LEBESGUE CRITERION FOR RIEMANN INTEGRABILITY

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Definition 1. *A gauge is a function $f : \Omega \subseteq \mathbb{R} \rightarrowtail \mathbb{R}_{>0}$. Given a gauge and interval, a tagged partition $\dot{\mathcal{P}} = \{[x_{i-1,x_i}], u_i\}_{i=1}^n$ is said to be δ-fine if for all indices i*

$$[x_{i-1}, x_i] \subseteq [u_i - \delta(u_i), u_i + \delta(u_i)]$$

Lemma 1 (Existence Lemma)**.** *Let $[a, b]$ be an interval and $\delta : [a, b] \rightarrowtail \mathbb{R}_{>0}$ a gauge on $[a, b]$. There exists a partition $\dot{\mathcal{P}}$ of $[a, b]$ which is δ-fine.*

Proof. Let $\mathcal{E}$ be the set of all $x \in [a, b]$ such that there exists a δ-fine partition of $[a, x]$. First, let us note that $\mathcal{E} \neq \emptyset$. In order to see this, let $x \in [a.b]$ such that $a \leq x \leq a + \delta(a)$; clearly $\{[a, x], a\}$ is a δ-fine partition since

$$[a, x] \subseteq [a - \delta(a), a + \delta(a)]$$

Moreover, we note that the set $\mathcal{E}$ is bounded above by b. Hence, this subset of $\mathbb{R}$ has a supremum $\mathcal{S} \in \mathbb{R}$. Indeed, since a and b are lower and upper bounds on $\mathcal{E}$, it follows that $a \leq \mathcal{S} \leq b$. We will show that $\mathcal{S} \in \mathcal{E}$ and that $\mathcal{S} = b$, thereby ensuring the existence of a δ-fine partition of $[a, b]$. Let us prove this.

(1) Suppose $\mathcal{S} \notin \mathcal{E}$. Then, since $\delta(\mathcal{S}) > 0$ we can find $e \in \mathcal{E}$ such that

$$\mathcal{S} - \delta(\mathcal{S}) < e < \mathcal{S}$$

since $\mathcal{S} \notin \mathcal{E}$. Now, $e \in \mathcal{E}$ implies there exists a δ-fine partition of $[a, e]$. Let $\dot{\mathcal{P}}$ be such a δ-fine partition. We then set $\dot{\mathcal{Q}} = \dot{\mathcal{P}} \cup \{[u, \mathcal{S}], \mathcal{S}\}$. Note that $\dot{\mathcal{P}}$ is δ-fine and that

$$[u, \mathcal{S}] \subseteq [\mathcal{S} - \delta(\mathcal{S}), \mathcal{S} + \delta(\mathcal{S})]$$

and thus $\dot{\mathcal{Q}}$ is a δ-fine partition of $[a, \mathcal{S}]$. Thus, $\mathcal{S} \in \mathcal{E}$. Contradiction.

(2) We prove $\mathcal{S} = b$. Suppose not, then clearly, $\mathcal{S} < b$. We may thus select some $w \in [a, b]$ with $\mathcal{S} < \delta(\mathcal{S})$. Indeed, by the previous item we have that $\mathcal{S} \in \mathcal{E}$ and thus the existence of some δ-fine $\dot{\mathcal{P}}$ of $[a, \mathcal{S}]$. Now we consider the partition $\dot{\mathcal{Q}} = \dot{\mathcal{P}} \cup \{[\mathcal{S}, w], \mathcal{S}\}$. This partition is also δ-fine, to see this note that

$$[\mathcal{S}, w] \subseteq [\mathcal{S} - \delta(\mathcal{S}), \mathcal{S} + \delta(\mathcal{S})]$$

Since $w \leq b$ we have found a δ-fine partition of $[a, w]$, whence we have $w \in \mathcal{E}$, and since $\mathcal{S}$ is an upperbound we have

$$w \leq \mathcal{S}$$

Contradiction.

