A CRASH COURSE IN PARTIAL DIFFERENTIAL EQUATIONS

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Contents

This collection of notes grew our of preparation for the final examination of an introductory course in partial differential differential equations at McGill university. This course does not assume familiarity with mathematical analysis beyond

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multivariate calculus, and as such most results will not be proven. Nonetheless, we try to emphasize rigor when it is possible.

The first part of these notes is qualitative: the results presented and the examples are, in some sense, computational and geared towards determining explicit solutions to various elementary/classical PDE. This includes the use of Fourier transforms and series together with the method of characteristics. Using these techniques, we will study the wave, transport and Schrödinger equations.

The second part consists of more qualitative results. We introduce random walks and give examples where they lead to diffusion equations. We study "generalized functions" and show how they can be used to define derivatives for all locally integrable functions–this also makes the notion of the Dirac delta "function" precise. Finally, we examine Harmonic functions in $\mathbb{R}^n$ and Green's function for the Laplacian.

Part 1. **Quantitative Analysis of Solutions**

As mentioned above, through this part of the notes we look for explicit solutions to PDE. Thus, we will always be assuming that the functions involved are smooth (or at least C^2). We begin with a quick discussion of the Fourier transform.

1. The Fourier Transform

The Fourier transform is just as useful in the study of ODE as it is in PDE theory. The idea is to *transform* a PDE into a simpler ODE. Once we have solved this ODE, (that is, if we can solve it) we perform an "inverse transform" to get back to the solution of our original PDE.

Definition 1. Let $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f, denoted $\hat{f}$, is the function $\mathbb{R}^n \to \mathbb{C}$ given by

$$\hat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\, \mathrm{d}\mathbf{x}. \tag{1.1}$$

Then operator $\mathcal{F}$ which takes $L^1(\mathbb{R}^n) \ni f \mapsto \hat{f}$ is a linear map.

There are several properties to be noted at this point. Suppose that $\mathbf{a} \in \mathbb{R}^n$, then for every $f \in L^1(\mathbb{R}^n)$ there holds

$$\widehat{f(\mathbf{x}-\mathbf{a})} = e^{-i\mathbf{x}\cdot\boldsymbol{\xi}} \hat{f}(\boldsymbol{\xi}) \quad \text{and} \quad [\widehat{e^{-i\mathbf{a}\cdot\mathbf{x}} f(\mathbf{x})}] = \hat{f}(\boldsymbol{\xi}-\mathbf{a}). \tag{F1}$$

If $c > 0$ is fixed, we define

$$g_c(\mathbf{y}) := \frac{1}{c^n} g\left(\mathbf{y}/c\right)$$

where $g : \mathbb{R}^n \to \mathbb{C}$ is given. Then, if $f \in L^1(\mathbb{R}^n)$

$$\widehat{f(c\mathbf{x})}(\boldsymbol{\xi}) = \left[\hat{f}\right]_c(\boldsymbol{\xi}) \quad \text{and} \quad \widehat{f_c}(\boldsymbol{\xi}) = \hat{f}(c\boldsymbol{\xi}). \tag{F2}$$

These properties are not hard to prove directly from the definition. Hence, we leave these as exercises to the interested reader. Given a Fourier transform $\hat{f}$, we can sometimes recover the original function f (when this is true is a topic best left to a course in analysis).

THEOREM (Fourier Inversion). *Let $f \in L^1(\mathbb{R}^n)$ be such that $\hat{f} \in L^1(\mathbb{R}^n)$. Then,*

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}} \, \mathrm{d}\boldsymbol{\xi}$$

for $\mathbf{x} \in \mathbb{R}^n$. We then adopt the notation $f = \left(\hat{f}\right)^\vee$.

Given two functions $f, g \in L^1(\mathbb{R}^n)$ define their convolution to be

$$(f * g)(\mathbf{x}) := \int_{\mathbb{R}} f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) \, \mathrm{d}\mathbf{y} = (g * f)(\mathbf{x}).$$

This communicates nicely with the Fourier transform since

$$\widehat{f * g}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}) \hat{g}(\boldsymbol{\xi}). \tag{F3}$$

Perhaps even more interesting is how the Fourier transform responds to differentiation. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, then

$$\widehat{\partial^\alpha f}(\boldsymbol{\xi}) = i^{|\alpha|} \boldsymbol{\xi}^\alpha \hat{f}(\boldsymbol{\xi}). \tag{F4}$$

If $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ then we define $\boldsymbol{\xi}^\alpha$ to be

$$\boldsymbol{\xi}^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

1.1. **Examples of Fourier Transforms in Practice.** Our examples will be based upon ODEs, as the principle is the same but the calculations are much less perverse. The identities (F1)-(F4) will be the main points to remember.

Example 1.1. Suppose that $y(x)$ is a smooth function that satisfies the following ODE

$$y^{(4)} + 4y^{(2)} + y = f(x),$$

where $f(x)$ is some continuous function of compact support. Use the Fourier transform to compute $\hat{y}(\xi)$ (note that the expression will involve $\hat{f}(\xi)$).

Solution. Taking the Fourier transforms of both sides of the ODE, we deduce from that linearity of $\mathcal{F}$ that

$$\mathcal{F}\left[y^{(4)}\right] + 4\mathcal{F}\left[y^{(2)}\right] + \mathcal{F}[y] = \mathcal{F}[f].$$

Using (F4), it follows that

$$i^4\xi^4\hat{y}(\xi) + 4i^2\xi^2\hat{y}(\xi) + \hat{y}(\xi) = \hat{f}(\xi).$$

Therefore,

$$\hat{y}(\xi) = \frac{\hat{f}(\xi)}{\xi^4 - 4\xi^2 + 1}.$$

□

Example 1.2. Show that the Fourier transform of

$$f(x) := \frac{e^{-|x|}}{2} \quad \text{is} \quad \frac{1}{1+\xi^2}.$$

Solution. This is a straightforward calculation. First, observe that f is in $L^1(\mathbb{R})$ and the question makes sense. For every $\xi \in \mathbb{R}$ there holds

$$\begin{aligned}\hat{f}(\xi) = \int_{\mathbb{R}} \frac{e^{-|x|}}{2} e^{-i\xi x}\,\mathrm{d}x &= \int_{-\infty}^{0} \frac{e^{x}}{2} e^{-i\xi x}\,\mathrm{d}x + \int_{0}^{\infty} \frac{e^{-x}}{2} e^{-ix\xi}\,\mathrm{d}x \\ &= \int_{-\infty}^{0} \frac{e^{x-i\xi x}}{2}\,\mathrm{d}x + \int_{0}^{\infty} \frac{e^{-(x+i\xi x)}}{2}\,\mathrm{d}x \\ &= \frac{1}{2}\left[\int_{-\infty}^{0} e^{x(1-i\xi)}\,\mathrm{d}x + \int_{0}^{\infty} e^{-x(1+i\xi)}\,\mathrm{d}x\right].\end{aligned}$$

We now compute

$$\int_{-\infty}^{0} e^{x(1-i\xi)}\,\mathrm{d}x = \left.\frac{e^{x(1-i\xi)}}{(1-i\xi)}\right|_{-\infty}^{0} = \frac{1}{1-i\xi} - \lim_{x\to-\infty} \frac{e^{x(1-i\xi)}}{1-i\xi} = \frac{1}{1-i\xi}.$$

Here we have used the fact that

$$\left|\frac{e^{x(1-i\xi)}}{1-i\xi}\right| = \left|e^{x}\frac{e^{-xi\xi}}{1-i\xi}\right| = \left|\frac{e^{x}}{1-i\xi}\right| \cdot \underbrace{\left|e^{-ix\xi}\right|}_{\leq 1}$$

to deduce that

$$\lim_{x\to-\infty} \frac{e^{x(1-i\xi)}}{1-i\xi} = 0.$$

A similar calculation gives

$$\int_{0}^{\infty} e^{-x(1+i\xi)}\,\mathrm{d}x = \left.\frac{e^{-x(1+i\xi)}}{-(1+i\xi)}\right|_{0}^{\infty} = \frac{1}{1+i\xi}.$$

This means that

$$\hat{f}(\xi) = \frac{1}{2}\left(\frac{1}{1-i\xi} + \frac{1}{1+i\xi}\right) = \frac{2}{2(1-i\xi)(1+i\xi)} = \frac{1}{1+i\xi-i\xi-i^2\xi^2} = \frac{1}{1+\xi^2}.$$

□

Example 1.3. Using the Fourier transform, solve the ODE

$$y''(x) - y(x) = f(x).$$

For some given function $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$.

Solution. As in the previous example, we take the Fourier transform of both sides of the equation to obtain the following

$$i^2x^2\hat{y}(\xi) - \hat{y}(\xi) = \hat{f}(\xi).$$

Or, rather,

$$-\hat{y}(\xi)(\xi^2+1) = \hat{f}(\xi)$$

which means

$$\hat{f}(\xi) = -\frac{\hat{f}(\xi)}{1+\xi^2}.$$

Let $g(x) := \dfrac{e^{-|x|}}{2}$; by the previous example we may write the equation above as

$$\hat{y}(\xi) = -\hat{f}(\xi)\hat{g}(\xi) = -\widehat{f * g}(\xi).$$

Taking the Fourier inverse of both sides yields:

$$y(x) = \hat{y}^\vee = -\left(\widehat{f * g}(\xi)\right)^\vee = (f * g)(x) = \frac{1}{2}\int_{\mathbb{R}} f(y)\exp(|x-y|)\,\mathrm{d}y.$$

□

2. Fourier Series

The reader familiar with Taylor series has a good idea of what is about to come. The Fourier series is a method for "decomposing" a sufficiently into an infinite sum of trigonometric functions.

The problem goes as follows: suppose we are given a continuous function ϕ on some interval $(0, l)$, we wish to find coefficients b_n such that

$$\phi(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{l}\right), \quad \forall x \in (0, l).$$

We mostly skip the theory of Fourier series; it is a very deep subject that we cannot even begin to cover in these short notes. However, we can give the values of b_n such that the above decomposition holds true.

DEFINITION 2. Let ϕ be defined on the interval $(0,l)$. The Fourier sine series for ϕ, on $(0,l)$, is defined by

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{l}\right), \quad \forall x \in (0,l)$$

where for $n \in \mathbb{N}$ we define

$$b_n := \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{\pi n x}{l}\right)\,\mathrm{d}x.$$

The following "orthogonality identity" will also be very useful when solving problems:

$$\int_0^l \sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi m x}{l}\right)\,\mathrm{d}x = \begin{cases} 0, & \text{if } n \neq m, \\ l/2, & \text{if } n = m. \end{cases} \tag{2.1}$$

We also point out that one can proceed similarly with cosines in order to obtain a cosine series for ϕ. This is summarized below.

