Real Analysis:
The Topology of Metric Spaces

Edward Chernysh

August 8, 2017

Contents

NOTE: In this text we denote an arbitrary metric space by $\mathbb{X}$ or, sometimes, $\mathbb{Y}$.

1 Open, Closed and Compact Sets

1.1 Definition

Definition 1.1.1 (Open Sets). *We say a subset $E \subseteq \mathbb{X}$ is open in $\mathbb{X}$ if for all $x \in E$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq E$; where*

$$B(x, \varepsilon) := \{y \in X \mid d(x, y) < \varepsilon\}$$

is called the open ball of radius ε centered at x.

Definition 1.1.2 (Closed Sets). *Let $F \subseteq \mathbb{X}$. We say F is closed in $\mathbb{X}$ if the complement of F denoted $F^c := \mathbb{X} \setminus F$ is open in $\mathbb{X}$.*

Example 1.1.1. *The set $(0, 1)$ is open in $\mathbb{R}$.*

Example 1.1.2. *The set $[0, 1]$ is closed in $\mathbb{R}$.*

1.2 Characterization of Closed Sets

Theorem 1.2.1 (Properties of Open Sets). *Let $\{G_i\}_{i \in I} \subseteq \mathbb{R}$ be a family of open sets (that is, each G_i is open). If this is the case:*

1. *$\cup_{i \in I} G_i$ is open in $\mathbb{X}$.*
2. *Any finite intersection $\cap_{n=1}^{N} G_n$ is open in $\mathbb{X}$.*

Proof.

1. Let $x \in \cup_{i \in I} G_i$, then $x \in G_i$ for some $i \in I$. Indeed, each G_i is open, so there exists $\varepsilon > 0$ such that $B(x.\varepsilon) \subseteq G_i$, but $G_i \subseteq \cup_{i \in I} G_i$ whence $B(x, \varepsilon) \subseteq \cup_{i \in I} G_i$.
2. Let $x \in \cap_{i=1}^{n} G_i$. Then $x \in G_i$ for all $i \in [1, n] \cap \mathbb{N}$. For any such open set we may find $\varepsilon_i > 0$ such that $B(x, \varepsilon_i) \subseteq G_i$. Now taking $\delta := \min\{\varepsilon_1, ..., \varepsilon_n\}$ we have $B(x, \delta) \subseteq B(x, \varepsilon_n)$ for all n. In other words, $B(x, \delta) \subseteq \cap_{i=1}^{n} G_i$.

□

Corollary 1.2.1 (Properties of Closed Sets). *Let $\{F_i\}_{i \in I} \subseteq \mathbb{X}$ be a family of closed sets (that is, each F_i is closed). If this is the case:*

1. *The intersection of closed sets is closed in $\mathbb{X}$.*
2. *A finite union of closed sets is closed in $\mathbb{X}$.*

Theorem 1.2.2 (Characterization of Closed Sets). *Let $F \subseteq \mathbb{X}$ be any non-empty set. The following are equivalent:*

1. *F is closed in $\mathbb{X}$.*

2. *Every convergent sequence in F converges in F.*

Proof. ($1 \implies 2$). Let $F \subseteq \mathbb{X}$ be closed and let $(x_n) \subseteq F$ be any convergent sequence. Indeed, $(x_n) \to x \in \mathbb{X}$. We show $x \in F$. Suppose not, then $x \notin F \implies x \in F^c$. By definition, F^c is open in $\mathbb{X}$, whence there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq F^c$. Fix such $\varepsilon > 0$, then since $(x_n) \to x$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(x_n, x) < \varepsilon$. In particular, for N one has $d(x_N, x) < \varepsilon \implies x_N \in B(x, \varepsilon) \subseteq F^c$ and hence $x_N \in F^c$. But we assumed $x_n \in F$ for all $n \in \mathbb{N}$. Contradiction.

Conversely suppose (2) holds but (1) does not. That is, F is not closed and hence F^c is not open. Thus for some $x \in F^c$ we have for all $\varepsilon > 0$ that $B(x, \varepsilon) \cap F \neq \emptyset$. Fix such x and set $\varepsilon = 1/n$, we thereby construct a sequence $(x_n) \subseteq B(x, 1/n) \cap F$. Indeed, x_n is a sequence in F. We claim $\lim x_n = x$. To see this, fix $\varepsilon > 0$ arbitrarily and pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, then for all $n \geq N$ $1/n < \varepsilon$ and hence $x_n \in B(x, 1/n)$ implies $d(x_n, x) < \varepsilon$. Thus $(x_n) \to x$, but by hypothesis this implies $x \in F$. Contradiction. □

1.3 Compactness and Consequences

Definition 1.3.1 (Compactness). *Let $K \subseteq \mathbb{X}$. We say K is compact in $\mathbb{X}$ if for any open cover $\{G_i\}_{i \in I}$ in $\mathbb{X}$ of K we may find a finite subcover $\{G_{i_1}, ..., G_{i_n}\}$ containing K.*

Theorem 1.3.1. *Let $K \subseteq \mathbb{X}$ be a compact set. Then K is closed and bounded in $\mathbb{X}$.*

Proof. Fix $a \in \mathbb{X}$, define for each $n \in \mathbb{N}$ the open set $G_n := B(a, n)$. It is easy to see that

$$\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} \{x \in \mathbb{X} \mid d(x, a) < n\} = \mathbb{X} \supseteq K$$

Thus, by Theorem (1.2.1) each G_n is open in $\mathbb{X}$ and hence $\{G_n\}_{n=1}^{\infty}$ defines an open cover of K. By compactness of $K \subseteq \mathbb{X}$ there exists a finite subcover of $\{G_n\}_{n=1}^{\infty}$ containing K. That is for some $N \in \mathbb{N}$,

$$K \subseteq G_1 \cup G_2 \cup ... \cup G_N$$

Note that, $G_{n+1} \supseteq G_n$ whence $K \subseteq G_N$. That is, for all $xinK$ we have $d(x, a) < N$. Hence K is bounded in $\mathbb{X}$.

