SOME SERIES OF EVEN POWER

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These notes discuss famous series that converge to surprising values. More precisely, we explicitly compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4}$. This is done by virtue of the Parseval Identity. Other than this identity, we do not require more than elementary integral calculus. Given two continuous functions ϕ, ψ defined on a bounded interval $[a, b]$ we will define

$$\langle \phi, \psi \rangle := \int_a^b \phi(x)\psi(x)\,\mathrm{d}x.$$

We give the following 'weak version' of Parseval's identity which will be sufficient for our purpose.

Theorem (Parseval)**.** *Let f be continuous on $[0,1]$. Define*

$$A_n := \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle}$$

where $\{X_n(x)\}_{n=1}^{\infty} = \{\sin(\pi n x)\}_{n=1}^{\infty}$ or $\{X_n(x)\}_{n=0}^{\infty} = \{\cos(\pi n x)\}_{n=0}^{\infty}$. Then,

$$\sum_{n}^{\infty} A_n^2 \int_0^1 {X_n}^2(x)\,\mathrm{d}x = \int_0^1 f(x)^2\,\mathrm{d}x.$$

Using this proposition, we will be able to compute two rather interesting and surprising infinite series.

Proposition 1.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof. Consider the function $f(x) = x$ on the interval $[0,1]$. We will make use of Parseval's identity. Since $f(x)$ is an odd function, it is wise to use the family $\{X_n(x)\}_{n=1}^{\infty} = \{\sin(\pi n x)\}_{n=1}^{\infty}$.

For $n \geq 1$, integration by parts will give

$$\langle X_n, X_n \rangle = \int_0^1 \sin^2(\pi n x)\,\mathrm{d}x = \frac{1}{2}.$$

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Furthermore,

$$\begin{aligned}\langle f, X_n\rangle = \int_0^1 x\sin(\pi n x)\,\mathrm{d}x &= -\frac{x\cos(\pi n x)}{\pi n}\bigg|_{x=0}^{x=1} + \frac{1}{\pi n}\int_0^1 \cos(\pi n x)\,\mathrm{d}x \\ &= \frac{(-1)^{n+1}}{\pi n} + \frac{\sin(\pi n x)}{(\pi n)^2}\bigg|_{x=0}^{x=1} \\ &= \frac{(-1)^{n+1}}{\pi n}.\end{aligned}$$

This means that, for each $n \geq 1$, one has

$$A_n^2 = \frac{1}{\pi^2 n^2}\cdot\frac{1}{1/2^2} = \frac{4}{\pi^2 n^2}. \tag{1}$$

An easy computation also gives

$$\int_0^1 f(x)^2\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}.$$

This together with Parseval's identity and (1) yields

$$\frac{1}{3} = \sum_{n=1}^{\infty} A_n^2 \int_0^1 X_n(x)^2\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\cdot\frac{1}{2} = \sum_{n=1}^{\infty}\frac{2}{\pi^2 n^2}.$$

A rearrangement then implies that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{2}\cdot\frac{1}{3} = \frac{\pi^2}{6}.$$

□

One can continue in this way to calculate

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}, \quad \text{for every } k \in \mathbb{N}.$$

We illustrate this method one final time, but instead using the cosine series for a function f.

Proposition 2.

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Proof. We instead consider the function $f(x) = x^2$ on $[0, 1]$. Since f is even, we will employ the cosine series for f. That is, we take

$$\{X_n\} = \{\cos(\pi n x)\}_{n=0}^{\infty}.$$

Therefore, the summation in Parseval's identity will begin at $n = 0$. This means we have to carry out the computation of A_0, which is easy:

$$A_0 := \frac{\int_0^1 x^2 \cos(\pi \cdot 0 \cdot x)\,\mathrm{d}x}{\int_0^1 \cos(\pi \cdot 0 \cdot x)^2\,\mathrm{d}x} = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}.$$

If instead $n \geq 1$, then an integration by parts will yield

$$\langle X_n, X_n \rangle = \int_0^1 \cos^2(\pi n x)\,\mathrm{d}x = \frac{1}{2}.$$

Furthermore, for all $n \geq 1$ there holds

$$\langle f, X_n \rangle = \int_0^1 x^2 \cos(\pi n x)\,\mathrm{d}x = \frac{2\cos(\pi n)}{\pi^2 n^2}$$

whence it follows that

$$A_n^2 = \frac{4}{\pi^4 n^4} \cdot 4 = \frac{16}{\pi^4 n^4}.$$

Thus, invoking Parseval's identity gives

$$\begin{aligned} \frac{1}{5} = \int_0^1 x^4\,\mathrm{d}x = \int_0^1 f(x)^2\,\mathrm{d}x &= A_0^2 + \sum_{n=1}^{\infty} A_n^2 \int_0^1 X_n(x)^2\,\mathrm{d}x \\ &= \frac{1}{3^2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{16}{\pi^4 n^4}. \end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{2\pi^4}{16}\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{\pi^4}{8} \cdot \frac{4}{45} = \frac{\pi^4}{90}.$$

□

Remark 1. By following the procedures used in the two examples given here, one can 'easily' compute $\zeta(2k)$, for any $k \in \mathbb{N}$, where $\zeta(s)$ is the Riemann zeta function. However, we do not know the explicit value of $\zeta(2k+1)$ for any $k \geq 1$.