DEFINITION 3. Given a function ϕ on $(0,l)$, we define its cosine Fourier series to be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{l}\right), \quad \forall x \in (0,l)$$

where we put

$$a_n := \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{\pi n x}{l}\right)dx, \quad \forall n \in \mathbb{N}_0.$$

Our analogous orthogonality holds for $n \in \mathbb{N}_0$ and is given below

$$\int_0^l \cos\left(\frac{\pi n x}{l}\right)\cos\left(\frac{\pi m x}{l}\right)\,\mathrm{d}x = \begin{cases} 0, & \text{if } n \neq m, \\ l/2, & \text{if } n = m. \end{cases} \tag{2.2}$$

2.1. The Full Fourier Series.

Suppose that we are instead given a function ϕ defined on $(-l,l)$ for some $l > 0$. The full Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{\pi n x}{l}\right) + b_n \sin\left(\frac{\pi n x}{l}\right)\right]$$

where

$$a_n := \frac{1}{l}\int_{-l}^l \phi(x)\cos\left(\frac{\pi n x}{l}\right) \quad \text{and} \quad b_n := \frac{1}{l}\int_{-l}^l \phi(x)\sin\left(\frac{\pi n x}{l}\right).$$

For the a_n, we allow $n \geq 0$ whilst the b_n's are only defined for $n \geq 1$.

2.2. **Eigenfunctions and the Spectrum of ∂^2.** This subsection is devoted to the eigenvalue problem

$$\begin{cases} X''(x) + \lambda X(x) \equiv 0, & \text{in } (0,l), \\ X(0) = X(l) = 0. \end{cases} \tag{2.3}$$

We seek to see which non-trivial functions solve the boundary value problem above. That is, we find the spectrum and eigenspaces of the operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on an open bounded interval. We denote this operator by ∂^2, for the sake of simplicity.

Suppose that $\lambda = 0$. Then the ODE simply becomes the statement that $X(x) = ax + b$ for some $a, b \in \mathbb{R}$. However, since $X(0) = 0$ we recover $b = 0$. Furthermore, $X(l) = al = 0$ implies $a = 0$. Therefore, $X \equiv 0$ which means that $0 \notin \sigma(\partial^2)$.

The case of $\lambda < 0$ is fairly simple. Since we wish to solve $X''(x) = -\lambda X(x)$, with $-\lambda > 0$, this amounts to $X(x)$ being of the form

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

for some constants $A, B \in \mathbb{R}$. The condition $X(0) = 0$ implies $A + B = 0$ whilst we must have

$$A^{\sqrt{-\lambda}l} + Be^{-\sqrt{-\lambda}l} = 0$$

since $X(l) = 0$. Therefore, we necessarily have

$$A\left(e^{\sqrt{-\lambda}l} - e^{-\sqrt{-\lambda}l}\right) = 0.$$

Since e^x is injective, we find that $A = -B = 0$ whence $X(x) \equiv 0$ once again. Therefore, ∂^2 has no negative eigenvalues.[1]

So far we have nothing, but there is one last possibility: $\lambda > 0$ (and we will find non-trivial eigenfunctions). In this case the possible solutions are all of the form

$$X(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right)$$

for some constants $A, B \in \mathbb{R}$. Notice that $X(0) = 0$ forces $B = 0$. Since $X(l) = 0$ we have

$$X(l) = \sin\left(\sqrt{\lambda}l\right) = 0$$

which occurs if and only if

$$\sqrt{\lambda}l = \pi n, \quad n \in \mathbb{Z}.$$

This implies that λ is always of the form

$$\frac{\pi^2 n^2}{l^2}, \quad n \in \mathbb{N}.$$

[1]Those of you with some background in functional analysis will not be surprised by this fact.

This gives us a countable family of eigenfunctions and eigenvalues:

$$X_n(x) = \sin\left(\frac{\pi n x}{l}\right), \quad \lambda_n = \frac{\pi^2 n^2}{l^2}.$$

Similarly, we consider the following Neumann eigenvalue problem

$$\begin{cases} X''(x) + \lambda X(x) \equiv 0, & \text{in } (0, l), \\ X'(0) = X'(l) = 0. \end{cases} \tag{2.4}$$

The case $\lambda = 0$ gives $X(x) = ax + b$. But, $X'(l)$ forces $a = 0$. Therefore, $X(x)$ is a constant function. If $\lambda < 0$ we once again find

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

for some constants A and B. Since $X'(0) = 0 = X'(l)$, we recover

$$\sqrt{-\lambda}A - B\sqrt{-\lambda} = 0,$$
$$\sqrt{-\lambda}Ae^{\sqrt{\lambda}l} - B\sqrt{-\lambda}e^{-\sqrt{\lambda}l} = 0.$$

The first equation gives $A = B$ whilst the second implies that

$$A = B = 0.$$

Hence, this problem has no negative eigenvalues. As in the case of (2.1), we proceed with the more interesting case of $\lambda > 0$. The general solution to (2.2) in this case will be of the form

$$X(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right).$$

Now, $X'(0) = 0$ means that

$$0 = A\sqrt{\lambda}\cos(0) - B\sqrt{\lambda}\sin(0)$$

whence $A = 0$. Since $X'(l) =$ we must have

$$\sin(\lambda l) = 0$$

which give us the same eigenvalues as before. Although we also allow for the case $n = 0$ as this gives us a non-trivial eigenfunction (any constant function). Therefore,

$$X_n(x) = \cos\left(\frac{\pi n x}{l}\right), \quad \lambda_n = \left(\frac{\pi n}{l}\right)^2$$

These properties will soon come in handy.

2.3. **Convergence Properties of Fourier Series.** We have defined multiple Fourier series, but we have not yet considered their behaviour relative to their generating function ϕ. Given a function $f : [-L, L] \to \mathbb{R}$, we say that f is piecewise smooth if f is bounded and there exists a finite set $\Xi \subset [-L, L]$ such that f is smooth on $[-L, L] \setminus \Xi$. If f is has a discontinuity point at x we also assume that both limits

$$f(x^+) := \lim_{\varepsilon \searrow 0} f(x+\varepsilon) \quad \text{and} \quad f(x^-) := \lim_{\varepsilon \searrow 0} f(x-\varepsilon)$$

exists and are finite.

THEOREM (Pointwise Convergence Theorem). *Let f be piecewise smooth on a non-trivial interval $(-l, l)$.*

(1) The full Fourier series of f will converge pointwise, to the periodic extension of $f(x)$, at all points where the periodic extension of $f(x)$ is continuous.
(2) More generally, if the periodic extension of f is discontinuous at a point x, then the full Fourier series of f converges pointwise to

$$\frac{f(x^+) + f(x^-)}{2}.$$

A stronger result holds when we assume that ϕ is nicer.

THEOREM (Uniform Convergence). *Suppose that ϕ is continuous and piecewise C^1 on $(-l, l)$ with $\phi(-l) = \phi(l)$. Then the classical full Fourier series converges uniformly and absolutely to the periodic extension of ϕ on all of $\mathbb{R}$.*

We conclude with the statement of Parseval's identity.

THEOREM (Parseval). *Let $\phi \in L^2((a, b))$ and let $\{X_n\}_n$ and A_n be respectively the eigenfunctions and Fourier coefficients for any Fourier series of ϕ. Then,*

$$\sum_n A_n^2 \int_a^b X_n(x)^2 \, \mathrm{d}x = \int_a^b \phi(x)^2 \, \mathrm{d}x$$

where

$$A_n = \frac{\langle \phi, X_n \rangle}{\|X_n\|^2}.$$

Here $\langle \cdot, * \rangle$ is the standard $L^2(X)$ inner product and $\|\cdot\|$ is the $L^2(X)$ norm.

2.4. **Examples.** Fourier series are useful in obtaining numerical solutions to PDE. However, they are also of interest in their own right. One can use Fourier series to establish some surprising facts such as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example 2.1. Define a function $\phi : [0, 4] \to \mathbb{C}$ by

$$\phi(x) := \begin{cases} 1, & 0 < x \leq 1, \\ 2, & 1 < x \leq 2, \\ 3, & 2 < x \leq 3, \\ 4, & 3 < x \leq 4. \end{cases}$$

Extend now ϕ periodically to $\mathbb{R}$, i.e. set $\phi(x+4) := \phi(x)$ and consider the full Fourier series for ϕ:

$$\frac{a_0}{2} + \sum_{n \in \mathbb{N}} \left[a_n \cos\left(\frac{\pi n x}{4}\right) + b_n \sin\left(\frac{\pi n x}{4}\right) \right].$$

(1) To what values will the Fourier series converge to at $x = 0, 1, 4, 7.4$ and 40?
(2) Does the Fourier series converge uniformly to ϕ?
(3) Compute a_0.

Solution. Clearly, $\phi(0) = 4$ but $\phi(0^-) = 4$ and $\phi(0^+) = 1$. Therefore, the full Fourier series will converge to

$$\frac{\phi(0^+) + \phi(0^-)}{2} = \frac{5}{2} \quad \text{at } x = 0.$$

At $x = 1$, we have $\phi(1^+) = 2$ but $\phi(1^-) = 1$ whence the Fourier series converges (pointwise) to

$$\frac{\phi(1^+) + \phi(1^-)}{2} = \frac{3}{2} \quad \text{at } x = 1.$$

Just as for $x = 0$, the Fourier series will converge to $5/2$ at $x = 4$. Since 7.4 lies in the interior of the interval $(7, 8)$, where ϕ is continuous and identical to 4, we conclude that the Fourier series converges to 4 at $x = 7.4$. Finally, at $x = 40$, we have the same case as at $x = 0$ (by periodicity).

As for (2), we argue by contradiction. If the Fourier series converges uniformly to ϕ, then it converges pointwise to $\phi(x)$ at every point x, which we know to be false by the first part.

For (3) we compute the value of a_0 directly. That is,

$$a_0 = \frac{1}{4} \int_{-4}^{4} \phi(x)\, \mathrm{d}x = \frac{1}{2} \int_{0}^{4} \phi(x)\, \mathrm{d}x = \frac{1+2+3+4}{2} = 5.$$

□

Perhaps a deterministic example would be helpful. We begin with a useful lemma.