It remains to show that K is closed. We demonstrate this by showing K^c is open in $\mathbb{X}$ as per Definition 1.1.2. Let $x \in K^c$. Define for each $n \in \mathbb{N}$ a non-empty set $H_n := \{y \in \mathbb{X} \mid d(x, y) > 1/n\}$. It is very easy to see that each H_n is open and hence $\{H_n\}_{n \in \mathbb{N}}$ defines a collection of open sets in $\mathbb{X}$. Moreover, $H_n \subset H_{n+1}$. Indeed,

$$H := \bigcup_{n=1}^{\infty} H_n = \mathbb{X} \setminus \{x\}$$

But $x \in K^c \iff x \notin K$, whence $K \subseteq H$ which is open by Theorem 1.2.1. Thus, $\{G_n\}_{n\in\mathbb{N}}$ is an open cover of K. Thus by the same argument as before for some $M \in \mathbb{N}$ we have $K \subseteq H_M = \{y \in \mathbb{X} \mid d(x,y) > 1/M\}$. Similarly this implies $K^c \supset \{y \in \mathbb{X} \mid d(x,y) < 1/M\}$. Indeed, taking $\varepsilon = 1/M > 0$ we see that $B(x,\varepsilon) \subseteq K^c$ and hence K^c is open since $x \in K^c$ is arbitrary. Ergo, K is closed in $\mathbb{X}$. This is what we had to show. □

1.4 Heine-Borel on $\mathbb{R}$

Theorem 1.4.1 (One Dimensional Heine Borel). *A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof. By Theorem 1.3.1 $\rightarrow$ is true in any metric space, thus it is sufficient to prove only $\leftarrow$. By way of contradiction suppose instead that K is not compact in $\mathbb{R}$, then there exists an open cover $\{G_i\}_{i\in I}$ containing K that has no finite-subcover of K. Take such a finite subcover in $\mathbb{R}$ of K. Since K is bounded there exists $R > 0$ such that $K \subseteq [-R, R]$. Denote this set by I_1. We partition I_1 into two equal sub-intervals $I_1' = [-R, 0]$ and $I_1'' = [0, R]$. Now, $K \cap I_1'$ or $K \cap I_1''$ is non-empty and not contained in any finite-subcover of $\{G_i\}_{i\in I}$. To see this we write K as follows:

$$K = K \cap I_1 = K \cap \left(I_1' \cup I_1''\right) = \left(K \cap I_1'\right) \cup \left(K \cup I_1''\right)$$

Let I_2 be any such sub-interval of I_1. We repeat this process, whence constructing a nested sequence of closed and bounded intervals I_n in $\mathbb{R}$ such that for any $n \in \mathbb{N}$ $K \cap I_n$ is non empty and not contained in any finite-subcover of $\{G_i\}_{i\in I}$. By the nesting theorem, we have that

$$\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$$

Let $z \in \cap_{n=1}^{\infty} I_n$. Then $z \in I_n$ for all n. We will now prove that $z \in K$. To see this from each $K \cap I_n \ni x_n$ is selected. This describes a sequence (x_n) living in K. Now, the length of I_n is $\frac{R}{2^{n-2}}$. Thus, since $x_n \in K \cap I_n$ and $z \in I_n$ for all n it s clear that

$$|x_n - z| \leq \frac{R}{2^{n-2}}$$

Obviously, $(x_n) \to z$, and since K is closed it follows from Theorem 1.2.2 that $z \in K$. Since $\{G_i\}_{i\in I}$ is an open cover of $K \ni z$, there exists $j \in I$ such that $z \in G_j$, but since G_j is open by Definition 1.1.1 there exists $\varepsilon > 0$ such that $B(z,\varepsilon) \subseteq G_j$. Now, if we take $N \in \mathbb{N}$ such that $\frac{R}{2^{N-2}} < \varepsilon$ (we can always do so by the Archimedean Principle) then for all $x \in K \cap I_N$ we have

$$|x - z| < \frac{R}{2^{n-2}} < \varepsilon \implies x \in B(z,\varepsilon) \subseteq G_j$$

Hence $K \cap I_N \subseteq G_j$, but this contradicts our construction of I_n. Hence we conclude K is compact in $\mathbb{R}$ if and only if it is closed and bounded. □

2 Interior, Closure and Boundary

2.1 Definition

Definition 2.1.1 (Closure). *Let $A \subseteq \mathbb{X}$. Then we define the closure of A denoted $cl(A)$ by:*

$$cl(A) := \bigcap_{\substack{F \subseteq \mathbb{X} \\ F \text{ is closed in } \mathbb{X} \\ F \supseteq A}} F$$

Definition 2.1.2 (Interior). *Let $A \subseteq \mathbb{X}$. We define the interior of A denoted $int(A)$ by*

$$int(A) := \bigcup_{\substack{\Theta \subseteq \mathbb{X} \\ \Theta \text{ is closed in } \mathbb{X} \\ \Theta \subseteq A}} \Theta$$

Definition 2.1.3 (Boundary). *The boundary of $A \subseteq \mathbb{X}$ denoted ∂A is given by*

$$\partial A := \{x \in \mathbb{X} \mid \forall \varepsilon > 0 : B(x, \varepsilon) \cap A \neq \emptyset \wedge B(x, \varepsilon) \cap A^c \neq \emptyset\}$$

2.2 Characterization of Closure, Boundary and Interior

Theorem 2.2.1 (Characterization of Closure 1).

$$cl(A) = \{x \in \mathbb{X} \mid \forall \varepsilon > 0 : B(x, \varepsilon) \cap A \neq \emptyset\}$$

Proof. Let $E = \{x \in \mathbb{X} \mid \forall \varepsilon > 0 : B(x, \varepsilon) \cap A \neq \emptyset\}$. We first prove E is a closed set in X. To see this, consider $E^c = \{x \in \mathbb{X} \mid \exists \varepsilon > 0 : B(x, \varepsilon \cap A = \emptyset)\}$. Let $x \in E$, then there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap A = 0$. We wish to show there exists $r_1 > 0$ such that $B(x, r) \subseteq E^c$, i.e for $y \in B(x, r)$ there exists $r_2 > 0$ such that $B(y, r_2) \cap A = \emptyset$. Take $r_1 = r_2 = \varepsilon/2$. Then, note that $d(x, y) < \varepsilon/2$ and for all $z \in B(y, r_2)$ one has $d(y, z) < \varepsilon/2$. Hence, $d(x, z) \leq d(x, y) + d(z, y) = \varepsilon$. Hence, $B(y, r_2) \subseteq B(x, \varepsilon)$. But we assumed $B(x, \varepsilon) \cap A = \emptyset$ and thus $B(y, r_2) \cap A = \emptyset$ and hence $B(x, r_1) \subseteq E^c$ and E^c is open by Definition 1.1.1.

It remains to show E is the smallest closed set containing A. Let $F \subseteq \mathbb{X}$ be any closed set containing A, that is, $A \subseteq F$. Then $A^c \supseteq F^c$, and $F^c \cap A = O$. Thus, since F^c is open for any $x \in F^c$ there exists $r > 0$ such that $B(x, \varepsilon) \subseteq F^c$ and hence $B(x, \varepsilon) \cap A = \emptyset$ as well. Especially, $x \in E^c$ and hence $F^c \subseteq E^c$. It follows immediately that $E \subseteq F$. This is what we had to prove. ☐

Theorem 2.2.2 (Characterization of Closure 2). *Let $\mathbb{X}$ be a metric space and $S \subseteq \mathbb{X}$. Then the following are equivalent:*

1. *S is closed in $\mathbb{X}$.*
2. *$cl(S) = S$.*

Proof. (1 $\implies$ 2). Suppose S is closed, then $S \subseteq cl(S)$ is immediate. Let $x \in cl(S)$, then for all $\varepsilon > 0$ the set $B(x, \varepsilon) \cap S \neq \emptyset$. Take $\varepsilon = 1/n$ for $n \in \mathbb{N}$, and take $x_n \in B(x, 1/n) \cap S \neq \emptyset$, thereby constructing a sequence $(x_n)_{n=1}^{\infty} \subset S$. We claim $\lim x_n = x$, to see this note that $d(x_n, x) < 1/n$ and hence $(x_n) \to x$. But S is closed in $\mathbb{X}$, and (x_n) is a convergent sequence living in S. It follows from Theorem 1.2.2 that $x = \lim x_n \in S$. This shows $cl(S) \subseteq S$, whence $S = cl(S)$.

