \sum_{j=1}^{n_i} (y_{i,j} - y_{i,j-1}) f(x_i^*) - \sum_{i=1}^{n} \sum_{j=1}^{n_i} (y_{i,j} - y_{i,j-1}) f(y_{i,j}^*) \right\| \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n_i} (y_{i,j} - y_{i,j-1}) \left\| f(x_i^*) - f(y_{i,j}^*) \right\|
\end{aligned}$$

where, of course, the x_i^* are tags from $\dot{\mathcal{P}}$ and the $y_{i,j}^*$ are those from $\dot{\mathcal{Q}}$. However, since the mesh of $\dot{\mathcal{P}}$ is strictly less than δ, the above implies that

$$\left\|S\left(f;\dot{\mathcal{P}}\right)-S\left(f;\dot{\mathcal{Q}}\right)\right\|<\frac{\varepsilon}{b-a}\sum_{i=1}^{n}\sum_{j=1}^{n_i}(y_{i,j}-y_{i,j-1})=\varepsilon.$$

This proves the lemma. □

We can now prove our cornerstone theorem.

Theorem 2.3. *Let $f:[a,b]\to\mathcal{X}$ be a continuous function. Then, f is extended Riemann integrable on $[a,b]$.*

Proof. This proof relies upon the previous lemma. Let $\varepsilon>0$ be given and fix $\delta>0$ as in the statement of our lemma. Let $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ be two tagged partitions of $[a,b]$, both having mesh strictly less than δ. Letting $\dot{\mathcal{P}}\vee\dot{\mathcal{Q}}$ denote their common refinement, we get

$$\begin{aligned}\left\|S\left(f;\dot{\mathcal{P}}\right)-S\left(f;\dot{\mathcal{Q}}\right)\right\|&\le\left\|S\left(f;\dot{\mathcal{P}}\right)-S\left(f;\dot{\mathcal{P}}\vee\dot{\mathcal{Q}}\right)\right\|\\&\quad+\left\|S\left(f;\dot{\mathcal{P}}\vee\dot{\mathcal{Q}}\right)-S\left(f;\dot{\mathcal{Q}}\right)\right\|\end{aligned}$$

whence

$$\left\|S\left(f;\dot{\mathcal{P}}\right)-S\left(f;\dot{\mathcal{Q}}\right)\right\|<2\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, Theorem 2.1 gives that f is integrable. This completes the proof. □

We finally know that continuous functions $[a,b]\to\mathcal{X}$ are (extended) Riemann integrable. Although this was a significant amount of work, it also serves as a reminder that our definition is "sane".

Lemma 2.4. *Let $f:[a,b]\to\mathcal{X}$ be integrable. If $a\le c\le b$ then*

$$\int_a^b f=\int_a^c f+\int_c^b f.$$

As the proof is tedious and technical, we choose to omit the proof. Of course, the proof of this identity for the classical Riemann integral can easily be adapted for the lemma above. We leave the details to the reader.

Let us now conclude with a theorem which sharpens the estimate proven in Theorem 1.4. Here, we will use the fact that continuous functions are Riemann integrable. We also recall that continuous functions $[a,b]\to\mathbb{R}$ are Riemann integrable in the standard sense. Alternatively, one can note that our definition with $\mathcal{X}=\mathbb{R}$ corresponds with the classical Riemann integral.

Theorem 2.5. *Let $f : [a, b] \to \mathcal{X}$ be continuous. Then,*

$$\left\| \int_a^b f \right\| \leq \int_a^b \|f\| \, \mathrm{d}x.$$

Proof. Other than the Riemann integrability of f on $[a, b]$, we only need the continuity of f to ensure that $\|f\|$ is Riemann integrable (in the classical sense) on $[a, b]$. Indeed, by composition, the map $\|f\| : [a, b] \to \mathbb{R}$ is continuous and hence *classically Riemann integrable* on $[a, b]$.