LEMMA 1. *Let $f : (-l, l) \to \mathbb{R}$ be an odd function, i.e. $f(-x) = -f(x)$. Then the full Fourier series of f is precisely the sine Fourier series.*

Proof. It suffices to check that $a_n = 0$ for all $n \geq 0$. If $n = 0$ then $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,\mathrm{d}x$ where

$$\int_{-l}^{l} f(x)\,\mathrm{d}x = \int_{-l}^{0} f(x)\,\mathrm{d}x + \int_{0}^{l} f(x)\,\mathrm{d}x = \int_{0}^{l} f(-x)\,\mathrm{d}x + \int_{0}^{l} f(x)\,\mathrm{d}x = 0.$$

Thus, $a_0 = 0$. If $n \geq 1$ then,

$$\begin{aligned}\int_{-l}^{l} f(x)\cos\left(\frac{\pi n x}{l}\right)\,\mathrm{d}x &= \int_{-l}^{0} f(x)\cos\left(\frac{\pi n x}{l}\right)\,\mathrm{d}x + \int_{0}^{l} f(x)\cos\left(\frac{\pi n x}{l}\right)\,\mathrm{d}x \\ &= \int_{-l}^{0} f(x)\cos\left(\frac{\pi n x}{l}\right)\,\mathrm{d}x + \int_{0}^{l} f(x)\cos\left(\frac{\pi n x}{l}\right)\,\mathrm{d}x \\ &= \int_{0}^{l} -f(x)\cos\left(\frac{\pi n x}{l}\right)\,\mathrm{d}x + \int_{0}^{l} f(x)\cos\left(\frac{\pi n x}{l}\right)\,\mathrm{d}x\end{aligned}$$

which vanishes. □

For the sake of consistency:

LEMMA 2. *Let $f : (-l, l) \to \mathbb{R}$ be an even function. The full Fourier series is the cosine series for f.*

Proof. Argue as in the previous lemma. □

Example 2.2. Define $\phi : (-1, 1) \to \mathbb{R}$ by $\phi(x) = x$. Determine the full Fourier series

Solution. Since ϕ is odd, we need only compute the Fourier sine series for ϕ. The series we seek looks like

$$\sum_{n=1}^{\infty} b_n \sin(\pi n x), \quad n \in \mathbb{N}.$$

Now, if $n \in \mathbb{N}$ is given we find that

$$\begin{aligned}b_n = \int_{-1}^{1} \phi(x)\sin(\pi n x)\,\mathrm{d}x &= \int_{-1}^{1} x\sin(\pi n x)\,\mathrm{d}x \\ &= -\left.\frac{x\cos(\pi n x)}{\pi n}\right|_{-1}^{1} + \frac{1}{\pi n}\int_{-1}^{1}\cos(\pi n x)\,\mathrm{d}x\end{aligned}$$

where

$$\int_{-1}^{1}\cos(\pi n x)\,\mathrm{d}x = \left.\frac{\sin(\pi n x)}{\pi n}\right|_{-1}^{1} = 0.$$

Therefore,

$$b_n = -\left.\frac{x\cos(\pi n x)}{\pi n}\right|_{-1}^{1} = -\frac{\cos(\pi n) + \cos(\pi n)}{\pi n} = -\frac{2(-1)^n}{\pi n} = \frac{2(-1)^{n+1}}{\pi n}.$$

It follows that the Fourier series we seek is given by

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \sin(\pi n x).$$

□

Example 2.3. Using Parseval's identity, show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof. Consider the function $f(x) = x$ on the interval $[0,1]$. We will make use of Parseval's identity. Since $f(x)$ is an odd function, it is wise to use the family $\{X_n(x)\}_{n=1}^{\infty} = \{\sin(\pi n x)\}_{n=1}^{\infty}$.

For $n \geq 1$, integration by parts will give

$$\langle X_n, X_n \rangle = \int_0^1 \sin^2(\pi n x)\,\mathrm{d}x = \frac{1}{2}.$$

Furthermore,

$$\begin{aligned}
\langle f, X_n \rangle = \int_0^1 x \sin(\pi n x)\,\mathrm{d}x &= -\frac{x\cos(\pi n x)}{\pi n}\Big|_{x=0}^{x=1} + \frac{1}{\pi n}\int_0^1 \cos(\pi n x)\,\mathrm{d}x \\
&= \frac{(-1)^{n+1}}{\pi n} + \frac{\sin(\pi n x)}{(\pi n)^2}\Big|_{x=0}^{x=1} \\
&= \frac{(-1)^{n+1}}{\pi n}.
\end{aligned}$$

This means that, for each $n \geq 1$, one has

$$A_n^2 = \frac{1}{\pi^2 n^2} \cdot \frac{1}{1/2^2} = \frac{4}{\pi^2 n^2}. \tag{2.5}$$

An easy computation also gives

$$\int_0^1 f(x)^2\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}.$$

This together with Parseval's identity and (2.5) yields

$$\frac{1}{3} = \sum_{n=1}^{\infty} A_n^2 \int_0^1 X_n(x)^2\,\mathrm{d}x = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cdot \frac{1}{2} = \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2}.$$

A rearrangement then implies that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{2} \cdot \frac{1}{3} = \frac{\pi^2}{6}.$$

□

Remark. By following the procedures used in the two examples given here, one can 'easily' compute $\zeta(2k)$, for any $k \in \mathbb{N}$, where $\zeta(s)$ is the Riemann zeta function. However, we do not know the explicit value of $\zeta(2k+1)$ for any $k \geq 1$.

2.5. **Separation of Variables.** The separation of variables algorithm is a procedure for solving PDE that makes use of Fourier series and the assumption that a solution $u(x,t)$ to a PDE in two variables is the product of two single variable functions. That is, we use the assumption $u(x,t) = X(x)T(t)$ for two functions X and T to reduce the PDE to a system of eigenvalue problems that we can solve using Fourier series. This is best demonstrated rather than explained.

Example 2.4. Let $\ell > 0$. Use separation of variables and Fourier series to solve the Schrödinger equation

$$\begin{cases} u_t = iu_{xx}, & (x,t) \in (0,\ell) \times (0,\infty), \\ u(0,t) = 0 = u(\ell,t), & t > 0, \\ u(x,0) = f(x), & x \in (0,\ell). \end{cases} \tag{2.6}$$

Solution. Suppose that $u(x,t)$ solves the problem above and assume that one may write $u(x,t) = X(x)T(t)$ for two functions X and T of a single variable. Then,

$$T'(t)X(x) = u_t(x,t) = iu_{xx}(x,t) = iX''(x)T(t), \quad (x,t) \in (0,\ell) \times (0,\infty).$$

This may be written as

$$\frac{T'(t)}{iT(t)} \equiv \frac{X''(x)}{X(x)}.$$

Since the left hand side of the above depends only on t, and the right side depends only on x, we conclude that the above expression is constant. It therefore reduces to the two problems

$$X'' + \lambda X \equiv 0, \tag{2.7}$$

$$T'(t) + i\lambda T(t) \equiv 0. \tag{2.8}$$

For any λ, solving (2.8) is no problem. Indeed, for any $\lambda \in \mathbb{R}$ we know that a solution $T(t)$ is

$$T(t) = e^{-\lambda i t}.$$

Notice that, to avoid the trivial solutions, the boundary condition

$$0 = X(0)T(t) = X(\ell)T(t), \quad \forall t > 0$$

makes (2.7) into an eigenvalue problem that we have studied before. We know that a basis for the solution space is

$$X_n(x) = \sin\left(\frac{\pi n x}{\ell}\right), \quad \lambda_n := \left(\frac{\pi n}{\ell}\right)^2.$$

For a fixed index n we may also take

$$T_n(t) = \exp\left(\frac{\pi^2 n^2 i t}{\ell^2}\right).$$

Hence, each $u_n(x,t) = X_n(x)T_n(t)$ will satisfy

$$u_t = i u_{xx} \equiv 0, \quad u(0,t) = u(\ell,t) = 0.$$

Therefore, an infinite sum of the form

$$u(x,t) := \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{\ell}\right) \exp\left(\frac{\pi^2 n^2 i t}{\ell^2}\right)$$

will also satisfy this system. Notice also that

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{\ell}\right)$$

and by choosing b_n to be the Fourier sine coefficients

$$b_n := \frac{2}{\ell}\int_0^{\ell} f(x) \sin\left(\frac{\pi n x}{\ell}\right)\, \mathrm{d}x, \quad n \in \mathbb{N},$$

we can make it so that $u(x,0) = f(x)$ for $x \in (0,\ell)$. □

Example 2.5. Using separation of variables, solve the Schrödinger with Neumann boundary conditions:

$$\begin{cases} u_t = i u_{xx}, & (x,t) \in (0,\ell) \times (0,\infty), \\ u_x(0,t) = 0 = u_x(\ell,t), & t > 0, \\ u(x,0) = f(x), & x \in (0,\ell) \end{cases} \tag{2.9}$$

with $\ell > 0$ fixed.

Solution. The procedure is just as in the previous example: we seek an infinite family of solutions that are orthogonal and form a basis for a solution space. Assume that $u(x,t)$ can be written as a product $X(x)T(t)$, for two functions of a single variable X and T. Then,

$$X(x)T'(t) = u_t = i u_{xx} = i X''(x) T(x).$$

Once again, this means that

$$\frac{T'(t)}{iT(t)} \equiv \frac{X''(x)}{X(x)}$$

is a constant. This reduces our PDE system to two eigenvalue problems

$$T'(t) + i\lambda T(t) \equiv 0, \tag{2.10}$$

$$X''(x) + \lambda X(x) \equiv 0. \tag{2.11}$$

This first equation is trivial to solve in general, and we focus on the second. Now, we know that

$$u_x(x,t) = X'(x)T(t)$$

so that the boundary conditions imply

$$X'(0) = X'(\ell) = 0.$$

This is once again a problem we have already solved! We know that one may choose for $n \in \mathbb{N}_0$ the following eigenvalues and eigenfunctions:

$$X_n(x) := \cos\left(\frac{\pi n x}{\ell}\right), \quad \lambda_n := \left(\frac{\pi n}{l}\right)^2.$$

For any such $n \in \mathbb{N}_0$, we solve for an associated $T_n(t)$ directly. We have the condition

$$T_n'(t) = -\lambda_n i T_n(t)$$

which implies that

$$T_n(t) = C_n \exp\left(-i\frac{\pi^2 n^2}{l^2}t\right).$$

For the sake of simplicity, we take $C_n = 1$ in the above. Hence, to each $n \in \mathbb{N}_0$ we associate a solution

$$u_n(x,t) := \cos\left(\frac{\pi n x}{\ell}\right)\exp\left(-i\frac{\pi^2 n^2}{l^2}t\right).$$

Observe that for any coefficients $\{b_n\}_{n=0}^\infty$ in $\mathbb{R}$, the function

$$u(x,t) := \sum_{n=0}^\infty b_n u_n(x,t)$$

will satisfy

$$u_t = iu_{xx} \qquad u_x(0,t) = u_x(\ell,t) \equiv 0,\ t > 0.$$

However, in taking $t = 0$ we recover

$$u(x,0) = \sum_{n=0}^\infty b_n \cos\left(\frac{\pi n x}{\ell}\right),$$

and we want this to evaluate to $f(x)$. This is simple, as we only need to choose the b_n to the coefficients for the cosine series of f. □

2.6. **Other Boundary Problems.** This method of using Fourier series is useful in solving various types of boundary value problems, and not only those of the Dirichlet and Neumann type. We summarize the strategy for various types of equations below.

$X(x)$ is a solution to $X''(x)+\lambda X(x) = 0$ on $(0, \ell)$. We list eigenfunctions and eigenvalues for particular boundary conditions.

- *Dirichlet Conditions*: $X(0) = 0 = X(\ell)$

$$X_n(x) = \sin\left(\frac{\pi n x}{\ell}\right), \quad \lambda_n := \left(\frac{\pi n}{\ell}\right)^2, \quad n \in \mathbb{N}.$$

  These correspond to the components of the Fourier sine series on $(0, \ell)$.
- *Neumann Conditions*: $X'(0) = 0 = X'(\ell)$:

$$X_n(x) = \cos\left(\frac{\pi n x}{\ell}\right), \quad \lambda_n := \left(\frac{\pi n}{\ell}\right)^2, \quad n \in \mathbb{N}_0.$$

  These correspond to the components of the Fourier cosine series on $(0, \ell)$.
- *Periodic Conditions*: $X(-\ell) = X(\ell)$ and $X'(-\ell) = X'(\ell)$:

$$X_n(x) = \cos\left(\frac{\pi n x}{\ell}\right), \quad \lambda_n := \left(\frac{\pi n}{\ell}\right)^2, \quad n \in \mathbb{N}_0$$

  and

$$Y_n(x) = \sin\left(\frac{\pi n x}{\ell}\right), \quad n \in \mathbb{N}.$$

  These components correspond to those of the full Fourier series on $(-\ell, \ell)$.