(2 $\implies$ 1). Suppose $cl(S) = S$. $cl(S)$ is closed by definition and hence S is closed by equality. □

Theorem 2.2.3 (Characterization of Interior/Boundary). *Let $\mathbb{X}$ be a metric space and let $A \subseteq \mathbb{X}$. Then, $int(A) \cup \partial A = cl(S)$.*

Proof. Let $x \in int(A) \cup \partial A$. $x \in int(A) \implies x \in A \implies x \in cl(A)$. On the other hand, $x \in \partial A$ implies by Definition 2.1.3 that for all $\varepsilon > 0$ both $B(x, \varepsilon) \cap A \neq \emptyset$ and $B(x, \varepsilon) \cap A^c \neq \emptyset$. In particular, $B(x, \varepsilon) \cap A \neq \emptyset$ which implies by Theorem 2.2.1 that $x \in cl(A)$.

Conversely let $x \in cl(A)$. By Theorem 2.2.1 for all $\varepsilon > 0$ one has that $B(x, \varepsilon) \cap A \neq \emptyset$. We distinguish two possible cases:

1. $x \in cl(A^c)$. If this is the case, then for all $\varepsilon > 0$ implies $B(x, \varepsilon) \cap A^c \neq \emptyset$, thus $x \in \partial A$.

2. $x \notin cl(A^c)$.In this case then by Theorem 2.2.1 there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap A^c = \emptyset$. In fact, this implies $B(x, \varepsilon) \subseteq A$. But $B(x, \varepsilon)$ is open and contained by A, whence $x \in int(A)$.

In either case $x \in int(A) \cup \partial A$. Thus $cl(A) \subseteq int(A) \cup \partial A$. This concludes the proof. □

2.3 Nesting and Intersection Property of Compact Sub-spaces

Theorem 2.3.1 (Nesting Property of Compact Sets). *Let $\mathbb{X}$ be a metric space, and let $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots$ be a sequence of non-empty compact subsets of $\mathbb{X}$. Then*

$$\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$$

Proof. We argue by contradiction. Suppose $\cap_{n \in \mathbb{N}} K_n = \emptyset$. Then

$$K_1 \cap \left(\bigcap_{n=2}^{\infty} K_n \right) = \emptyset$$

$$\implies K_1 \subseteq \left(\bigcap_{n=2}^{\infty} K_n \right)^c = \bigcup_{n=2}^{\infty} K_n^c$$

By Theorem 1.3.1 each K_n is closed and bounded, it follows then from Definition 1.1.2 that each K_n^c is open in $\mathbb{X}$. Thus, $\{K_n^c\}_{n=2}^{\infty}$ forms an open-cover of K_1 which is assumed to be compact. Thus, we may pickup a finite-subcover, such that $K_1 \subseteq \{K_n\}_{n=2}^{N}$. But $K_{n+1} \subseteq K_n \iff K_n^c \subseteq K_{n+1}^c$ which implies

$$K_1 \subseteq K_N^c \iff K_1 \cap K_N = \emptyset$$

Contradiction. □

Corollary 2.3.1 (Nested Interval Property). *Let I_n be a nested sequence of closed and bounded intervals in $\mathbb{R}$. Then*

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Theorem 2.3.2. *Let $K_1 \cap K_2 \cap ... \cap K_n$ be non-empty for any finite n. Suppose K_n is also compact in $\mathbb{X}$ for any n. Then,*

$$\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$$

Proof. The proof is identical to Theorem 2.3.1. □

2.4 Inheritance of Compactness and Completeness

Theorem 2.4.1 (Weak Inheritance of Compactness). *Let $\mathbb{X}$ be a metric space and let $K \subseteq \mathbb{X}$ be compact. If $S \subseteq K$ is closed, then S is compact in $\mathbb{X}$.*

Proof. Let $\{G_i\}_{i \in I}$ be any open cover of K. We will show Definition 1.3.1 holds. By definition $S^c = \mathbb{X} \setminus S$, thus $S \cup S^c = \mathbb{X}$. Since $S \subseteq \{G_i\}_{i \in I}$ we have $\mathbb{X} = \{\{G_i\}_{i \in I}, S^c\}$. Indeed, S is closed and S^c is open in $\mathbb{X}$. Thus this is a collection of open sets in $\mathbb{X}$. Since $K \subseteq X$ we have $K \subseteq \{\{G_i\}_{i \in I}, S^c\}$ and hence $\{\{G_i\}_{i \in I}, S^c\}$ is an open cover of K. By compactness of K we can pick up a finite subcover such that

$$K \subseteq G_1 \cup ... G_n \cup S^c$$

Since $S \subseteq K$:

$$\implies S \subseteq G_1 \cup ... G_n \cup S^c$$

Clearly, $S \cap S^c = \emptyset$ and hence

$$S \subseteq G_1 \cup ... \cup G_n$$

Thus, we have found a finite subcover of S. □

Theorem 2.4.2 (Inhertiance of Completeness). *Let $\mathbb{X}$ be a complete metric space and $K \subseteq \mathbb{X}$ closed. Then (K, d) is a complete metric space where d is the metric that $\mathbb{X}$ is equipped with.*

Proof. Let (x_n) be a Cauchy sequence in K. Then (x_n) is also a Cauchy sequence in $\mathbb{X}$, which is assumed to be complete and hence (x_n) converges to some $x \in \mathbb{X}$. But by Theorem 1.2.2 $x \in K$. □

3 Sequential Compactness

3.1 Definition and Example

Definition 3.1.1 (Sequential Compactness). *Let $\mathbb{X}$ be a metric space and $K \subseteq \mathbb{X}$. K is called sequentially compact if every sequence (x_n) has a subsequence $(x_{n_k}) \to x \in K$.*

Example 3.1.1. *Let Z be any closed and bounded set in $\mathbb{R}$. Then by Bolzano-Weirstrauss Z is sequentially compact.*

Lemma 3.1.1. *Let (x_n) be a Cauchy sequence in $\mathbb{X}$, then if (x_n) has a subsequence that converges to $x \in \mathbb{X}$ then the entire sequence converges to x.*