Let $\left\{\dot{\mathcal{P}}_n\right\}_{n=1}^{\infty}$ be a sequence of partitions of $[a, b]$ whose mesh tends to zero as $n \to \infty$. Then, since $f \in \mathcal{R}([a, b], \mathcal{X})$, we have

$$\int_a^b f = \lim_{n\to\infty} S\left(f; \dot{\mathcal{P}}_n\right).$$

By continuity of the norm, this extends to

$$\left\| \int_a^b f \right\| = \lim_{n\to\infty} \left\| S\left(f; \dot{\mathcal{P}}_n\right) \right\|.$$

Fix an index $n \in \mathbb{N}$ and let

$$a = x_0 < \cdots < x_m = b$$

denote the end points of the sub-intervals in $\dot{\mathcal{P}}_n$, and let x_j^* be the tag from the j^{th} interval. Clearly,

$$\left\| S\left(f; \dot{\mathcal{P}}_n\right) \right\| \leq \sum_{j=1}^{m} (x_j - x_{j-1}) \left\| f(x_j^*) \right\|$$

which is itself a Riemann sum for the function $\|f(\cdot)\|$ on $[a, b]$. Since $\|f\|$ is Riemann integrable on $[a, b]$, putting all the pieces together gives

$$\left\| \int_a^b f \right\| = \lim_{n\to\infty} \left\| S\left(f; \dot{\mathcal{P}}_n\right) \right\| \leq \lim_{n\to\infty} S\left(\|f\|; \dot{\mathcal{P}}_n\right) = \int_a^b \|f\|.$$

This completes the proof. $\square$

3. Transformations of Riemann Integrable Functions

In this subsection, we fix Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ over the same ground field $\mathbb{K}$. Before giving a useful proposition, we recall some definitions from functional analysis. We say a linear operator $T : \mathcal{X} \to \mathcal{Y}$ is *bounded* if there exists $M > 0$ such that

$$\|T(x)\| \leq M \|x\|, \quad \forall x \in \mathcal{X}.$$

One can show that a linear operator $T : \mathcal{X} \to \mathcal{Y}$ is bounded if and only if it is continuous. Commonly, one writes $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ to denote the set of all bounded linear operators $\mathcal{X} \to \mathcal{Y}$. Obviously, $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ is a vector space over $\mathbb{K}$. One can make $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ into a normed space by defining for each $T \in \mathscr{L}(\mathcal{X}, \mathcal{Y})$ the norm

$$\|T\| := \inf \{M \geq 0 : \|T(x)\| \leq M \|x\|, \forall x \in \mathcal{X}\}.$$

It can be shown that the infimum above is achieved since the set

$$\{M \geq 0 : \|T(x)\| \leq M \|x\|, \forall x \in \mathcal{X}\}$$

is non-empty, closed, and bounded from below. Thus, $\|Tx\| \leq \|T\| \|x\|$ for all $x \in \mathcal{X}$. Actually, since $\mathcal{Y}$ is a Banach space, the space $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ is also a Banach space.

With these formalities out of the way, we state the following.

Proposition 3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $A : [a, b] \to \mathscr{L}(\mathcal{X}, \mathcal{Y})$ continuous. Fix $x \in \mathcal{X}$. The function*

$$A(\cdot)x : [a, b] \longrightarrow \mathcal{Y}, \quad t \mapsto A(t)(x)$$

is continuous, and hence Riemann integrable, on $[a, b]$. Furthermore,

$$\int_a^b A(t)x = \left(\int_a^b A(t)\right) x.$$

Proof. First, fix $t, s \in [a, b]$. Then, $A(t) - A(s)$ is itself a bounded linear operator $\mathcal{X} \to \mathcal{Y}$. As a result, we get that

$$\|A(t)x - A(s)x\| \leq \|A(t) - A(s)\| \|x\|.$$

Since A is continuous, we see from the above that $A(\cdot)x$ is a continuous function $[a, b] \to \mathcal{Y}$. In particular, $A(\cdot)x$ is Riemann integrable. Next, we prove the identity. Let $\left\{\dot{\mathcal{P}}_n\right\}_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a, b]$ whose mesh tends to zero as $n \to \infty$. Since $A(\cdot)x$ is Riemann integrable on $[a, b]$, we have that

$$\int_a^b A(t)x = \lim_{n\to\infty} S\left(A(\cdot)x; \dot{\mathcal{P}}_n\right).$$