3. Wave Equations

We now turn our attention towards a different PDE that is ever-present in physics: the wave equation. We focus on the spatial dimensions 1 and 3. In one dimension, the wave equation is given by

$$u_t \equiv c^2 u_{xx},$$

where u is a C^2 function of two variables x and t. Here $c \neq 0$ is real and is modeled by the ratio T/ρ, where T is the uniform tension in a string and ρ is the volume density. In three dimensions, this generalizes to

$$u_t \equiv c^2 \Delta u.$$

where Δ is the 3-dimensional Laplacian. Recall that in $\mathbb{R}^n$ the Laplacian Δ is the linear operator described by

$$\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

It will often be convenient to denote $\frac{\partial}{\partial x_j}$ by ∂_j and the reader should keep track of this notation. The Laplacian will also play a crucial role in the next *part* of this text. We begin by considering the 1 dimensional wave equation and its corresponding initial value problems. Throughout this section you may assume all functions are smooth (or, say, C^2).

3.1. **The One-Dimensional Variant.** Here we are interested in solutions to the one dimensional wave equation

$$u_t \equiv c^2 u_{xx}, \quad c \neq 0. \tag{3.1}$$

The associated initial value problem goes as

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R}, t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{R}, \\ u_t(x,0) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{3.2}$$

THEOREM (d'Alembert). *Let u be a solution to (3.1). Then*

(1) There exist functions f and g such that

$$u(x,t) = f(x+ct) + g(x-ct).$$

(2) If u satisfies the IVP (3.2) then

$$u(x,t) = \frac{\phi(x+ct) - \phi(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\tau) \, \mathrm{d}\tau.$$

The formula above continues to hold (to a certain degree) when we introduce a "source term" to the IVP (3.2). Consider the IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} \equiv f(x,t), & x \in \mathbb{R}, t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{R}, \\ u_t(x,0) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{3.3}$$

Then we have the following analogue of Theorem 3.1.

THEOREM. *If u solves (3.3) then*

$$u(x,t) = \frac{\phi(x+ct) - \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\tau)\,\mathrm{d}\tau + \frac{1}{2c}\int_{\mathfrak{D}} f(y,s)\,\mathrm{d}A.$$

Here $\mathfrak{D}$ is the domain of dependence associated with (x,t).

The domain of dependence is the triangle in the (x,t)-plane with coordinates

$$\{(x-ct,0),(x+ct,0),(x,t)\}.$$

This also extends to problems involving *semi-infinite* strings. These are fixed at one end (traditionally, at $x = 0$). The corresponding problem for a semi-infinite string is given below

$$\begin{cases} v_{tt} \equiv c^2 v_{xx}, & x \geq 0, t > 0, \\ v(x,0) = \phi(x), & x \geq 0, \\ v_t(x,0) = \psi(x), & x \geq 0, \\ v(0,t) = 0, & t \geq 0. \end{cases} \tag{3.4}$$

THEOREM. *If v satisfies (3.4) then*

$$u(x,t) = \frac{1}{2}\begin{cases} \phi(x+ct) + \phi(x-ct) + \dfrac{1}{c}\displaystyle\int_{x-ct}^{x+ct} \psi(\tau)\,\mathrm{d}\tau, & x \geq ct, \\ \phi(x+ct) - \phi(ct-x) + \dfrac{1}{c}\displaystyle\int_{ct-x}^{x+ct} \psi(\tau)\,\mathrm{d}\tau, & 0 < x \leq ct. \end{cases}$$

To help drill all of this in, we now provide some examples of solved exercises.

3.2. Examples for the 1D Wave Equation.

Example 3.1. Let $\mathbf{E}(x,y,z,t)$ and $\mathbf{B}(x,y,z,t)$ be smooth electric and magnetic fields, respectively. They are governed by Maxwell's equations:

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \times \mathbf{B} = \mu\varepsilon\partial_t E, \quad \nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0.$$

in the above μ and ε are constants (that you likely encountered in physics). Show that if

$$\mathbf{E}(x,y,z,t) = (0,0,E(x,t)) \quad \text{and} \quad \mathbf{B}(x,y,z,t) = (0,B(x,t),0)$$

then both $E(x,t)$ and $B(x,t)$ satisfy the wave equation:

$$\partial_{tt} u - c^2 \partial_{xx} u \equiv 0, \quad c = (\mu\varepsilon)^{-1/2}.$$

Solution. We first relate $B(x,t)$ to $E(x,t)$ using Maxwell's equations. Observe that

$$(0, -B_t(x,t), 0) = -\partial_t \mathbf{B}(x,y,z,t) = \nabla \times \mathbf{E}(x,y,z,t) = (0, -E_x(x,t), 0).$$

Thus, $\boxed{B_t(x,t) = E_x(x,t)}$. In like,

$$\mu\varepsilon(0,0,E_t(x,t)) = \nabla \times \mathbf{B}(x,y,z,t) = (0,0,B_x(x,t)).$$

Thus, $\boxed{\mu\varepsilon E_t(x,t)} = B_x(x,t)$. Finally,

$$E_{xx}(x,t) = \partial_x B_t(x,t) = B_{tx}(x,t) = \partial_t B_x(x,t) = \mu\varepsilon\partial E_t(x,t) = \mu\varepsilon E_{tt}(x,t).$$

Similarly, $B(x,t)$ satisfies the wave equation. ☐

Example 3.2. Fix a time $t > 0$ and assume $\phi_{1,2}, \psi_{1,2}$ are bounded functions defined on $\mathbb{R}$. Let u_i, for $i = 1,2$, denote the solution to

$$\partial_{tt}u_i - c^2\partial_{xx}u \equiv 0, \quad u_i(x,0) = \phi_i(x), \quad \partial_t u_i(x,0) = \psi(x).$$

Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\phi_1 - \phi_2\| < \delta \text{ and } \|\phi_1 - \phi_2\| < \delta \implies \|u_1 - u_2\|_\infty < \varepsilon.$$

Solution. Applying d'Albembert's formula, we find that

$$u_1(x,t) = \frac{1}{2}\left(\phi_1(x+ct) + \phi_1(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_1(\sigma)\,\mathrm{d}\sigma$$
$$u_2(x,t) = \frac{1}{2}\left(\phi_2(x+ct) + \phi_2(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_2(\sigma)\,\mathrm{d}\sigma.$$

This implies, in particular, that for $t > 0$ fixed:

$$|u_1(x,t) - u_2(x,t)| \leq \frac{|\phi_1(x+ct) - \phi_2(x+ct)| + |\phi_1(x-ct) - \phi_2(x-ct)|}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} |\psi_1(\sigma) - \psi_2(\sigma)|\,\mathrm{d}\sigma.$$

Especially,

$$|u_1(x,t) - u_2(x,t)| \leq \|\phi_1 - \phi_2\|_\infty + \frac{1}{2c}\int_{x-ct}^{x+ct} \|\psi_1 - \psi_2\|_\infty\,\mathrm{d}\sigma.$$

Fix $\varepsilon > 0$ and take $\delta > 0$ such that $\delta(1+t) < \varepsilon$. Then, for each x:

$$|u_1(x,t) - u_2(x,t)| \leq \delta + \frac{\delta}{2c}\cdot 2ct = \delta(1+t) < \varepsilon.$$

Take now the supremum over $x \in \mathbb{R}$; this yields

$$\|u_1 - u_2\|_\infty \leq \delta(1+t) < \varepsilon.$$

☐

Example 3.3. Consider a vibrating infinite string with initial disturbance at $t = 0$ in the intervals $[1, 2]$ and $[4, 5]$. At $t = 10$, at which positions will one feel these disturbances?

Solution. Applying d'Alembert's formula with $t = 10$ shows that we wish to know for which x:

$$[x - 10, x + 10] \cap ([1, 2] \cap [4, 5]) \neq \varnothing.$$

First, note that $[x - 10, x + 10]$ intersects $[1, 2]$ if and only if

$$x + 10 \geq 1 \quad \text{and} \quad x - 10 \leq 2.$$

That is, if and only if $x \in [-9, 12]$. Similarly, $[x - 10, x + 10]$ intersects $[4, 5]$ if and only if $x \in [-6, 15]$. Hence, we feel the disturbance for $x \in [-9, 15]$. □

Example 3.4. Consider the 1D wave equation $u_t t = u_x x$ for $(x, t) \in \mathbb{R} \times (0, \infty)$ together with the initial constraints

$$\phi(x) = 0, \quad u_t(x, 0) = \psi(x)$$

where

$$\psi(x) = \begin{cases} 1, & \text{if } |x - 3| \leq 1 \text{ or } |x + 3| \leq 1, \\ 0, & \text{else.} \end{cases}$$

(i) At the time $t = 1$ where on the string is the displacement non-zero?
(ii) At time $t = 10$ which points have maximal displacement?
(iii) Compute $u(0, t)$.

Solution. Before we proceed, we note that by d'Alembert's formula any solution is of the form

$$u(x, t) = \frac{1}{2} \int_{x-t}^{x+t} \psi(\sigma) \, d\sigma.$$

(i). At time $t = 1$ we obtain $u(x, 1) = \frac{1}{2} \int_{x-t}^{x+t} \psi(\sigma) \, d\sigma$ which is non-zero if and only if

$$(x - 1, x + 1) \cap \{x : |x - 3| \leq 1 \text{ or } |x + 3| \leq 1\} \neq \varnothing.$$

Note that $|x - 3| \leq 1$ if and only if$-1 \leq x - 3 \leq 1$ which is equivalent to saying $2 \leq x \leq 4$. Similarly, $|x + 3| \leq 1$ if and only if $-4 \leq x \leq -2$. Therefore, we seek points x such that

$$(x - 1, x + 1) \cap [[-4, -2] \sqcup [2, 4]] \neq \varnothing.$$

Clearly, $(x - 1, x + 1) \cap (-4, -2)$ is non-empty if and only if $x - 1 < -2$ and $x + 1 > -4$. That is, if and only if $x \in (-5, -1)$. In like, $(x - 1, x + 1)$ intersects $(2, 4)$ if and only if $x - 1 < 4$ and $x + 1 > 2$. Thus, if and only if $x \in (1, 5)$.

(ii) The solution $u(x, 10)$ will have maximal displacement whenever $(x-10, x+10)$ covers $[[-4,-2] \sqcup [2,4]]$. This will occur if and only if $x - 10 \leq -4$ and $x + 10 \geq 4$. Thus, if and only if $x \in [-6, 6]$.