Proof. Let $\varepsilon > 0$ be given, then since $(x_{n_k}) \to x$ there exists $N_1 \in \mathbb{N}$ such that for all $n_k \geq N_1$ we have $d(x_{n_k}, x) < \varepsilon/2$. Similarly since (x_n) is Cauchy there exists $N_2 \in \mathbb{N}$ such that for all $n, m \geq N_2$ we have $d(x_n, x_m) < \varepsilon/2$. Now, define $N := \max\{N_1, N_2\}$ and take $n, n_k \geq N$. Then,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Theorem 3.1.1 (Completeness of Sequentialy Compact Metric Spaces). *Let $\mathbb{X}$ be a sequentially compact metric space. Then $\mathbb{X}$ is a complete metric space.*

Proof. Let (x_n) be a Cauchy sequence in $\mathbb{X}$, since $\mathbb{X}$ is sequentially compact there is a subsequence $(x_{n_k}) \to x \in \mathbb{X}$. By Lemma 3.1.1 this space is complete. □

Theorem 3.1.2 (Compactness impliess Sequential Compactness). *Let $\mathbb{X}$ be a metric space and let $K \subseteq \mathbb{X}$ be compact. Then K is sequentially compact.*

Proof. Let (x_n) be any sequence in K. For $n \in \mathbb{N}$ we define the set (and subsequence) $S_n := \{x_j \mid j \geq n\}$. It is trivial to show that $S_n \supseteq S_{n+1}$. Each $cl(S_n)$ is closed by definition and also a subset of K since K is closed in $\mathbb{X}$ by Theorem 1.3.1. Indeed, each $cl(S_n)$ is compact in $\mathbb{X}$ by Theorem 2.4.1, whence it follows from Theorem 2.3.1

$$\bigcap_{n=1}^{\infty} cl(S_n) \neq \emptyset$$

Pick x in this intersection, then $x \in cl(S_n)$ for all n. By Theorem 2.2.1 for all $\varepsilon > 0$, $B(x, \varepsilon) \cap S_n \neq \emptyset$. Select $x_{n_1} \in B(x, 1) \cap S_1$. Take $x_{n_2} \in B(x, 1/2) \cap S_2$ and so on, thereby constructing a subsequence (x_{n_k}) of (x_n) such that $d(x_{n_k}, x) < 1/k \leq 1/n_k$ for all n_k. Hence, $\lim x_{n_k} = x$. Since x_{n_k} is a convergent sequence in K which is compact and hence closed by Theorem 1.3.1 it follows from Theorem 1.2.2 that $x \in K$. Hence (x_n) has a convergent subsequence in K. □

3.2 Lebesgue Numbers and Finite Sequential Covers

Lemma 3.2.1 (Existence of Lebesgue Numbers). *Let $K \subseteq \mathbb{X}$ be sequentially compact. Then for any open cover of $\{G_i\}_{i \in I}$ K we have*

$$(\exists \delta > 0)(\forall x \in K)(\exists i \in I)(B(x, \delta) \subseteq G_i)$$

Remark: Such a value of δ is called a *Lebesgue* number of the cover.

Proof. By way of contradiction suppose not, then for some open cover $\{G_i\}_{i \in I}$

$$(\forall \delta > 0)(\exists x \in K)(\forall i \in I)(B(x, \delta) \not\subseteq G_i)$$

Take $\delta = 1/n$, then we can construct a sequence (x_n) in K such that for all $i \in I$ one has $B(x, 1/n) \not\subseteq G_i$. By sequential compactness of K there exists a subsequence $(x_{n_k}) \to x \in K$. Indeed, there exists some $j \in I$ such that $x \in G_j$, since this is an open set there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq G_j$. Indeed, for such an $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n_k \geq N$ one has both

1. $d(x_{n_k}, x) < \varepsilon/2$
2. $1/n_k < \varepsilon/2$

Let $n_k > N$, then for $z \in B(x_{n_k}, 1/n_k)$ we have

$$d(z, x) \leq d(z, x_{n_k}) + d(x_{n_k}, x) < \frac{1}{n_k} + \frac{\varepsilon}{2} < \varepsilon$$

Whence $B(x_{n_k}, 1/n_k) \subseteq B(x, \varepsilon) \subseteq G_j \implies B(x_{n_k}, 1/n_k) \subseteq G_j$. This contradicts our construction and hence our assumption that some cover does not have a Lebesgue Number. As a result, we conclude that the Lemma holds true. □

Lemma 3.2.2 (Finite Sequential Cover Lemma). *Let $K \subseteq \mathbb{X}$ be sequentially compact, then*

$$(\forall \delta > 0)(\exists \{x_1, ..., x_n\} \subset K)\,(K \subseteq B(x_1, \delta) \cup ... \cup B(x_n, \delta))$$

Proof. We argue by contradiction, suppose not. Then, for some $\delta > 0$ given any finite collection of elements in K $\{x_1, ..., x_n\}$ we have

$$B(x_1, \delta) \cup ... \cup B(x_n, \delta) \not\supseteq K$$

Fix such $\delta > 0$, then take $x_1 \in K$. Indeed, by hypothesis $K \not\subseteq B(x_1, \delta)$. Hence there exists $x_2 \in K$ such that $x_2 \notin B(x_1, \delta)$. Again, $K \not\subseteq B(x_1, \delta) \cup B(x_2, \delta)$. We proceed inductively, thereby constructing a sequence $(x_n) \subset K$ such that for all $n > m \in \mathbb{N}$

$$x_n \notin B(x_m, \delta)$$

By assumption, K is sequentially compact and hence has a convergent subsequence $(x_{n_k}) \to x \in K$. Especially this sequence is Cauchy in K. Taking $\varepsilon = \delta > 0$, we see that for all $k > m \geq N \in \mathbb{N}$ one has $d(x_{n_k}, x_{n_m}) < \delta$. Since $k > m$ it follows that

$$x_{n_k} \in B(x_{n_m}, \delta)$$

Contradiction. □

3.3 The Equivalence Theorem

Theorem 3.3.1 (Equivalence Theorem). *Let $\mathbb{X} \supseteq K$. The following are equivalent:*

1. *K is compact.*
2. *K is sequentially compact.*

Remark: These are **not** equivalent in topological spaces.