Thus, we have both $\mathcal{S} \in \mathcal{E}$ and $\mathcal{S} = b$. Hence $b \in \mathcal{E}$ and there exists a δ-fine partition of $[a, b]$. □

Theorem 1 (Riemann's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. The following are equivalent.*

1. *f is Riemann integrable on $[a, b]$.*
2. *For all $\epsilon > 0$ there exists a partition $\mathcal{W} = \{I_i\}_{i=1}^n$ of $[a, b]$ such that for any tagged partitions $\dot{\mathcal{Q}}$, $\dot{\mathcal{P}}$ with the same sub-intervals as $\mathcal{W}$ one has*
$$\left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| < \epsilon$$
3. *For all $\epsilon > 0$ there exists $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ with*
$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} \text{ and } M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$$
one has
$$\sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \epsilon$$

Proof. ($1 \implies 2$). Let $\epsilon > 0$, by Cauchy's First Criterion there exists $\delta > 0$ such that for any two tagged partition $\dot{\mathcal{Q}}, \dot{\mathcal{P}}$ with $\left\|\dot{\mathcal{Q}}\right\| < \delta$, $\left\|\dot{\mathcal{P}}\right\| < \delta$ implies
$$\left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| < \epsilon$$

Let $\mathcal{W}$ be any partition of $[a, b]$ with $\|W\| < \delta$. Then, for any tagged partition $\dot{\mathcal{P}}, \dot{\mathcal{Q}}$ with the same endpoints as $\mathcal{W}$ we must have $\left\|\dot{\mathcal{Q}}\right\| < \delta$ and $\left\|\dot{\mathcal{P}}\right\| < \delta$ whence by Cauchy's First Criterion it follows
$$\left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| < \epsilon$$

($2 \implies 3$). Let $\epsilon > 0$ be given, by our hypothesis in (2) we can select a partition $\mathcal{W}$ of $[a, b]$ such that for any two tagged partitions $\dot{\mathcal{P}}, \dot{\mathcal{Q}}$ with the same end points as $\mathcal{W} = \{[x_{i-1}, x_i]\}_{i=1}^n$ one has
$$\left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| < \frac{\epsilon}{2} \tag{1}$$
Indeed, f is bounded on $[a, b]$ and hence on any $[x_{i-1}, x_i]$, thus f achieves infimum and supremum on each sub-interval. We may thus set m_i and M_i as in (3). Thus, for a given $\epsilon > 0$ and i we can find $u_i, v_i \in [x_{i-1}, x_i]$ satisfying
$$f(u_i) < m_i + \frac{\epsilon}{4(b-a)} \text{ and } f(v_i) > M_i - \frac{\epsilon}{4(b-a)}$$
$$\iff -f(u_i) > -m_i - \frac{\epsilon}{4(b-a)} \text{ and } f(v_i) > M_i - \frac{\epsilon}{4(b-a)}$$

Whence we infer for $1 \leq i \leq n$:
$$f(v_i) - f(u_i) + \frac{\epsilon}{2(b-a)} > (M_i - m_i)$$
$$\implies \left(f(v_i) - f(u_i) + \frac{\epsilon}{2(b-a)}\right)(x_i - x_{i-1}) > (M_i - m_i)(x_i - x_{i-1})$$

Since i is arbitrary within $[1, n] \cap \mathbb{N}$ we may sum over such i preserving the inequality:

$$\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < \sum_{i=1}^{n}\left(f(v_i) - f(u_i) + \frac{\epsilon}{2(b-a)}\right)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n}\left(f(v_i) - f(u_i)\right)(x_i - x_{i-1}) + \frac{\epsilon}{2} = S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}}) + \frac{\epsilon}{2}$$

Where $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], v_i\}$ and $\dot{\mathcal{Q}} = \{[x_{i-1}, x_i], u_i\}$. However, both of these are tagged partitions with the same endpoints as $\mathcal{W}$, thus by hypothesis

$$\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}}) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