(iii). The "difficulty" lies in computing $\int_{-t}^{t} \psi(\sigma)\,\mathrm{d}\sigma$. If $t \leq 2$ then clearly $(-t, t)$ does not intersect $(-4,-2) \sqcup (2,4)$ which implies that $u(0,t) = 0$. On the other-hand, if $t \geq 4$ then $(-t, t)$ engulfs $(-4,-2) \sqcup (2,4)$ which would imply that

$$\frac{1}{2}\int_{-t}^{t} \psi(\sigma)\,\mathrm{d}\sigma = \frac{2+2}{2} = 2.$$

Suppose now that $t \in (2,4)$. Then,

$$u(0,t) = \frac{t - 2 + (-2+t)}{2} = t - 2.$$

That is,

$$u(0,t) = \begin{cases} 0, & \text{if } t \leq 2, \\ t-2, & \text{if } 2 < t < 4, \\ 2, & \text{if } t \geq 4. \end{cases}$$

□

3.3. **The 3-Dimensional Wave Equation.**

This section mostly lists results for functions $u(\mathbf{x}, t)$, with $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$, that satisfy the 3D wave equation

$$u_{tt} = c^2 \Delta u \tag{3.5}$$

and the corresponding initial value problem

$$\begin{cases} u_{tt} \equiv c^2 u_{xx}, & \mathbf{x} \in \mathbb{R},^3\ t > 0, \\ u(\mathbf{x}, 0) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3, \\ u_t(\mathbf{x}, 0) = \psi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases} \tag{3.6}$$

As we shall soon see, solution to (3.2) behave differently than those to (3.1).

THEOREM (Kirchoff). *Let u solve* (3.6). *Then*

$$u(\mathbf{x}, t) = \frac{1}{4\pi c^2 t^2} \iint_{\partial B(\mathbf{x}, t)} [\phi(\mathbf{y}) + (\mathbf{y} - \mathbf{x}) \cdot \nabla\phi(\mathbf{y}) + t\psi(\mathbf{y})]\ \mathrm{d}\mathbf{S}_{\mathbf{y}}.$$

We illustrate this with some examples.

Example 3.5. Assume a phenomenon propagates in 3-dimensional space according to the equation $u_{tt} - \Delta u \equiv 0$. Suppose a disturbance at $x = 0$ creates an initial change in velocity according to

$$\psi(\mathbf{x}) = \begin{cases} 1, & \text{if } |\mathbf{x}| \leq 1, \\ 0, & \text{else.} \end{cases}.$$

Evaluate $u(\mathbf{0}, t)$ for $t \geq 0$

Solution. Kirchoff's formula yields that for $(x, t) \in \mathbb{R} \times (0, \infty)$:

$$u(\mathbf{0}, t) = \frac{1}{4\pi t} \iint_{\partial B(\mathbf{0},t)} \psi(\mathbf{y})\, \mathrm{d}\mathbf{S} = \frac{1}{4\pi t} \iint_{\partial B(\mathbf{0},t) \cap \overline{B(\mathbf{0},1)}} \mathrm{d}\mathbf{S}$$

Thus, since $\partial(\mathbf{0}, t)$ does not intersect $\overline{B(\mathbf{0}, 1)}$ for $t > 1$, it follows that $u(\mathbf{0}, t)$ vanishes for $t > 1$. Otherwise, if $0 < t < 1$:

$$u(\mathbf{0}, t) = \frac{4\pi t^2}{4\pi t} = t.$$

□

Example 3.6. Suppose that the propagation due to a pressure disturbance in 3D is modeled by the 3D wave equation: $u_{tt} = \Delta u$. At time $t = 0$ an explosion occurs at position $\mathbf{x} = 0$ inducing the initial conditions:

$$\phi(\mathbf{x}) \equiv 0, \quad \psi(\mathbf{x}) = \mathbb{1}_{|\mathbf{x}| \leq 1}(\mathbf{x}).$$

(i) What is the value of $u(\mathbf{x}, 10)$ where $\mathbf{x} = (10, 0, 0)$?
(ii) At $t = 10$, what is the value of u at the point $\mathbf{x} = (20, 8, 17)$?
(iii) At what times will $u((20, 20, 20,), t)$ be non-zero?

Solution. Before we proceed, we apply Kirchoff's formula to obtain an explicit representation of u:

$$u(\mathbf{x}, t) = \frac{1}{4\pi t} \iint_{\partial B(\mathbf{x},t)} \psi(\mathbf{y})\, \mathrm{d}\mathbf{S}.$$

(i). Directly taking $\mathbf{x} = (10, 0, 0)$ and $t = 10$:

$$(\mathbf{x}, t) \mapsto \frac{1}{40\pi} \int_{\partial B((10,0,0),10) \cap \overline{B(\mathbf{0},1)}} \mathrm{d}\mathbf{S}$$

(ii). We shall show that $|\mathbf{y} - \mathbf{x}| = 10$ implies that $|\mathbf{y}| > 1$. This will imply that u evaluates to zero at $(\mathbf{x}, 10)$. Indeed,

$$|\mathbf{x}| - |\mathbf{y}| \leq 10 \implies |\mathbf{y}| > 1.$$

(iii). It is easier to determine at which points the displacement vanishes. This is equivalent to saying that

$$|\mathbf{y} - \mathbf{x}| = t \implies |\mathbf{y}| \geq 1.$$

This can occur only if

$$0 \leq t <\leq \sqrt{1200} - 1, \quad t \geq \sqrt{1200} + 1.$$

□

4. Method of Characteristics

The method of characteristics is a computational approach to solving transport-type PDE in $\mathbb{R}^n$. This is difficult to explain, and we instead provide a simple (but detailed) example as an introduction for the reader.

Example 4.1. Determine a general solution $u(x, y)$ to the PDE

$$au_x + bu_y \equiv, \quad (x, y) \in \mathbb{R}^2$$

where $a, b \in \mathbb{R} \setminus \{0\}$.

Solution. Notice that the PDE may alternatively be written as $(a, b) \cdot \nabla u \equiv 0$. This is precisely the statement that any solution u is constant in the direction (a, b)! Alternatively, this means the value of u is constant along any curve parametrized by

$$(x(s), y(s))$$

such that $\dot{x}(s) = a$ and $\dot{y}(s) = b$. All such curves are of the form

$$(x(s), y(s)) = (as + c_1, bs + c_2)$$

for some constants c_1 and c_2. Now, any such curve can also be written as

$$y(x) = \frac{b}{a}x + C.$$

Now, any solution u will be constant along any such curve. Furthermore, the value of u at a point (x, y) depends only on which of these curves the point (x, y) is. Since such curves are determined completely and uniquely by the trailing constant C it follows that

$$u(x, y) = f(bx - ay)$$

for some function f of a single variable. □

Of course, there is a more algorithmic approach to solving such equations that does not rely on geometric intuition. We illustrate this below in several examples.

Example 4.2. Use the method of characteristics to solve

$$u_t + uu_x \equiv 1, \quad u(x, 0) = x, \text{ in } t \geq 0.$$

Solution. As in the previous problem, we first find curves along which we have information. Define

$$\dot{t}(s) = 1, \quad \dot{x}(s) = z(s)$$

where $z(s) = u(x(s), t(s))$ in terms of some dummy variable s. Then,

$$\dot{z}(s) = u_t \dot{t}(s) + u_x \dot{x}(s) \equiv 1.$$

Therefore, $z(s) = s + A$ for some constant A. Choose $t(s) = s$ so that $t(0) = 0$. Then, we necessarily have

$$x(s) = \frac{s^2}{2} + As + B.$$

Now,

$$z(0) = A = u(x(0), t(0)) = x(0) = B.$$

Thus,

$$x(s) = x(t) = \frac{t^2}{2} + A(t+1).$$

Now, $z(s) = u(x(s), t) = t + A$. The above, however, states that

$$\frac{2x - t^2}{2(t+1)} = A$$

and thus

$$u(x,t) = t + \frac{2x - t^2}{2(t+1)}.$$

□

Part 2. Qualitative Behaviour of Solutions

Throughout this part of the text, we focus on properties governing the behaviour of solutions to certain simple PDE. Also, this part will make heavy use of mathematical analysis and will also include derivations of PDE. We begin considering random walks and their relationship with diffusion equations. Following this, we introduce *distributions* and discuss weak solutions to partial differential equations.

We then consider harmonic functions in $\mathbb{R}^n$ and state (without proof, unfortunately) some of their properties. A brief study of Green's functions and fundamental solutions will follow afterwards.

5. Random Walks and Diffusion Equations

First let us fix a sufficiently smooth function $f : (a, b) \to \mathbb{R}$ and let $x \in (a, b)$ be given. By considering the Taylor expansion of f, one can check that

$$\lim_{\triangle x \to 0} \frac{f(x + \triangle x) + f(x - \triangle x) - 2f(x)}{2\triangle x} = f''(x).$$

Setting. Consider a random walk on a line with time and distance increments $\triangle x$ and $\triangle t$ and let $p(x, t)$ denote the probability of being at the position x at the given time t. Suppose that at a time t we move to the right by $\triangle x$ with probability $\frac{1}{2}$ and move to the left by $\triangle x$ with probability $\frac{1}{2}$. If as our increments are such that

$$\frac{(\triangle x)^2}{\triangle t} = 4, \tag{†}$$

determine a diffusion equation that governs $p(x, t)$.

The probability of being at a position x at a time $t + \triangle t$, for $\triangle t > 0$, is given formally as $p(x, t+\triangle t)$. Now, we can only reach this position if we are at a position $(x \pm \triangle x)$ at time t. If we are at position $x+\triangle x$ then we have a 1/2 chance of moving to the left and so forth. Thus,

$$p(x, t + \triangle t) = \frac{1}{2} p(x - \triangle x, t) + \frac{1}{2} p(x + \triangle x, t).$$

In this case, we find that

$$\frac{p(x, t + \triangle t) - p(x, t)}{\triangle t} = \frac{p(x - \triangle x, t) + p(x + \triangle x, t) - 2p(x, t)}{2\triangle t}.$$

By letting $\triangle x, \triangle t \to 0$, we will extract a differential equation from the above. This is where we use the information encoded in the ratio

$$\frac{(\triangle x)^2}{\triangle t} = 4.$$

It then follows that

$$\frac{p(x, t + \triangle t) - p(x, t)}{\triangle t} = 2\frac{p(x - \triangle x, t) + p(x + \triangle x, t) - 2p(x, t)}{(\triangle x)^2}.$$

Therefore,

$$\begin{aligned} p_t(x, t) &= \lim_{\triangle t \to 0} \frac{p(x, t + \triangle t) - p(x, t)}{\triangle t} \\ &= \lim_{\triangle x \to 0} 2\frac{p(x - \triangle x, t) + p(x + \triangle x, t) - 2p(x, t)}{(\triangle x)^2} \\ &= 2p_{xx}(x, t). \end{aligned}$$

We conclude that such a random walk can be modeled with a diffusion equation.

A somewhat more involved example is in order.

Example 5.1. Let g be defined as in the previous problem and consider the smooth (you do not have to prove this) function $f(x) := \int_{\mathbb{R}} \frac{g(x-y)}{1+y^6}\,\mathrm{d}y$. Compute $f^{(4)}$.