However, it is not difficult to see that

$$S\left(A(\cdot)x; \dot{\mathcal{P}}_n\right) = \left(S\left(A(\cdot); \dot{\mathcal{P}}_n\right)\right)(x)$$

for each $n \in \mathbb{N}$. Moreover, the function $A(\cdot)$ is continuous and takes values in a Banach space $\mathscr{L}(\mathcal{X}, \mathcal{Y})$. Therefore, $A(\cdot)$ is integrable and $\int_a^b A(t)$ will be a bounded linear operator $\mathcal{X} \to \mathcal{Y}$. In particular, we get that

$$\lim_{n\to\infty} S\left(A(\cdot); \dot{\mathcal{P}}_n\right) = \int_a^b A(t) \tag{3.1}$$

in the space $\mathscr{L}(\mathcal{X},\mathcal{Y})$. But, if a sequence $\{T_n\}_{n=1}^{\infty}$ of bounded linear operators in $\mathscr{L}(\mathcal{X},\mathcal{Y})$ converges to an element $T\in\mathscr{L}(\mathcal{X},\mathcal{Y})$, then $T_n(x)$ converges to $T(x)$ in $\mathcal{Y}$ for each fixed $x\in\mathcal{X}$. Indeed, this follows from the inequality

$$\|T_n(x)-T(x)\|\leq\|T_n-T\|\,\|x\|\,.$$

Finally, putting all the pieces together, we infer that

$$\int_a^b A(t)x=\lim_{n\to\infty}S\left(A(\cdot);\dot{\mathcal{P}}_n\right)(x)=\left(\int_a^b A(t)\right)(x).$$

□

The following proposition admits a similar method of proof.

Proposition 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces over a field $\mathbb{K}$. Assume that $f:[a,b]\to\mathcal{X}$ is continuous and let $A:\mathcal{X}\to\mathcal{Y}$ be a bounded linear operator. There holds the following,*

$$\int_a^b A(f)=A\left(\int_a^b f\right).$$

Proof. Clearly, the composite

$$[a,b]\xrightarrow{f}\mathcal{X}\xrightarrow{A}\mathcal{Y}$$

is continuous, and hence $A(f)$ is Riemann integrable on $[a,b]$. Let $\{\dot{\mathcal{P}}_n\}_{n=1}^{\infty}$ be a sequence of tagged partitions for the interval $[a,b]$ whose mesh tends to 0 as $n\to\infty$. Using the integrability of $A(f)$, one has

$$\int_a^b A(f)=\lim_{n\to\infty}S\left(A(f);\dot{\mathcal{P}}_n\right).$$

By the linearity of the operator A, it is easy to see that for each n

$$S\left(A(f);\dot{\mathcal{P}}_n\right)=A\left(S\left(f;\dot{\mathcal{P}}_n\right)\right).$$

Invoking the integrability of f, we simultaneously obtain

$$\lim_{n\to\infty}S\left(f;\dot{\mathcal{P}}_n\right)=\int_a^b f.$$

Finally, the continuity of A yields

$$\int_a^b A(f)=\lim_{n\to\infty}S\left(A(f);\dot{\mathcal{P}}_n\right)=\lim_{n\to\infty}A\left(S\left(f;\dot{\mathcal{P}}_n\right)\right)=A\left(\int_a^b f\right).$$

This concludes the proof. □

3.1. **The Continuity of the Riemann Integral.** Let $f : [a, b] \to \mathcal{X}$ be a continuous map. Here, $[a, b] \subset \mathbb{R}$ is a non-trivial compact interval. For each $c \in [a, b]$, we will adopt the convention that $\int_c^c f = 0$. As pointed out at the start of this chapter, we can certainly make sense of this when viewing $\{c\}$ as the trivial interval $[c, c]$. What we show here, is that this "special case" does not affect the continuity of the Riemann integral.