($3 \implies 1$). Fix $\epsilon > 0$ and let $\mathcal{P} = \{[x_{i-1}, x_i]\}$ be a partition satisfying our hypothesis in (3). Let $A_i := [x_{i-1}, x_i)$ for $1 \leq i \leq n-1$ and $A_n := [x_{n_1}, x_n]$. On each A_i set $\alpha := m_i$ and $\omega := M_i$. Hence, on $[a, b]$ we have that $\alpha(x) \leq f(x) \leq \omega(x)$. Moreover, since α, ω are step functions we can express them as follows:

$$\alpha(x) = \sum_{i=1}^{n} m_i \chi_{A_i} \quad \text{and} \quad \omega(x) = \sum_{i=1}^{n} M_i \chi_{A_i}$$

Indeed, these step functions are Riemann Integrable with integrals:

$$\int_a^b \alpha = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

$$\int_a^b \omega = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Moreover, by hypothesis in (3)

$$\int_a^b \omega - \alpha = \sum_{i=1}^{n}\left(M_i - m_i\right)\left(x_i - x_{i-1}\right) < \epsilon$$

Thus, by Cauchy's Second Criterion $f \in \mathcal{R}([a, b])$. □

Theorem 2 (Lebesgue)**.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\mathcal{Z}$ be the set of all discontinuity points of f on $[a, b]$. The following are equivalent.*

(1) *$f \in \mathcal{R}([a, b])$*
(2) *$\mathcal{Z}$ is a null set.*

Proof. ($1 \implies 2$). It is easy to show that f is discontinuous at x if and only if it's oscillation (denoted $w(f; x)$) is non-zero at this point. Thus, we may express $\mathcal{Z}$ as follows

$$\mathcal{Z} = \{x \in [a, b] \mid w(f; x) > 0\} \tag{2}$$

We now define for $k \in \mathbb{N}$ the following set, whereby we may express $\mathcal{Z}$ in an equivalent, but more convenient form

$$H_k := \{x \in [a,b] \mid w(f;x) \geq \frac{1}{2^k}\} \tag{3}$$

$$\mathcal{Z} = \bigcup_{k \in \mathbb{N}} H_k \tag{4}$$

By criterion (3) of the previous Theorem, there exists a partition $\mathcal{P} = \{x_{i-1}, x_i\}_{i=1}^n$ such that for M_i, m_i defined as in (3) we have

$$\sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \frac{\epsilon}{4^k} \tag{5}$$

Without loss of generality, we can show H_k is null for arbitrary $k \in \mathbb{N}$, since the union of 0-measure sets also has measure 0. Fix $k \in \mathbb{N}$ and consider those i for which $H_k \cap (x_{i-1}, x_i) \neq \emptyset$. Select $x \in H_k \cap (x_{i-1}, x_i)$, then there exists $r > 0$ such that $(x - r, x + r) \subseteq (x_{i-1, x_i}) \subseteq [a,b]$ and $w(f;x) \geq 1/2^k$. It can be easily shown that

$$\frac{1}{2^k} \leq w(f;x) = \inf_{\delta > 0}\{W(f;V_\delta(x))\} \leq W(f;V_r(x)) \leq W(f;[a,b]) = M_i - m_i$$

Which implies

$$\frac{1}{2^k} \leq M_i - m_i \tag{6}$$

We now consider the set $S := \{i \in \mathbb{N} \cap [1,n] \mid H_k \cap (x_{i-1}, x_i) \neq \emptyset\}$. Indeed, it is easily seen that H_k can be covered by

$$H_k \subseteq \bigcup_{i \in S} ((x_{i-1}, x_i) \cup \{x_{i-1}, x_i\})$$

$$\subseteq \bigcup_{i \in S} (x_{i-1}, x_i) \cup \bigcup_{j=1}^{n} \left(x_j - \frac{\epsilon}{4(n+1)}, x_j + \frac{\epsilon}{4(n+1)}\right)$$

We take note that S is a finite subset of $\mathbb{N}$ and $\{j\}$ is of countable order. Thus we have covered H_k with countably many open intervals. We now show they have length less than ϵ. First,

$$\frac{1}{2^k} \sum_{i \in S} \Delta x_i \leq \sum_{i \in S} (M_i - m_i)\, \Delta x_i < \frac{\epsilon}{4^k}$$

$$\implies \sum_{i \in S} \Delta x_i < \frac{\epsilon}{2^k} \leq \frac{\epsilon}{2}$$

Similarly,

$$\sum_{j=1}^{n} \frac{\epsilon}{2(n+1)} < \frac{\epsilon}{2}$$

Whence, we have that $\ell(H_k) = 0$ and hence $\ell(\mathcal{Z}) = 0$.