Solution. Denote by $h(x) := 1/(1+x^6)$. Then $f(x) = (h * g)(x) = (g * h)(x)$. In more explicit terms, this is to say that

$$f(x) = \int_{\mathbb{R}} \frac{g(y)}{1+(x-y)^6}\,\mathrm{d}y.$$

Using that f is smooth, we may differentiate under the integral sign to obtain

$$\begin{aligned} f^{(4)}(x) = \int_{\mathbb{R}} \frac{\partial^4}{\partial x^4}\left[\frac{g(y)}{1+(x-y)^6}\right]\,\mathrm{d}y &= \int_{\mathbb{R}} g(y)\frac{\partial^4}{\partial x^4}\left[\frac{1}{1+(x-y)^6}\right]\,\mathrm{d}y \\ &= \left\langle g^{(4)}, h(x-y)\right\rangle \end{aligned}$$

which, by the previous problem, is precisely

$$\frac{12}{1+x^6}.$$

□

Example 5.2. Consider a random walk where at integer multiples of $\triangle t$ we move to the left by $\triangle x$ with probability 1/4, move to the right by $\triangle x$ with probability 1/4, and stay at our position with probability 1/2. Let $p(x,t)$ be the probability of having the position x at a time t and express p as a solution to a diffusion equation. You may assume that

$$\frac{(\triangle x)^2}{\triangle t} \to \sigma^2 \qquad \text{as } \triangle x, \triangle t \to 0.$$

Solution. Fix a pair (x,t) and observe that the setup gives us the following:

$$p(x, t+\triangle t) = \frac{1}{2}p(x,t) + \frac{1}{4}p(x-\triangle x, t) + \frac{1}{4}p(x+\triangle x, t).$$

Hence,

$$\begin{aligned} \frac{p(x,t+\triangle t) - p(x,t)}{\triangle t} &= \frac{p(x-\triangle x,t) + p(x+\triangle x,t)}{4\triangle t} - \frac{p(x,t)}{2\triangle t} \\ &= \frac{p(x+\triangle x,t) + p(x-\triangle x,t) - 2p(x,t)}{4\triangle t}. \end{aligned}$$

This means that, as $\triangle x$ and $\triangle t$ decrease to 0,

$$\begin{aligned} p_t(x,t) &= \frac{\sigma^2}{4} \lim_{\triangle x \to 0} \frac{p(x+\triangle x,t)+p(x-\triangle x,t)-2p(x,t)}{(\triangle x)^2} \\ &= \frac{\sigma^2}{4} p_{xx}(x,t). \end{aligned}$$

This yields the following diffusion equation:

$$\boxed{\partial_t p(x,t) \equiv \frac{\sigma^2}{4} \partial_{xx} p(x,t).}$$

□

This gives us an intuition behind diffusion type equations.

5.0.1. *Random Walk and Small Drifts*. Here we consider a random walk with a small drift. Fix grid sizes $\triangle t, \triangle x > 0$ and let $\delta > 0$ be small[2] We consider the random walk where at each integer multiple of $\triangle t$, we move to the right by $\triangle x$ with probability $\frac{1}{2} + \delta$, and move to the left by $\triangle x$ with probability $\frac{1}{2} - \delta$. Assume also that at $t = 0$ we are at $x = 0$. Suppose, in addition, that we have some fixed $\sigma^2 > 0$ and $\alpha > 0$ such that

$$\sigma^2 = c\frac{(\triangle x)^2}{2\triangle t} \quad \text{and} \quad \delta = \alpha \triangle x.$$

We seek to model this setting with a diffusion equation. We are given the relation

$$p(x,t+\triangle t) = \left(\frac{1}{2}+\delta\right) p(x-\triangle x,t) + \left(\frac{1}{2}-\delta\right) p(x+\triangle x,t).$$

This allows us to write

$$\begin{aligned} \frac{p(x,t+\triangle t)-p(x,t)}{\triangle t} &= \frac{(1+2\delta)p(x-\triangle x,t)+(1-2\delta)p(x+\triangle x,t)-2p(x,t)}{2\triangle t} \\ &= \frac{\sigma^2}{c(\triangle x)^2}\left(p(x-\triangle x,t)+p(x+\triangle x,t)-2p(x,t)\right) \\ &\quad + \frac{2\sigma^2\delta}{c(\triangle x)^2}\left(p(x-\triangle x,t)-p(x+\triangle x,t)\right). \end{aligned}$$

Now, this first term tends to

$$\frac{\sigma^2}{c} p_{xx}(x,t)$$

[2]How small is more or less up to you, so long as δ is much less than 1/2.

whilst the second term is equal to

$$\frac{2\alpha\sigma^2}{c} \cdot \frac{p(x-\triangle x,t) - p(x,t) + p(x,t) - p(x+\triangle x,t)}{\triangle x}$$

which tends to

$$\frac{2\alpha\sigma^2}{c} \cdot (-2p_x(x,t)) = \frac{-4\alpha\sigma^2}{c} p_x(x,t).$$

We therefore recover the diffusion equation

$$cp_t(x,t) = \sigma^2 p_{xx}(x,t) - 4\alpha\sigma^2 p_x(x,t).$$

6. Distributions: Generalized Functions

This section is devoted to the development of the theory of distributions on $\mathbb{R}^n$. These *generalized functions* will open doors and broaden our horizons. To be more precise, they will redefine what it means to be a solution to a PDE.

Consider the set of all smooth functions of compact support $\mathbb{R}^n \to \mathbb{C}$, it is a vector space over the complex numbers. Let us denote this vector space by $\mathcal{D}(\mathbb{R}^n)$, where the dimension $n \in \mathbb{N}$ is understood. Clearly, the space $\mathcal{D}(\mathbb{R}^n)$ is infinite dimensional. We topologize $\mathcal{D}(\mathbb{R}^n)$ by endowing the space with the sup-norm:

$$\|\cdot\|_\infty := \sup_{\mathbf{x}\in\mathbb{R}^n} |\cdot(\mathbf{x})| < \infty.$$

Let now $\{\varphi_k\}_{k=1}^\infty$ be a sequence of functions in $\mathcal{D}(\mathbb{R}^n)$ and let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We say that $\varphi_k \to \varphi$ in $\mathcal{D}(\mathbb{R}^n)$ provided each of the following hold true:

(1) There exists a compact set $\Lambda \subset \mathbb{R}^n$ such that $\mathrm{supp}(\varphi_k) \Subset \Lambda$ for all $k \in \mathbb{N}$;
(2) For every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ one has

$$\|\partial^\alpha \varphi_k - \partial^\alpha \varphi\|_\infty \xrightarrow{k\to\infty} 0.$$

In this case, we shall write $\varphi = \lim_{k\to\infty} \varphi_k$. This notion of convergence provides us with a way to define the continuity of functionals $F : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$. Indeed, a functional F on $\mathcal{D}(\mathbb{R}^n)$ is called continuous if

$$\lim_{k\to\infty} F(\varphi_k) = F(\varphi)$$

for every sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{D}(\mathbb{R}^n)$ converging to $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Definition 4. A distribution is a continuous linear functional $F : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$.

This new notion allows us to give meaning to the Dirac delta "function". Indeed, we let $\delta_0 : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$ be given by $\delta_0(\varphi) := \varphi(0)$. It is easy to check that this is a continuous linear functional (and thus a distribution).

Perhaps a more interesting example is, actually, a large class of examples. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then f induces a distribution $\langle f, \cdot \rangle$ defined as

$$\langle f, \varphi \rangle := \int_{\mathbb{R}^n} f(\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

This notation is actually very convenient. Given a distribution $F \in D(\mathbb{R}^n)$, we will use the notation $\langle F, \varphi \rangle$ to denote $F(\varphi)$. The space of all distributions is denoted $\mathcal{D}^*(\mathbb{R}^n)$ and is, itself, a complex vector space. If f is a locally integrable function and $\langle F, \cdot \rangle$ a distribution, we say $f = F$ in the sense of distributions if

$$\langle F, \varphi \rangle = \langle f, \varphi \rangle\,, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Further, we say two functions f and g are equal in the sense of distributions if $g = \langle f, \cdot \rangle$ in the sense of distributions.

Definition 5. Let $\{F_k\}_{k=1}^{\infty}$ be a sequence of distributions. We say that F_k converges to a distribution F if

$$\lim_{k\to\infty} \langle F_k, \varphi \rangle = \langle F, \varphi \rangle\,, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

We say a sequence of functions $\{f_k\}_{k=1}^{\infty}$ converges to a function f in the sense of distributions if $\langle f_k, \cdot \rangle$ converges to $\langle f, \cdot \rangle$ in the sense of distributions. Finally, a sequence of functions converges to a distribution F if $\langle f_k, \cdot \rangle \to F$ as distributions.

Example 6.1. The sequence of hat functions is a family of locally integrable functions defined by

$$f_k(x) := \begin{cases} k/2, & \text{if } |x| \le 1/k, \\ 0, & \text{else.} \end{cases}$$

Prove that $f_k \to \delta_0$ as $k \to \infty$, in the sense of distributions.

Solution. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be given. Notice that

$$\begin{aligned} |\langle f_k, \varphi \rangle - \langle \delta_0, \varphi \rangle| = \left| \int_{-1/k}^{1/k} \varphi(x)\,\mathrm{d}x - \varphi(0) \right| &= \left| \int_{-1/k}^{1/k} \frac{k}{2}\varphi(x)\,\mathrm{d}x - \int_{-1/k}^{1/k} \frac{k}{2}\varphi(0)\,\mathrm{d}x \right| \\ &\le \frac{k}{2} \int_{-1/k}^{1/k} \sup_{|x|\le 1/k} |\varphi(x) - \varphi(0)|\,\mathrm{d}x \\ &= \sup_{|x|\le 1/k} |\varphi(x) - \varphi(0)|\,. \end{aligned}$$

By continuity of φ, this last term tends to zero as $k \to \infty$. Hence, $f_k \to \delta_0$ as $k \to \infty$, in the sense of distributions. □

6.1. **Differentiation of Distributions.** Perhaps the greatest strength of distributions is their differentiability. It is well known that not every function is differentiable. In fact, there exist everywhere continuous functions that are no-where differentiable. This is not the case for distributions. Let $\langle F, \cdot \rangle$ be a distribution on $\mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index. We define the α-derivative of $\langle F, \cdot \rangle$ to be the following distribution:

$$\langle \partial F, \varphi \rangle := (-1)^{|\alpha|} \langle F, \partial^\alpha \rangle$$

where $\varphi \in \mathcal{D}(\mathbb{R}^n)$. It is easy to check that $\langle \partial^\alpha, \cdot \rangle$ it itself a well defined distribution. We say that a distribution $\langle G, \cdot \rangle$ is the α-derivative of $\langle F, \cdot \rangle$ (in the sense of distributions) provided

$$\langle \partial^\alpha F, \varphi \rangle = \langle G, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Example 6.2. Consider the Heaviside function

$$H(x) := \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

Clearly, H is locally integrable. Physicists and engineers frequently argue that the derivative of H is δ_0, because of the "jump" at $x = 0$. We will prove that $H' = \delta_0$ in the sense of distributions.