Proof. By Theorem 3.1.2 ($1 \implies 2$) is true. Thus we only show ($2 \implies 1$). Let $K \subseteq \mathbb{X}$ be sequentially compact and let $\{G_i\}_{i \in I}$ be any open cover of K. Then by Lemma 3.2.1 there exists $\delta > 0$ such that for any $x \in K$ $B(x, \delta) \subseteq G_{i_0}$ for some $i_0 \in I$. By Lemma 3.2.2 for such a delta there exists a finite collection $\{x_1, ..., x_n\} \subseteq K$ such that $K \subseteq B(x_1, \delta) \cup ... \cup B(x_n, \delta)$. Thus,

$$K \subseteq B(x_1, \delta) \cup ... \cup B(x_n, \delta) \subseteq G_{i_1} \cup ... \cup G_{i_n}$$

Hence we have found a finite sub-cover and K is compact in $\mathbb{X}$. □

4 Basic Heine-Borel in $\mathbb{R}^n$

Lemma 4.0.1. *Let $\mathbb{X} := (\mathbb{R}^d, \|\cdot\|_1)$ for $d \in \mathbb{N}$. A sequence $(x^{(n)})$ converges in $\mathbb{X}$ if and only if each component sequence $(x_k^{(n)})_{k=1}^d$ converges in $\mathbb{R}$.*

Proof. Let $(x^{(n)}) = (x_1^{(n)}, x_2^{(n)}, ..., x_d^{(n)})$ be a sequence in $\mathbb{R}^d$ that converges to a vector $x \in \mathbb{R}^d$. We claim each component sequence converges to the limit's respective component. To see this let $\varepsilon > 0$ be given, the for some $N \in \mathbb{N}$ one has for all $n \geq N$:

$$\sum_{k=1}^{d} \left| x_k^{(n)} - x_k \right| < \varepsilon$$

Indeed, for any $k \in [1, d] \cap \mathbb{N}$ we have for all $n \geq N$:

$$\left| x^{(n)_k} - x_k \right| < \varepsilon \implies \lim x_k^{(n)} = x_k \in \mathbb{R}$$

Conversely suppose the kth component of $(x_k^{(n)})$ converges to x_k. We show the sequence converges to $x = (x_1, ..., x_d)$. Indeed, for each $k \in [1, d] \cap \mathbb{N}$ we can find $N_k \in \mathbb{N}$ such that for all $n \geq N_k$ we have

$$\left| x_k^{(n)} - x_k \right| < \frac{\varepsilon}{d}$$

Take $N = \max_{i \in \{1,...,d\}} \{N_i\}$. Then for all $n \geq N$ one has:

$$\sum_{k=1}^{d} \left| x_k^{(n)} - x_k \right| < \sum_{k=1}^{d} \frac{\varepsilon}{d} = \varepsilon$$

Hence,

$$\lim_{n\to\infty} x^{(n)} = x$$

□

Theorem 4.0.1 (Basic Heine-Borel in $\mathbb{R}^n$). *Let $K \subseteq \mathbb{R}^n$ equipped with metric induced by the one norm $\|\cdot\|_1$. Then K is compact if and only if K is closed and bounded.*

Proof. One direction is always true by Theorem 1.3.1. By Theorem 3.3.1 it is sufficient to show K is sequentially compact.Let $(x^{(n)})$ be a sequence in $\mathbb{R}^d$. Then, $(x_1^{(n)})$ is a sequence in $\mathbb{R}$. Since $(x^{(n)})$ is bounded by M for any i we have $\left|x_i^{(n)}\right| \leq M$ which describes a closed and bounded subset of $\mathbb{R}$. Thus, by Bolzano-Weirstraus this sequence has a subsequence $x_1^{n_{j_1}} \to x_1 \in \mathbb{R}$. We look at the second component subsequence (along these same indices) $x_2^{n_{j_1}}$, indeed this is a sequence in $\mathbb{R}$ and hence by Bolzanno Weirstrauss since K is closed and bounded it has a subsequence that converges to $x_2 \in \mathbb{R}$. Along these indices our original subsequence $x_1^{n_{j_1}}$ stil converges to x_1 since it is a subsequence of a convergent sequence. Proceeding in this way we construct a subsequence with limit $x \in \mathbb{R}^d$.

□

5 Continuity on Metric Spaces

Definition 5.0.1. *Let $y \in \Omega \subseteq \mathbb{X}$, we say $f : \Omega \subseteq X \to \mathbb{Y}$ is continuous at y provided*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \Omega)(d_X(x, y) < \delta \implies d_Y(f(x), f(y) < \varepsilon)$$

5.1 Sequential Crtierion

Theorem 5.1.1 (Sequential Criterion for Continuity). *Let $f : \mathbb{X} \to \mathbb{Y}$ and $y \in \mathbb{X}$. TFAE:*

1. *f is continuous at y.*
2. *For any sequence (x_n) living in $\mathbb{X}$ that converges to y, the sequence $f(x_n)$ converges to $f(y)$.*

Proof. ($1 \implies 2$). Let $(x_n) \to y$ be a sequence in $\mathbb{X}$. Let $\varepsilon > 0$ be given, then by continuity of f at y there exists $\delta > 0$ such that for any $x \in \mathbb{X}$ with $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$. Since $(x_n) \to y$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ one has $d_X(x_n, y) < \delta$. Then, for all $n \geq N$ $d_Y(f(x_n), f(y)) < \varepsilon$ and hence $\lim f(x_n) = f(y)$.

($2 \implies 1$). Suppose (2) holds but (1) fails. So, there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists some $x \in \mathbb{X}$ such that $d_X(x, y) < \delta$ but $d_Y(f(x), f(y)) \geq \varepsilon$. Fix such $\varepsilon > 0$ and take $\delta = 1/n$, thus constructing a sequence (x_n) living in K such that for $n \in \mathbb{N}$: $d_X(x_n, y) < 1/n$. Clearly, $(x_n) \to y$, whence by hypothesis $\lim f(x_n) = f(y)$. So for ε there exists $M \in \mathbb{N}$ such that $n \geq M$ implies $d_Y(f(x_n), f(y)) < \varepsilon$. Contradiction. □

5.2 Topological Criterion

Lemma 5.2.1. *Let $f : \mathbb{X} \rightarrowtail \mathbb{Y}$ and $x \in \mathbb{X}$. TFAE*

1. *f is continuous at x.*
2. *For any open set $V \subset \mathbb{Y}$ containing $f(x)$ there exists an open set $U \subseteq \mathbb{X}$ such that $f(U) \subseteq V$.*

Proof. ($1 \implies 2$). Suppose f is continuous at $x \in X$. Let $V \subset Y$ be some open set containing $f(x)$. Then, since V is open we can find $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq V$. By continuity of f on X, given such an $\varepsilon > 0$ we can select $\delta > 0$ such that for any $y \in X$ with $d_X(x, y) < \delta$ implies $d_Y(f(y), f(x)) < \varepsilon$. But $d_Y(f(y), f(x)) < \varepsilon$ implies $f(y) \in B(f(x), \varepsilon) \subseteq V \implies f(y) \in V$. Consider now the open ball $B(x, \delta) \subseteq X$. For all $y \in B(x, \delta)$ one has $d_X(y, x) < \delta$ which implies $f(y) \in V$ by what was previously shown. Let $U = B(x, \delta)$ which is open. Then for any $y \in U$ it follows $f(y) \in V$ and hence $f(U) \subseteq V$.