Theorem 3.3. *Let $f : [a, b] \to \mathcal{X}$ be Riemann integrable. Consider the function*

$$F : [a, b] \to \mathcal{X}, \quad F(t) := \int_a^t f(t).$$

Then, $F(t)$ is well defined, continuous, and satisfies $F(a) = 0$.

Proof. Since f is Riemann integrable, it is bounded on $[a, b]$. By definition, we also have $F(a) = 0$. Now, to prove the continuity of the integral. We first check continuity at a. Let $t \geq a$; if $t = a$ then

$$\left\| \int_a^t f \right\| = 0$$

by definition. If instead $t \in (a, b]$, then

$$\left\| \int_a^t f(t) \right\| \leq (t - a) \sup_{t \in [a,b]} \|f(x)\| =: M|t - a|,$$

where $M > 0$ is defined in the obvious way. Notice that the above actually holds even when $t = a$. This shows that $\int_a^t f$ is continuous at $t = a$. Similarly, one can show that $\int_a^t f$ is a continuous function at $t = b$. If $c \in (a, b)$ the argument is a little different. Let $\varepsilon > 0$ be given and choose M such that

$$\sup_{t \in [a,b]} \|f(t)\| < M < \infty.$$

Pick $\delta > 0$ satisfying the inequality

$$\delta < \frac{\varepsilon}{M}.$$

If $0 < t - c < \delta$, then

$$\left\| \int_a^t f - \int_a^c f \right\| = \left\| \int_c^t f \right\| \leq M|t - c| < \varepsilon.$$

If $-\delta < t - c < 0$ one similarly has

$$\left\| \int_a^t f - \int_a^c f \right\| = \left\| \int_t^c f \right\| \leq M|c - t| < \varepsilon.$$

Since $\left\| \int_a^t f - \int_a^c f \right\| = 0$ when $t = c$, the continuity of $F(t)$ on $[a, b]$ follows. $\square$

Actually, one can partially strengthen the above.

Corollary 3.4. *Let $f : [a,b] \to \mathcal{X}$ be Riemann integrable and fix $c \in [a,b]$. The function defined by*

$$F : [a,b] \to \mathcal{X}, \quad F(t) := \int_c^t f$$

is continuous.

Proof. By the previous theorem, the function

$$F_1(t) := \begin{cases} 0, & t \le c, \\ \int_c^t f, & t > c \end{cases}$$

is continuous on $[a,b]$. By the same token, so is the map

$$F_2(t) := \begin{cases} 0, & t \ge c, \\ \int_t^c f, & t < c. \end{cases}$$

The result then follows from the identity

$$F(t) = F_1(t) - F_2(t), \quad t \in [a,b].$$

□

4. The Fundamental Theorem of Calculus

So far, our main goal has been to develop a generalized calculus for functions taking values in Banach spaces. So far, we have only discussed the integration portion of such a theory. In this section, we discuss *a* notion of differentiation and prove a generalization of the fundamental theorem of calculus. In later sections, we will discuss alternative derivatives/differentials and it is important that the reader distinguish these various definitions.

Throughout this section, $\mathcal{X}$ denotes a Banach space and I is a bounded interval in $\mathbb{R}$. We recall some asymptotic notation.

Definition 5. A function $f : I \to \mathcal{X}$ is called *differentiable* at a point $c \in I$ if there exists $\lambda \in \mathcal{X}$ such that

$$f(t) - f(c) = \lambda(t-c) + o\left(|t-c|\right)$$

as $t \to c$, with $t \in I$. That is, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f(t) - f(c) - \lambda(t-c)\| \le \varepsilon|t-c|, \quad \text{for all } t \in I \text{ with } |t-c| < \delta.$$

In this case, we write $f'(c) = \lambda$ and call this the *derivative* of f at c. The map f is called differentiable on I if f is differentiable at all points $c \in I$.

Our definition of differentiability appears to be substantially different from the "standard definition" one encounters in calculus. Below, we show that this is not at all the case – we have simply chosen to give the definition in a form that is more common in functional analysis. After all, one will not always be able to work with functions defined on an interval. For such functions, the formulation above is easier to extend.