Solution. We fix a function $\varphi \in \mathcal{D}(\mathbb{R})$. Then,

$$\begin{aligned} \langle H', \varphi \rangle = - \langle H, \varphi' \rangle = - \int_{\mathbb{R}} H(x)\varphi'(x)\,\mathrm{d}x &= - \int_0^\infty \varphi'(x)\,\mathrm{d}x \\ &= - \lim_{x \to \infty} \varphi(x) + \varphi(0) \\ &= \varphi(0) \\ &= \langle \delta_0, \varphi \rangle . \end{aligned}$$

Since $\alpha \in \mathcal{D}(\mathbb{R})$ was arbitrary, we are done. □

PROPOSITION 1. *Let $\{F_k\}_{k=1}^\infty$ be a sequence of distributions converging to a distribution F. Then, for every multi-index α, $\partial^\alpha F_k$ converges to $\partial^\alpha F$.*

Proof. If α is given and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is fixed, one has for each $k \in \mathbb{N}$

$$\langle F_k^\alpha, \varphi \rangle = (-1)^{|\alpha|} \langle F_k, \partial^\alpha \phi \rangle$$

where $\partial^\alpha \phi \in \mathcal{D}(\Omega)$. Letting $k \to \infty$, we obtain

$$\lim_{k \to \infty} \langle F_k^\alpha, \varphi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \varphi \rangle = \langle \partial^\alpha F, \varphi \rangle .$$

This completes the proof. □

Example 6.3. Define

$$g(x) := \begin{cases} x^3, & \text{if } x \geq 0, \\ -x^3, & \text{if } x < 0. \end{cases}$$

Calculate $g^{(4)}$ in the sense of distributions. **Hint**: recall that distributional derivatives agree with the regular derivative for $C^1(\mathbb{R})$ functions.

Solution. It is easy to check that $g \in C^2(\mathbb{R})$. This implies that,

$$g'' = \begin{cases} 6x, & \text{if } x \geq 0, \\ -6x, & \text{if } x < 0. \end{cases}$$

Now, let $\phi \in \mathcal{D}(\mathbb{R})$ and calculate

$$\begin{aligned} \left\langle g^{(3)}, \phi \right\rangle = -\left\langle g'', \phi' \right\rangle &= \int_{-\infty}^{0} 6x\phi'(x)\,\mathrm{d}x - \int_{0}^{\infty} 6x\phi'(x)\,\mathrm{d}x \\ &= 6x\phi(x)|_{-\infty}^{0} - \int_{-\infty}^{0} 6\phi(x)\,\mathrm{d}x - 6x\phi(x)|_{0}^{\infty} + \int_{0}^{\infty} 6\phi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} 6\phi(x)\,\mathrm{d}x \end{aligned}$$

where we have used that ϕ has compact support. This implies that, in the sense of distributions,

$$g^{(3)} = \begin{cases} 6, & \text{if } x \geq 0, \\ -6, & \text{if } x < 0. \end{cases}$$

Once again taking the derivative (in the sense of distributions), we obtain for each $\phi \in \mathcal{D}(\mathbb{R})$:

$$\left\langle g^{(4)}, \phi \right\rangle = -\int_{0}^{\infty} 6\phi'(x)\,\mathrm{d}x + \int_{-\infty}^{0} 6\phi'(x)\,\mathrm{d}x = 6\phi(0) + 6\phi(0)$$

whence $g^{(4)} = 12\delta_0$, where δ_0 is the Dirac delta 'function'. □

Example 6.4. Let g be defined as in the previous problem and consider the smooth (you do not have to prove this) function $f(x) := \int_{\mathbb{R}} \frac{g(x-y)}{1+y^6}\,\mathrm{d}y$. Compute $f^{(4)}$.

Solution. Denote by $h(x) := 1/(1+x^6)$. Then $f(x) = (h * g)(x) = (g * h)(x)$. In more explicit terms, this is to say that

$$f(x) = \int_{\mathbb{R}} \frac{g(y)}{1+(x-y)^6}\,\mathrm{d}y.$$

Using that f is smooth, we may differentiate under the integral sign to obtain

$$f^{(4)}(x) = \int_{\mathbb{R}} \frac{\partial^4}{\partial x^4}\left[\frac{g(y)}{1+(x-y)^6}\right] \mathrm{d}y = \int_{\mathbb{R}} g(y)\frac{\partial^4}{\partial x^4}\left[\frac{1}{1+(x-y)^6}\right] \mathrm{d}y$$
$$= \left\langle g^{(4)}, h(x-y)\right\rangle$$

which, by the previous problem, is precisely

$$\frac{12}{1+x^6}.$$

□

Example 6.5. Let $\langle F, \cdot\rangle$ be a distribution and $g \in C^\infty(\mathbb{R}^n)$. The map

$$\langle gF, \varphi\rangle := \langle F, g\varphi\rangle$$

is again a distribution. If Δ denotes the 3-dimensional Laplacian, show that

$$|\mathbf{x}|^2 \Delta\delta_0 = 6\delta_0.$$

Solution. Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ be given. Then,

$$\left\langle |\mathbf{x}|^2 \Delta\delta_0, \varphi\right\rangle = \left\langle \delta_0, \Delta\left(|\mathbf{x}|^2 \varphi(\mathbf{x})\right)\right\rangle = \sum_{j=1}^{3} \left\langle \delta_0, \partial_j^2(x_1^2+x_2^2+x_3^2)\varphi(\mathbf{x})\right\rangle.$$

Fix an index j; we calculate

$$\partial_j(x_1^2+x_2^2+x_3^2)\varphi(\mathbf{x}) = 2x_j\varphi(\mathbf{x}) + (x_1^2+x_2^2+x_3^2)\varphi_{x_j}(\mathbf{x}).$$

Once again differentiating yields

$$\partial_j^2(x_1^2+x_2^2+x_3^2)\varphi(\mathbf{x}) = 2\varphi(\mathbf{x}) + 4x_j\varphi_{x_j}(\mathbf{x}) + (x_1^2+x_2^2+x_3^2)\varphi_{x_jx_j}(\mathbf{x}).$$

It follows from this that

$$\left\langle |\mathbf{x}|^2 \Delta\delta_0, \varphi\right\rangle = \sum_{j=1}^{3} \left\langle \delta_0, \partial_j^2(x_1^2+x_2^2+x_3^2)\varphi(\mathbf{x})\right\rangle = \sum_{j=1}^{3} 2\varphi(\mathbf{0}) + 0 + 0 = 6\varphi(\mathbf{0}).$$

This is precisely the statement $\left\langle |\mathbf{x}|^2 \Delta\delta_0, \varphi\right\rangle = \langle \delta_0, \varphi(\mathbf{x})\rangle$. □

Example 6.6. If $f : \mathbb{R}^n \to \mathbb{R}$ is not sufficiently smooth, there is no guarantee that $f_{xy} = f_{yx}$. Show that if $f \in L^1_{\text{loc}}(\mathbb{R}^2)$, then $f_{xy} = f_{yx}$ in the sense of distributions.

Solution. Let us fix $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and consider

$$\begin{aligned}\langle f_{xy}, \varphi\rangle = (-1)^2 \langle f, \varphi_{xy}\rangle &= \iint_{\mathbb{R}^2} f(\mathbf{x})\varphi_{xy}(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \iint_{\mathbb{R}^2} f(\mathbf{x})\varphi_{yx}(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \langle f_{yx}, \varphi\rangle .\end{aligned}$$

□

Example 6.7. Consider the function

$$f(x,y) := \begin{cases} 1, & \text{if } x^3 \leq y, \\ 0, & \text{else.}\end{cases}$$

Compute f_{xy} in the sense of distributions.

Solution. Fix a function $\varphi \in \mathcal{D}(\mathbb{R}^2)$ and write

$$\begin{aligned}\langle f_{xy}, \varphi\rangle = \langle f, \varphi_{xy}\rangle = \iint_{\mathbb{R}^2} f(x,y)\varphi_{xy}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{-\infty}^{\infty}\int_{x^3}^{\infty} \varphi_{xy}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \varphi_x(x,x^3)\,\mathrm{d}x.\end{aligned}$$

□

DEFINITION 6. Let L be a linear differential operator and u locally integrable. We say $Lu = f$ in the sense of distributions if

$$\langle Lu, \varphi\rangle = \langle f, \varphi\rangle , \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

In this case, u is called a weak solution to the PDE $Lu = f$.

Example 6.8. Let $a, b \in \mathbb{R} \setminus \{0\}$ and suppose f is $C^1(\mathbb{R})$. Show that the function

$$u(x,y) := f(bx - ay)$$

solves the PDE

$$au_x + bu_y \equiv 0$$

in the sense of distributions.

Solution. Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$ be given and consider

$$\langle au_x + bu_y, \varphi\rangle = -\langle u, a\varphi_x\rangle - \langle u, b\varphi_y\rangle . \tag{6.1}$$

Also, since φ has compact support, integration by parts implies

$$\begin{aligned}\langle u, a\varphi_x\rangle &= a\int_{\mathbb{R}}\int_{-\infty}^{\infty} f(bx-ay)\varphi_x(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &= ab\int_{\mathbb{R}}\int_{-\infty}^{\infty} f'(bx-ay)\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Furthermore,

$$\begin{aligned}\langle u, b\varphi_y\rangle &= b\int_{\mathbb{R}}\int_{-\infty}^{\infty} f(bx-ay)\varphi_y(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= -ab\int_{\mathbb{R}}\int_{-\infty}^{\infty} f'(bx-ay)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

It follows now from (6.1) that

$$\langle au_x + bu_y, \varphi\rangle = 0$$

whence we conclude that $au_x + bu_y = 0$ in the sense of distributions. □

7. Harmonic Functions and The Laplace Equation

Fix $n \in \mathbb{N}$ and recall that we denote by Δ the Laplacian in $\mathbb{R}^n$. Throughout what remains of this document, $\Omega \Subset \mathbb{R}^n$ will denote a bounded, open, connected set with smooth boundary.

Definition 7. A function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is called harmonic if $\Delta u \equiv 0$ in Ω.

There are many astounding properties involving harmonic functions. Before we state these (unfortunately, without proof), we require some notation. If E is a Lebesgue measurable set of positive (but finite!) measure and f is locally integrable, we define

$$⨍_E f(x)\,\mathrm{d}x := \frac{1}{m(E)}\int_E f(x)\,\mathrm{d}m.$$

Harmonic functions satisfy the mean value property, as we shall see.

Theorem (Mean Value Criterion). *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. The following are equivalent:*

(1) u is harmonic in Ω;
(2) For every open ball $B(\mathbf{x},\rho) \Subset \Omega$ we have

$$u(\mathbf{x}) = ⨍_{\partial B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}\mathbf{S} = ⨍_{B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}y.$$

This property allows one to conclude the following (although the result requires a proof).

THEOREM (Strong Maximum Principle). *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω. If u achieves an extremumin Ω, then u is constant on $\overline{\Omega}$.*

An immediate corollary of this is the so-called *weak maximum principle*.

COROLLARY 1. *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω. Then*

$$\max_{\partial\Omega} u(\mathbf{x}) = \max_{\overline{\Omega}} u(\mathbf{x}).$$

Example 7.1. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a function that satisfies the mean value property in Ω for shells, i.e.