($2 \implies 1$). Let $\varepsilon > 0$ be given and fix $x \in X$, we show there exists $\delta > 0$ such that for all $y \in X$ with $d_X(y, x) < \delta \implies d_Y(f(y), f(x)) < \varepsilon$. Now, $V = B(f(x), \varepsilon)$ is an open set in Y containing $f(x)$, so our hypothesis in (2) implies that we can find an open set U in X containing x such that $f(U) \subseteq V$. Now since $x \in U$ and U is open, there exists $\delta > 0$ such that $B(x, \delta) \subseteq U \implies f(B(x, \delta)) \subseteq V = B(f(x), \varepsilon)$. So $y \in B(x, \delta) \implies f(y) \in B(f(x), \varepsilon)$, i.e $d_Y(f(y), f(x)) < \varepsilon$. However, $d_X(y, x) < \delta$ implies $y \in B(x, \delta)$ which implies $d_Y(f(y), f(x)) < \varepsilon$. Since we chose $\varepsilon > 0$ arbitrarily it follows that

$$(\forall \varepsilon > 0)(\exists \delta > 0)(y \in X)(d_X(y, x) < \delta \implies d_Y(f(y), f(x) < \varepsilon)$$

□

Theorem 5.2.1 (Topological Criterion for Continuity). *Let $f : \mathbb{X} \rightarrowtail \mathbb{Y}$ and $x \in \mathbb{X}$. TFAE*

1. *f is continuous.*
2. *For any open set $V \subset \mathbb{Y}$ the set $f^{-1}(V) = \{x \in \mathbb{X} \mid f(x) \in V\}$ is open in $\mathbb{X}$.*

Proof. ($1 \implies 2$). Suppose f is continuous on X and let $V \subset Y$ be an open set. If $f^{-1}(V) = \emptyset$ then there is nothing to show. So, suppose $f^{-1}(V) \neq \emptyset$. Let $c \in f^{-1}(V)$, hence $f(c) \in V$, and V is assumed to be open in (Y, d_Y). So by Lemma 5.2.1 we can find an open set U_c containing c in (X, d_X) such that $f(U_c) \subseteq V$. Or equivalently, $U_C \subseteq f^{-1}(V)$. Moreover, since this holds for any $c \in f^{-1}(V)$ we have:

$$\bigcup_{c \in f^{-1}(V)} U_c \subseteq f^{-1}(V) \tag{1}$$

However by construction of our collection $\{U_c\}_{c\in f^{-1}(V)}$ it follows from the fact that $c \in f^{-1}(V) \implies c \in U_c$

$$\bigcup_{c\in f^{-1}(V)} U_c \supseteq f^{-1}(V) \tag{2}$$

Denote $\cup_{c\in f^{-1}(V)} U_c$ by U. Combining (1) and (2) we see that $f^{-1}(V) = U$. Since each U_c is open by Lemma 5.2.1, and U is the union of open sets and hence open itself, $f^{-1}(V)$ must also be open by equality.

$(2 \implies 1)$. Conversely suppose for any open set $V \subset Y$ the set $f^{-1}(V)$ is open in (X, d_X). Let $c \in X$, and let $V \subset Y$ be any open set containing $f(c)$. This is an open set in Y and hence the hypothesis implies the set $U = f^{-1}(V)$ containing c is open in X. Thus, $f(U) = V$, which implies $f(U) \subseteq V$. By Lemma 5.2.1 f is continuous at c, and since c was chosen arbitrarily we have that f is continuous on X. □

5.3 Preservation of Compactness and Extrema Theorem

Theorem 5.3.1 (Preservation Theorem). *Let $\mathbb{X}, \mathbb{Y}$ be metric spaces equipped with $d_X : X \to \mathbb{R}_+$ and $d_Y : Y \to \mathbb{R}_+$ respectively. Let $f : \mathbb{X} \to \mathbb{Y}$ be continuous, then if $K \subseteq \mathbb{X}$ is compact so is $f(K)$ in $\mathbb{Y}$.*

Proof. Let (y_n) be a sequence in $f(K)$, then for each y_n there exists $x_n \in K$ such that $f(x_n) = y_n$. Especially, this defines a sequence living in K which is assumed to be compact and hence sequentially compact by Theorem 3.1.2. Hence we can find a subsequence $(x_{n_k}) \to x \in K$. By continuity of f over K, $\lim f(x_{n_k}) = f(x) \in f(K)$. Moreover, $f(x_{n_k}) = y_{n_k} \to y = f(x)$ is a subsequence of our original sequence (y_n). Hence, $f(K)$ is sequentially compact in $\mathbb{Y}$ and compact by Theorem 3.3.1. □

Theorem 5.3.2. *Let $f : \mathbb{X} \to \mathbb{R}$ be continuous, if $K \subseteq \mathbb{X}$ is compact, then $f(K)$ has a supremum and infimum in $f(K)$.*

Proof. By the previous Theorem $f(K)$ is compact in $\mathbb{R}$, and is hence closed and bounded by Theorem 1.3.1. Indeed, by completeness of $\mathbb{R}$ there exists $m = \inf\{f(K)\}$ and $M = \sup\{f(K)\}$ in $\mathbb{R}$. We claim $m \in f(K)$ and $M \in f(K)$. To see this, consider M, and let $\varepsilon = 1/n$, by property of infimum, there exists $y_n \in f(K)$ such that $M1/n < y_n \leq M$. By the squeeze theorem, $\lim y_n = y = M$. But (y_n) lives in $f(K)$ whence we have $y \in f(K)$ by Theorem 1.2.2. □

Corollary 5.3.1. *Let $f : \mathbb{X} \to \mathbb{R}$ be continuous and $K \subseteq \mathbb{X}$ be compact, then f attains absolute max and min on K.*

5.4 Uniform Continuous Mappings

Theorem 5.4.1. *Let $f : K \subseteq \mathbb{X} \to \mathbb{Y}$ be continuous and K a compact set. Then, f is uniformly continuous on K. Namely*

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x, y \in K)(d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Proof. We argue by contradiction and hence assume it's negation. That is, for some $\varepsilon > 0$ one has for all $\delta > 0$ some pair $(x, y) \subset K$ such that $d_X(x, y) < \delta$ but $d_Y(f(x), f(y)) \geq \varepsilon$. We fix such $\varepsilon > 0$ and take $\delta = 1/n$ for $n \in \mathbb{N}$. Thus constructing two sequences $\{(x_n), (y_n)\}$ in K. Indeed, K is compact in $\mathbb{X}$ and by *The Equivalence Theorem* (3.3.1) K is sequentially compact and hence (x_{n_k}) and (y_{n_k}) are convergent subsequences of each respective parent. Let x denote the limit of x_{n_k}, we claim $(y_{n_k}) \to x$ as well. Let $\varepsilon > 0$ be given, then there exists $N_1 \in \mathbb{N}$ such that for all $n_k \geq N_1$ one has $d_X(x_{n_k}, x) < \varepsilon/2$. Take $N_2 \in \mathbb{N}$ such that $1/N_2 < \varepsilon/2$, indeed for all $n_k \geq N_2$ we also have $1/n_k < \varepsilon/2$. Now consider for $n_k \geq N := \max\{N_1, N_2\}$

$$d_X(y_{n_k}, x) \leq d_X(y_{n_k}, x_{n_k}) + d_X(x_{n_k}, x) < \frac{1}{n_k} + \frac{\varepsilon}{2} < \varepsilon$$