Lemma 4.1. *Let $I \subseteq \mathbb{R}$ be an interval and fix a point $c \in I$. A function $f : I \to \mathcal{X}$ is differentiable at c, with derivative $f'(c) \in \mathcal{X}$, if and only if*

$$\lim_{\substack{t \to c \\ t \in I}} \frac{f(t) - f(c)}{t - c}$$

exists an is equal to $f'(c)$.

Proof. Suppose first that $f'(c)$ exists according to the definition. Given $\varepsilon > 0$, we choose $\delta > 0$ such that

$$\|f(t) - f(c) - (t - c)f'(c)\| \leq \frac{\varepsilon}{2}|t - c|$$

for all $t \in I$ with $|t - c| < \delta$. If $0 < |t - c| < \delta$, we then have

$$\|f(t) - f(c) - (t - c)f'(c)\| \leq \frac{\varepsilon}{2}|t - c| < \varepsilon|t - c|.$$

For such $t > c$, we divide through by $|t - c| = (t - c) > 0$ to obtain

$$\left\|\frac{f(t) - f(c)}{t - c} - f'(c)\right\| < \varepsilon.$$

If instead $-\delta < t - c < 0$, we divide both sides by $|t - c| = (c - t)$ yielding

$$\begin{aligned}\left\|\frac{f(t) - f(c)}{c - t} - \frac{t - c}{c - t}f'(c)\right\| &= \left\|-\frac{f(t) - f(c)}{t - c} + f'(c)\right\| \\ &= \left\|\frac{f(t) - f(c)}{t - c} - f'(c)\right\| < \varepsilon.\end{aligned}$$

Hence,

$$\lim_{\substack{t \to c \\ t \in I}} \frac{f(t) - f(c)}{t - c} = f'(c) \quad \text{in } \mathcal{X}.$$

Conversely, suppose this last equation holds true for f. Fixing $\varepsilon > 0$, we find $\delta > 0$ such that

$$\left\|\frac{f(t) - f(c)}{t - c} - f'(c)\right\| < \varepsilon$$

for all $t \in I$ with $0 < |t-c| < \delta$. Clearly, for these t:

$$\left\| \frac{f(t)-f(c)}{t-c} - f'(c) \right\| = \frac{1}{|t-c|} \|f(t)-f(c)-(t-c)f'(c)\| < \varepsilon.$$

But this implies that

$$\|f(t)-f(c)-(t-c)f'(c)\| < \varepsilon|t-c|.$$

Since equality holds for $t=c$, it follows that

$$\|f(t)-f(c)-(t-c)f'(c)\| \leq \varepsilon|t-c|$$

for all $t \in I$ with $|t-c| < \delta$. □

It is not difficult to see that the usual addition rule for derivatives continues to hold in this abstract structure. Indeed, this is a simple application of the triangle inequality. Similarly, scalar multiplication commutes with differentiation. We summarize these with the following proposition.

Proposition 4.2. *Let $f, g : I \to \mathcal{X}$ be differentiable at a point $c \in I$. Then,*

(1) $f+g$ is differentiable at c with derivative $(f+g)'(c) = f'(c) + g'(c)$.
(2) If $\gamma \in \mathbb{K}$, the function $\gamma \cdot f(t)$ is differentiable at c with derivative equal to $c \cdot f'(c)$.
(3) Suppose in addition that $\mathcal{X}$ is a Banach algebra over $\mathbb{K}$ with multiplication operation $*$. Then $f * g$ is differentiable at $c \in I$ with derivative equal to:

$$(f*g)'(c) = f'(c) * g(c) + f(c) * g'(c).$$

Proof. The proof is left as a routine exercise to the reader. □

As we should expect, differentiable functions are at least continuous.