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}\mathbf{S}, \quad \forall B(\mathbf{x},\rho) \Subset \Omega.$$

Prove that for all $B(\mathbf{x},\rho) \Subset \Omega$ one also has

$$u(\mathbf{x}) = \fint_{B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

Solution. Let $B(\mathbf{x},\rho) \Subset \Omega$ and note that

$$\int_{B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_0^\rho \left(\int_{\partial B(\mathbf{x},\gamma)} u(\mathbf{y})\,\mathrm{d}\mathbf{S} \right) \mathrm{d}\gamma$$

where, by assumption,

$$\int_{\partial B(\mathbf{x},\gamma)} u(\mathbf{y})\,\mathrm{d}\mathbf{S} = 4\pi\gamma^2 u(\mathbf{x}).$$

Therefore,

$$\int_{B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}\mathbf{y} = 4\pi u(\mathbf{x}) \int_0^\rho \gamma^2\,\mathrm{d}\gamma = \frac{4\pi\rho^3}{3} u(\mathbf{x}).$$

This is equivalent to saying

$$\fint_{B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}\mathbf{y} = u(\mathbf{x})$$

and the proof is complete. □

Example 7.2. Let $\mathbb{D}_2$ denote the open ball of radius 2, centered at the origin, in $\mathbb{R}^2$. Let $\Omega \Supset \mathbb{D}_2$ be a domain and suppose $u(r,\theta) \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω. Assume $u(2,\theta) = 3\sin(2\theta) + 1$.

(1) What is the maximum value of u in $\overline{\mathbb{D}_2}$?
(2) Compute $u(\mathbf{0})$.

Solution. For the first part, we need only note that u is harmonic in $\mathbb{D}_2$ and continuous on $\overline{\mathbb{D}_2}$. Thus, the weak maximum principle implies that

$$\max_{\overline{\mathbb{D}_2}} u(r,\theta) = \max_{\partial\mathbb{D}_2} u(r,\theta).$$

It therefore suffices to maximum $u(2,\theta) = 3\sin(2\theta)+1$ for $0 \le \theta < 2\pi$. Obviously, this function will achieve its maximum at $\theta = \pi/4$ where u will take on the value of 4. Therefore, the maximum value of u on $\overline{\mathbb{D}_2}$ is 4. For the second part, we apply the mean value theorem to deduce that

$$\begin{aligned} u(\mathbf{0}) = \fint_{\partial\mathbb{D}_2} u(2,\theta)\,\mathrm{d}\theta &= \frac{1}{4\pi}\int_0^{2\pi} [3\sin(2\theta)+1]\,\mathrm{d}\theta \\ &= \frac{2\pi}{4\pi} \\ &= \frac{1}{2}. \end{aligned}$$

□

8. Fundamental Solutions & Green's Functions

We once again return to the notion of the Laplacian. The fundamental solution to the Laplacian in $\mathbb{R}^n$, is a locally integrable function $\Phi(\mathbf{x})$ on $\mathbb{R}^n$ such that

$$\Delta\Phi = \delta_0 \quad \text{in the sense of distributions.}$$

We will only consider the case $n=1$ and $n=3$. For $n=3$ this is given by

$$\Phi(\mathbf{x}) := -\frac{1}{4\pi|\mathbf{x}|}.$$

We will write $\Phi(\mathbf{x},\mathbf{x}_0)$ to denote the function $\Phi(\mathbf{x}-\mathbf{x}_0)$. The following representation formula is sometimes useful in obtaining explicit solutions to differential equations:

Theorem. *Given $f \in C_c^\infty(\mathbb{R}^3)$. Then*

$$u(\mathbf{x}) = \Phi(\mathbf{x}-\mathbf{y})f(\mathbf{y})\,\mathrm{d}y$$

satisfies $\Delta u \equiv f$.

Definition 8. Suppose $\Omega \Subset \mathbb{R}^3$ is a domain and choose $\mathbf{x}_0 \in \Omega$. The Green's function for Ω with source point $\mathbf{x}_0$ is a function $G(\mathbf{x},\mathbf{x}_0)$ defined on $\Omega \setminus \{\mathbf{x}_0\}$ such that

(1) The function
$$H(\mathbf{x}) = G(\mathbf{x}-\mathbf{x}_0) - \Phi(\mathbf{x}-\mathbf{x}_0)$$
is smooth and harmonic in Ω;

(2) $G(\mathbf{x}, \mathbf{x}_0)$ vanishes on $\partial\Omega$.

In practice how can one find the Green's function? This is usually done by following the steps:

(1) Fix $\mathbf{x}_0 \in \Omega$ and find a smooth harmonic function $H(\mathbf{x}, \mathbf{x}_0)$ (in Ω) equal to $-\Phi(\mathbf{x} - \mathbf{x}_0)$ on $\partial\Omega$.
(2) Define
$$G(\mathbf{x}, \mathbf{x}_0) = H(\mathbf{x}, \mathbf{x}_0) + \Phi(\mathbf{x} - \mathbf{x}_0).$$

Careful! This is easier said than done. To illustrate this procedure we shall give one easy example in $\mathbb{R}$.

Example 8.1. Let $L = \partial^2$ be the second derivative operator on $\mathbb{R}$ (i.e. the one dimensional Laplacian).

(1) Show that
$$\Phi(\mathbf{x}) := \frac{|x|}{2}$$
is the Fundamental solution for L.
(2) Find a Green's function for $(-1, 1)$.

Solution. The first part is straightforward, let $\varphi \in \mathcal{D}(\mathbb{R})$ and note that
$$\left\langle \partial^2\Phi, \varphi \right\rangle = \int_{-\infty}^{\infty} \frac{|x|}{2}\varphi''(x)\,\mathrm{d}x = \frac{-1}{2}\int_{-\infty}^{0} x\varphi''(x)\,\mathrm{d}x + \frac{1}{2}\int_{0}^{\infty} x\varphi''(x)\,\mathrm{d}x.$$
Now, since φ has compact support
$$\int_{-\infty}^{0} x\varphi''(x)\,\mathrm{d}x = -\int_{-\infty}^{0} \varphi'(x)\,\mathrm{d}x = -\varphi(0).$$
Likewise,
$$\int_{0}^{\infty} x\varphi''(x)\,\mathrm{d}x = -\int_{0}^{\infty} \varphi'(x)\,\mathrm{d}x = \varphi(0).$$
This implies that
$$\left\langle \partial^2\Phi, \varphi \right\rangle = \varphi(0) = \langle \delta_0, \varphi \rangle .$$
This establishes (1). For the second part, let $x_0 \in (-1, 1)$ be fixed. At $x = 1$ we have
$$\Phi(1 - x_0) = \frac{|1 - x_0|}{2} = \frac{1 - x_0}{2}$$
whilst at $x = -1$
$$\Phi(-1 - x_0) = \frac{1 + x_0}{2}.$$

Consider the function

$$H(x, x_0) := -\frac{1 - x_0 x}{2}$$

which is harmonic (and smooth) in $\mathbb{R}$. Notice also that $H(1, x_0) = -\Phi(1 - x_0)$ and $H(-1, x_0) = -\Phi(-1 - x_0)$. Thus, the Green's function is

$$G(x, x_0) := \frac{|x - x_0| - (1 - x_0 x)}{2}.$$

$\square$

Example 8.2. Let $\Omega \Subset \mathbb{R}^3$ be a domain.[3] Let $G(\mathbf{x}, \mathbf{x}_0)$ be the Green's function for Δ in Ω. What can we say about the sign of G in Ω?

Solution. Let us fix $\mathbf{x}_0 \in \Omega$; the Green's function with singularity $\mathbf{x}_0$ will be of the form

$$G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}) = H(\mathbf{x}) + \Phi(\mathbf{x} - \mathbf{x}_0)$$

for some smooth harmonic function H in Ω equal to $\Phi(\mathbf{x} - \mathbf{x}_0)$ on $\partial\Omega$. Let $\varepsilon > 0$ with $\varepsilon \ll 1$. Consider the domain

$$\Omega_\varepsilon := \Omega \setminus \overline{B(\mathbf{x}_0, \varepsilon)}.$$

Obviously, for each such $\varepsilon > 0$, the Green's function $G(\mathbf{x}, \mathbf{x}_0)$ is harmonic in Ω_ε. Notice that on $\partial\Omega$, the Green's function vanishes. Furthermore, for all ε sufficiently small, on

$$\partial B(\mathbf{x}_0, \varepsilon) \subset \partial\Omega$$

the Green's function is negative since $H(\mathbf{x})$ is bounded but $\Phi(\mathbf{x} - \mathbf{x}_0)$ tends to $-\infty$ as $\mathbf{x} \to \mathbf{x}_0$. Hence, we can choose $\varepsilon \ll 1$ such that $G(\mathbf{x}, \mathbf{x}_0) \leq -M$, for $M > 0$ large, whenever $\mathbf{x} \in \overline{B(\mathbf{x_0}, \varepsilon)}$. In particular, the Green's function is negative on a portion of $\partial\Omega_\varepsilon$. The weak maximum principle then implies that $G(\mathbf{x}, \mathbf{x}_0)$ is negative in Ω_ε. We conclude that G is negative in Ω for each fixed $\mathbf{x}_0 \in \Omega$.

$\square$

Example 8.3. Let $\mathbf{u}(x, y) := (u_1(x, y), u_2(x, y))$ be a smooth vector field. What does it mean for this vector field to solve $\text{div}(\mathbf{u}) = 0$ in the sense of distributions? Moreover, if $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, define

$$\mathbf{u}(x, y) := \begin{cases} \mathbf{a}, & y \geq f(x), \\ \mathbf{b}, & y < f(x) \end{cases}$$

where $f \in C^\infty(\mathbb{R})$ is given. Show that $\text{div}(\mathbf{u}) \neq 0$ in the sense of distributions.

[3]An open, bounded, connected subset of $\mathbb{R}^3$ having smooth boundary.

Solution. First, let $\mathbf{u}$ be some smooth vector field. Then, for any $\varphi \in \mathcal{D}(\mathbb{R}^2)$ there holds

$$\begin{aligned}\langle \operatorname{div}(\mathbf{u}), \varphi\rangle = \langle \partial_x u_1 + \partial_y u_2, \varphi\rangle &= \langle \partial_x u_1, \varphi\rangle + \langle \partial_y u_2, \varphi\rangle \\ &= -\langle u_1, \varphi_x\rangle - \langle u_2, \varphi_y\rangle \\ &= -\iint_{\mathbb{R}^2} (\mathbf{u}\cdot\nabla\phi)\ \mathrm{d}A.\end{aligned}$$

Therefore, $\operatorname{div}(\mathbf{u}) = 0$ in the sense of distributions if this last line evaluates to 0 for every $\varphi \in \mathcal{D}(\mathbb{R}^2)$. For the second part, we check

$$\langle \operatorname{div}(\mathbf{u}), \varphi\rangle = -\iint_{\mathbb{R}^2} (\mathbf{u}\cdot\nabla\phi)\,\mathrm{d}A = -\int_{\mathbb{R}}\int_{f(x)}^{\infty} \mathbf{a}\cdot\nabla\phi\,\mathrm{d}A - \int_{\mathbb{R}}\int_{-\infty}^{f(x)} \mathbf{b}\cdot\nabla\phi\,\mathrm{d}A$$

which does not, in general, vanish. □