Hence, $\lim y_{n_k} = x$. By continuity of f over K, $\lim f(x_{n_k}) = f(x)$ and $\lim f(y_{n_k}) = f(x)$. Thus by the same argument as above there exists $N \in \mathbb{N}$ such that for all $n_k \geq N$ we have for any $\varepsilon > 0$:

1. $d_Y(f(x_{n_k}), f(x)) < \varepsilon/2$
2. $d_Y(f(y_{n_k}), f(x)) < \varepsilon/2$

For such n_k:

$$d_Y(f(x_{n_k}), f(y_{n_k})) \leq d_Y(f(x_{n_k}), f(x)) + d_Y(f(x), f(y_{n_k})) < \varepsilon$$

Contradiction. □

Theorem 5.4.2 (Preservation of Cauchy Sequences). *Uniformly continuous maps send Cauchy sequences to Cauchy sequences in any metric space.*

Proof. Let $\mathbb{X} = (X, d_X), \mathbb{Y} = (Y, d_Y)$ be two metric spaces and $f : \mathbb{X} \to \mathbb{Y}$ a uniformly continuous function. Let (x_n) be a Cauchy sequence in $\mathbb{X}$, we will show $f(x_n)$ is Cauchy in $\mathbb{Y}$.

Let $\varepsilon > 0$ be given, since f is uniformly continuous, one has the existence of some $\delta > 0$ such that given any $x, y \in X$, $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$. Fix such $\delta > 0$. Since (x_n) is Cauchy in $\mathbb{X}$, it follows that there exists $N \in \mathbb{N}$ such that $n, m \geq N$ implies $d_X(x_n, x_m) < \delta$. Thus, for $n, m \geq N$ we have $d_X(x_n, x_m) < \delta$ implies $d_Y(f(x_n), f(x_m)) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary we conclude that $(f(x_n))$ is Cauchy in $\mathbb{Y}$. □

5.5 Extension by Uniform Continuity

Definition 5.5.1. *Let $S \subseteq \mathbb{X}$, we say S is dense in $\mathbb{X}$ if $cl(S) = X$. Or, equivalently, if for all $x \in X$ there exists a sequence (x_n) in S that converges to x.*

Theorem 5.5.1. *Let $S \subseteq \mathbb{X}$ be a dense set, $\mathbb{Y}$ a complete metric space and $f : S \to \mathbb{Y}$ be a uniformly continuous function, there exists a unique uniformly continuous function $\overline{f} : \mathbb{X} \to \mathbb{Y}$ such that $\overline{f} = f(x)$ for all $x \in S$.*

Proof. Let us first prove uniqueness. Let $\overline{f}, \overline{g}$ be two uniform extension to f. Take $x \in X$ and let $(x_n) \to x$ be a sequence in $\mathbb{X}$. Then $\overline{f}(x_n) = f(x_n) = \overline{g}(x_n)$ for all x_n. Thus, since $\overline{f}, \overline{g}$ are continuous we must have

$$\overline{f}(x) = \lim f(x_n) = \overline{g}(x) \ \ \forall x \in X$$

Now we construct such a function.

Claim. Let (x_n) be a sequence in S with $\lim x_n = x \in \mathbb{X}$. The sequence $f(x_n)$ is Cauchy in $\mathbb{Y}$.

Claim. If $(x_n) \subseteq \mathbb{X}$ and $(x_n) \to x \in X$ the limit $\lim f(x_n)$ exists in $\mathbb{Y}$.

Claim. Let $x \in \mathbb{X}$ and $(x_n), (y_n)$ be two sequence in S that converge to x. Then $\lim f(x_n) = \lim f(y_n)$.

We now construct our extension. For each $x \in \mathbb{X}$ we select sequence (x_n) in S and set $\overline{f}(x) := \lim f(x_n)$. By a previous claim, this is well defined since it is independent of the choice of sequence. Moreover, if $x \in S$, since f is uniformly continuous

$$\lim f(x_n) = f(x)$$

So, $\overline{f}(x) = f(x)$ by contstruction of $\overline{f}$. Thus, $\overline{f}$ is an extension of f to $\mathbb{X}$. We now show that $\overline{f}$ is uniformly continous on $\mathbb{X}$. Let $\varepsilon > 0$, since f is uniformly continuous on S, for z, z' and some $\delta > 0$ we have $d_X(z, z') < \delta$ implies $d_Y(f(z), f(z')) < \varepsilon/3$.

Let $x, y \in \mathbb{X}$ with $d_X(x, y) < \delta$, and take $(x_n) \to x, (y_n) \to y$ sequence in S that converge to x and y respectively. Then,

$$\overline{f}(x) = \lim f(x_n) \text{ and } \overline{f}(y) = \lim f(y_n)$$

Let $N > 0$ be such that for all $n \geq N$ one has both

1. $d_Y(f(x_n), \overline{f}(x)) < \varepsilon/3$
2. $d_Y(f(y_n), \overline{f}(y)) < \varepsilon/3$

Let N' be such that $\forall n \geq N'$ we have

1. $d_X(x_n, x) < \delta/2$
2. $d_X(y_, y) < \delta/2$

Indeed, for $n \geq N'$ we have $d_X(x_n, y_n) \leq d_X(x_n, x) + d_X(x, y_n) \leq d_X(x_n, x) + d_X(x, y) + d_X(y, y_n) < 2\delta$. Moreover, given $n \in \mathbb{N}$ we have

$$d_Y(\overline{f}(x), \overline{f}(y)) \leq d_Y(\overline{f}(x), f(x_n)) + d_Y(f(x_n), f(y_n)) + d_Y(f(y_n), \overline{f}(y)$$

Taking $n \geq \max\{N, N'\}$ it follows:

$$d_Y(\overline{f}(x), \overline{f}(y)) < \varepsilon$$

□

6 The General Heine-Borel Theorem

6.1 Norm Equivalence Lemmas

Lemma 6.1.1. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Then there exists $D > 0$ such that for all $x \in \mathbb{R}^n$*

$$\|x\| \leq \|x\|_1$$

Proof. Let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{R}^n$. Then given a vector $x \in \mathbb{R}^n$ we can express x as a linear combination of basis vectors

$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$$

$$\implies \|x\| = \|x_1 e_1 + x_2 e_2 + \ldots + x_n e_n\| \leq \|x_1 e_1\| + \ldots + \|x_n e_n\|$$
$$= |x_1| \|e_1\| + \ldots + |x_n| \|e_n\|$$