Proposition 4.3. *Let $f : I \to \mathcal{X}$ be differentiable at a point $c \in I$. Then, f is continuous at c.*

Proof. First, let $\delta > 0$ be such that

$$\|f(t)-f(c)-\lambda(t-c)\| \leq |t-c|, \quad \text{for all } t \in I \text{ with } |t-c| < \delta.$$

If $t \in I$ satisfies $|t-c| < \delta$, the triangle inequality gives

$$\|f(t)-f(c)\| \leq |t-c| + \|\lambda(t-c)\| = |t-c| + \|\lambda\|\,|t-c|.$$

Let $\varepsilon > 0$ be given; since the right hand side of the above equation tends to 0 as $t \to c$, we can choose $\delta' > 0$ such that

$$|t-c| + \|\lambda\|\,|t-c| < \varepsilon, \quad \text{whenever } |t-c| < \delta'.$$

Take δ'' to be the minimum of δ and δ'. For each $t \in I$ satisfying $|t-c| < \delta''$, all of this gives

$$\|f(t) - f(c)\| \leq |t-c| + \|\lambda\| \, |t-c| < \varepsilon.$$

We infer that f is continuous at the point c. $\square$

4.1. **The Fundamental Theorem of Calculus: Form 1.** In this subsection, we prove a version of the fundamental theorem of calculus. The proof is is standard, but the result will be used to deduce the second (and more applicable) form of the fundamental theorem. This will be accomplished in the subsequent subsection. For now, we handle the following.

Theorem 4.4. *Suppose that $\mathcal{X}$ is a Banach space over $\mathbb{K}$ and let $f : [a,b] \to \mathcal{X}$ be continuous. Define*

$$F : [a,b] \to \mathcal{X}, \quad F(x) := \int_a^t f.$$

Then, F is continuously differentiable on $[a,b]$ and satisfies the rule $F' \equiv f$ on $[a,b]$.

Proof. Let us fix x, y with $a \leq x < y \leq b$. Then, using the additivity of the integral, we get that

$$F(y) - F(x) = \int_a^y f - \int_a^x f = \int_x^y f.$$

Moreover, if the integral is taken with respect to a variable t,

$$\int_x^y f(x) \, \mathrm{d}t = (y-x) f(x).$$

Therefore,

$$\|F(y) - F(x) - (y-x)f(x)\| = \left\| \int_x^y (f(t) - f(x)) \, \mathrm{d}t \right\| \tag{4.1}$$

$$\leq \|x - y\| \sup_{x \leq t \leq y} \|f(t) - f(x)\| \, . \tag{4.2}$$

Being continuous on $[a,b]$, the function f is also uniformly continuous on this interval. The above therefore implies that

$$\lim_{y \searrow x} \frac{F(y) - F(x)}{y - x}$$

exists and is equal to $f(x)$. Let us now treat the "left hand derivative". Of course, the above already covers the case $x = a$ and as such we may as well assume that $x > a$. Let y be such that $a \leq y < x$. As above, we get

$$F(y) = \int_a^y f \quad \text{and} \quad F(x) = \int_a^x f$$

so that $F(x) - F(y) = \int_y^x f$. Or, rather,

$$F(y) - F(x) = -\int_y^x f.$$

Therefore,

$$\begin{aligned} \|F(y) - F(x) - (y-x)f(x)\| &= \left\| -\int_y^x f - (y-x)f(x) \right\| \\ &= \left\| \int_y^x f - (x-y)f(x) \right\| \\ &\le (x-y) \sup_{y \le t \le x} \|f(t) - f(x)\|. \end{aligned}$$

Thus, the same arguments used in (4.1)-(4.2) apply to give

$$\lim_{y \nearrow x} \frac{F(y) - F(x)}{y - x} = f(x).$$

This proves the claim. □

By way of a similar argument, one can establish the following. We leave the details of the proof as an exercise to the reader.

Proposition 4.5. *Suppose that $\mathcal{X}$ is a Banach space over $\mathbb{K}$ and let $f : [a,b] \to \mathcal{X}$ be continuous. Define*

$$F : [a,b] \to \mathcal{X}, \quad F(x) := \int_t^b f.$$

Then, F is continuously differentiable on $[a,b]$ and satisfies $F' \equiv -f$ on $[a,b]$.