Pick $D := \max_{1 \leq i \leq n} \|e_i\|$. Thus,

$$\|x\| \leq D \sum_{i=1}^{n} |x_i| = D \|x\|_1$$

□

Lemma 6.1.2. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and take $x \in \mathbb{R}^n$. Then there exists $C > 0$ such that*

$$C \|x\|_1 \leq \|x\|$$

Proof. Let $S := \{x \in \mathbb{R}^n \mid \|x\|_1 = 1\}$. This sphere is bounded and closed in $\mathbb{R}^n$ with respect to the metric d_1 induced by $\|\cdot\|_1$. By the *Basic Heine-Borel* Theorem 4.0.1 S is compact. Consider the following map

$$f : \mathbb{R}^n \to \mathbb{R}, x \mapsto \|x\|$$

This function is continuous on $(\mathbb{R}, d_1)$, to see this fix $\varepsilon > 0$ and consider:

$$|\|x\| - \|y\|| \leq \|x - y\| \leq D \|x - y\|_1$$

take $\delta = \varepsilon/D$ and the result follows. By compactness of S, this function f attains an absolute minimum at some $x_\star \in S$. But $x_\star \in S \implies \|x_\star\|_1 = 1$. Hence $x_\star \neq 0$ and thus $\|x_\star\| \neq 0$. Set $C = \|x_\star\| > 0$.

Suppose $x \in \mathbb{R}^n$ with $x \neq 0$. Then

$$\frac{x}{\|x\|_1} \in S$$

since

$$\left\| \frac{x}{\|x\|_1} \right\|_1 = \frac{\|x\|_1}{\|x\|_1} = 1$$

Thus,

$$\left\| \frac{x}{\|x\|_1} \right\| \geq C$$

or equivalently $\|x\| \geq C \cdot \|x\|_1$. □

6.2 Proof of Heine-Borel

Proposition 6.2.1. *Let $\mathbb{X} = (\mathbb{R}^n, \|\cdot\|_1)$ and $\mathbb{Y} = (\mathbb{R}^n, \|\cdot\|)$. A sequence $(x_n) \subset \mathbb{R}^n$ converges in $\mathbb{X}$ if and only if it converges in $\mathbb{Y}$.*

Proof. Suppose $x_n \to x$ in $\mathbb{X}$. Let $\varepsilon > 0$ be given, take $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\|x_n - x\|_1 < \frac{\varepsilon}{D}$$

Where $D > 0$ is as in Lemma 6.1.1. In such a case, for all $n \geq N$

$$\|x_n - x\| \leq D \|x_n - x\|_1 < D \cdot \frac{\varepsilon}{D} = \varepsilon$$

Conversely suppose $x_n \to x$ in $\mathbb{Y}$. Then for any $\varepsilon > 0$ and $C > 0$ as in Lemma 6.1.2 there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$\|x_n - x\| < C\varepsilon$$

By this same lemma for any $x \in \mathbb{R}^n$ we have $\|x\|_1 \leq \frac{\|x\|}{C}$. Thus, for $n \geq N$

$$\|x_n - x\|_1 \leq \frac{\|x_n - x\|}{C} < \varepsilon$$

□

Theorem 6.2.1 (The n-Dimensional Heine-Borel Theorem)**.** *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$. Then K is compact in $(\mathbb{R}^n, \|\cdot\|)$ if and only if it is closed and bounded in this same metric space.*

Proof. One direction is true as always by Theorem 1.3.1. Thus we only assume K is closed and bounded. Let (x^n) be a sequence in $\mathbb{R}^n$. By Theorem 4.0.1 K is compact in $\mathbb{R}^n$ with respect to the one-norm and is hence also sequentially compact by Theorem 3.1.2. Thus this same sequence has a subsequence that converges to $x \in \mathbb{R}^n$ with respect to the one norm. By Lemmas 6.1.2 and 6.1.1 there exists $C, D \in \mathbb{R}_{>0}$ such that for any $x \in \mathbb{R}^n$

$$C \|x\|_1 \leq \|x\| \leq D \|x\|_1$$

Hence by Proposition 6.2.1 a sequence converges in $\mathbb{R}^n$ with respect to $\|\cdot\|$ if and only if it converges with respect to $\|\cdot\|_1$. Hence this same convergent subsequence must converge in $\mathbb{R}^n$ wrt to $\|\cdot\|$. Hence $(\mathbb{R}^n, \|\cdot\|)$ is sequentially compact and compact by Theorem 3.3.1. □

7 Banach Fixed Point Theorem

Definition 7.0.1. *Let $\mathbb{X}$ be a metric space, a function $f : X \to X$ is called a contraction on $\mathbb{X}$ if there exists $C \in (0, 1)$ such that for all $x, y \in \mathbb{X}$ one has $d(f(x), f(y)) \leq C(x, y)$.*

Theorem 7.0.1 (Banach Fixed Point Theorem). *Let $\mathbb{X} = (X, d)$ be a complete metric space and $f : X \to X$ a contraction. Then f has a unique fixed point in X.*

Proof. Let $x_0 \in X$ and set $x_{n+1} := f(x_n)$. Thus constructing a sequence (x_n) in $\mathbb{X}$. We claim this sequence is Cauchy, let us prove this. Consider $N > 0$ and $n, m > N$, wlog take $n > m$. Then

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_m) + ... + d(x_{m+1}, x_m)$$

For any k one has

$$d(x_k, x_{k-1}) \leq C^{k-1} d(x_1, x_0)$$

Which implies

$$d(x_n, x_m) \leq C^{n-1} d(x_1, x_0) + ... + C^m d(x_1, x_0)$$

$$= C^m d(x_1, x_0)(1 + ... + C^{n-m-1} = C^m d(x_1, x_0) \frac{1 - C^{n-m}}{1 - C} \leq C^m d(x_1, x_0) \frac{1}{1 - C}$$

Moreover since $m \geq N$ and $C \in (0, 1)$ we have

$$d(x_n, x_m) \leq \frac{C^N}{1 - C} d(x_1, x_0)$$

Since $\lim C^N = 0$ for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$ we have

$$C^N \frac{d(x_1, x_0)}{1 - C} < \varepsilon$$

Hence $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$ and (x_n) is indeed Cauchy. By completeness it converges to $x \in X$. Since f is a contraction it is uniformly continuous on X. Thus,

$$\lim f(x_n) = f(x)$$

by continuity. However we also have $\lim x_n = \lim x_{n+1} = x$. By uniqueness of limit $x = f(x)$.

We show this point is unique. Suppose not, then take such $x \neq y \in X$. Then $d(x, y) > 0$, but $d(x, y) = d(f(x), f(y)) \leq Cd(x, y)$. Since $d(x, y) > 0$ we have $C > 1$. Contradiction.
□

Example 7.0.1. *Let $X = C([0, 1])$. Fix kernel $k(x, y) \equiv 1$ for $(x, y) \in [0, 1] \times [0, 1]$. By the Banach Fixed Point theorem there exists a unique solution to:*

$$f(X) = \lambda \int_0^1 f(y)dy + g(x)$$