4.2. **The Fundamental Theorem of Calculus: Form 2.** This section is devoted to a proof of the following identity:

$$u(b) - u(x) = \int_a^b u'.$$

Of course, we will need to place additional regularity conditions on this function u. At the very least, we need u to be differentiable with a derivative that is extended Riemann integrable.

Let $f : [a,b] \to \mathcal{X}$ be differentiable. Then, $f'(t)$ exists for all $t \in [a,b]$. Thus, we can view $f'(t)$ as a function $[a,b] \to \mathcal{X}$. We will say that f is *continuously differentiable* if f' exists and is continuous on $[a,b]$. Notice then that f' must also be Riemann integrable on $[a,b]$. More generally, for $k \ge 1$, we define

$$C^k([a,b], \mathcal{X}) := \left\{ f : [a,b] \to \mathcal{X} : f^{(k)} \text{ exists and is continuous} \right\}.$$

Here, we will mostly be interested in the space $C^1([a,b],\mathcal{X})$. Our main goal here is to establish the following:

Theorem 4.6. *Let $\mathcal{X}$ be a Banach space over $\mathbb{R}$ or $\mathbb{C}$ and suppose $f:[a,b]\to\mathcal{X}$ is continuously differentiable on $[a,b]$. Then, there holds the following*

$$\int_a^b f' = f(b) - f(a). \tag{4.3}$$

The proof will require a "lemma" that is of significant interest in its own right. Of course, the statement will not be the least bit surprising. However, the fact that the claim holds true is certainly reassuring.

Proposition 4.7. *Let $f:[a,b]\to\mathcal{X}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f'(t)=0$ for all $t\in(a,b)$, then, f is constant.*

Proof. Let $a<c<d<b$; we will show that $f(c)=f(d)$. It would then follow by continuity of f is constant. Let $\varepsilon>0$ be given and define

$$E := \{t\in[c,d] : \|f(t)-f(c)\| \le \varepsilon(t-c)\}.$$

The set E is non-empty since $c\in E$ by definition. Let s denote $\sup E$. Since $s=\sup E$, there exists a sequence of elements (t_n) in E converging to s as $n\to\infty$. Invoking the continuity of f and the norm, it is easy to see that

$$\|f(s)-f(c)\| = \lim_{n\to\infty}\|f(t_n)-f(c)\| \le \lim_{n\to\infty}\varepsilon(t_n-c) = \varepsilon(s-c).$$

Hence, $s\in E$ as well. Suppose, by way of contradiction, that $s<d$. Using that $f'(s)=0$, we can find $t\in(s,d)$ such that

$$\|f(t)-f(s)\| = \|f(t)-f(s)-f'(s)(t-s)\| \le \varepsilon(t-s)$$

implying that

$$\begin{aligned}\|f(t)-f(c)\| \le \|f(t)-f(s)\| + \|f(s)-f(c)\| &\le \varepsilon(t-s)+\varepsilon(s-c)\\ &= \varepsilon(t-c).\end{aligned}$$

That is, $t\in E$ which contradicts the upper-bound property of s. As a result, we conclude that $d=s$ and that

$$\|f(d)-f(c)\| \le \varepsilon(d-c).$$

Since $\varepsilon>0$ was taken arbitrarily, the proof is complete. ☐

Having dispensed with this technicality, we can prove (4.3).

Proof of Theorem 4.6. Consider the function $G:[a,b]\to\mathcal{X}$ defined by the quantity

$$G(x) := \left(\int_a^x f'\right) - f(x).$$

By Theorem 4.4, $G(x)$ is continuously differentiable on $[a, b]$ and satisfies

$$G'(x) = f'(x) - f'(x) = 0$$

for each $x \in [a, b]$. Invoking the previous proposition, we see that $G(x)$ must be constant. That is,

$$-f(a) = G(a) = G(b) = -f(b) + \int_a^b f'.$$

The proof is now complete. $\square$

From this one can easily prove the so called "mean value inequality". We leave its proof as an exercise to the reader.

Corollary 4.8. *Let $f : [a, b] \to \mathcal{X}$ be continuously differentiable. Then,*

$$\|f(b) - f(a)\| \leq (b - a) \cdot \|f'\|_\infty .$$

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