(2 $\implies$ 1). We will establish criterion (3) of Riemann's Criteria. Suppose now that $\mathcal{Z}$ has measure 0 and let $\epsilon > 0$ be given. We can hence find countably many *disjoint* open intervals $\{J_k\}_{k\in\mathbb{N}} = \{(a_k, b_k)\}_{k\in\mathbb{N}}$ such that

$$\mathcal{Z} \subseteq \bigcup_{k\in\mathbb{N}} J_k \quad \text{and} \quad \sum_{k\in\mathbb{N}} \ell(J_K) < \frac{\epsilon'}{2M} \tag{7}$$

$$\text{Where we bound } f \text{ by } M \in \mathbb{R}_{>0} \text{ and } \epsilon' := \frac{\epsilon}{b-a+1} \tag{8}$$

We now construct a gauge δ on $[a, b]$ as follows. We distinguish two cases.

(1) $x \notin \mathcal{Z}$. Then x is a continuity point of f. So, there exists $\delta(x) > 0$ such that $y \in [a, b]$ with $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon'/2$. Let us define

$$M_x := \sup_{y\in V_\delta(x)} |f(y)| \quad \text{and} \quad m_x := \inf_{y\in V_\delta(x)} |f(y)|$$

Then for $y, y' \in V_\delta(x)$ we have

$$|f(y) - f(y')| \leq |f(x) - f(y)| + |f(y') - f(y)| < \epsilon'$$

Whereby it follows $M_x - m_x = \sup_{y,y'\in V_\delta(x)} |f(y) - f(y')| \leq \epsilon'$

We select such $\delta(x)$ as our gauge-point.

(2) $x \in \mathcal{Z}$. Then, $x \in J_k$ for some unique index k. Since J_k is open, there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq J_k$. If this is the case, then $\Delta x \leq \ell(J_k)$. We take such a δ as out gauge-point.

In this way we have defined a gauge $\delta(x)$ on $[a, b]$. However, we may take $\delta/2$ as our gauge. By our existence lemma, we can find a partition $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], u_i\}_{i=1}^n$ that is δ-fine. For this partition set M_i, m_i as in (3) of Riemann's Criterion. We will show (3) holds true. Note that for each tag u_i we have exactly one of the following:

(1) $u_i \notin \mathcal{Z}$. In this case, then $[x_{i-1}, x_i] \subseteq [u_i - \delta(u_i)/2, u_i + \delta(u_i)/2] \cap [a, b] \subseteq V_{\delta(u_i)}(u_i)$. Whence,

$$M_i - m_i \leq M_{u_i} - m_{u_i} \leq \epsilon'$$

(2) $u_i \in \mathcal{Z}$. Then, $(x_{i-1}, x_i) \subseteq (u_i - \delta(u_i), u_i + \delta(u_i)) \subseteq J_k$, thus yielding $\Delta x_i \leq \ell(J_k)$.

Define $S := \{i \mid u_i \notin \mathcal{Z}\}$ and $S' := \{i \mid u_i \in \mathcal{Z}\}$. We write

$$\begin{aligned}
\sum_{i=1}^n (M_i - m_i)\Delta x_i &= \sum_{i\in S} (M_i - m_i)\Delta x_i + \sum_{i\in S'} (M_i - m_i)\Delta x_i \\
&\leq \epsilon' \sum_{i\in S} \Delta x_i + 2M \sum_{i\in S'} \ell(J_k) \leq \epsilon' \sum_{i=1}^n \Delta x_i + 2M \sum_{k=1}^\infty \ell(J_k) \\
&< \epsilon'(b-a) + 2M\left(\frac{\epsilon'}{2M}\right) = \epsilon'(b-a+1) = \epsilon
\end{aligned}$$

This is what we had to show. □

References.

(1) Bartle, Robert Gardner, and Donald R. Sherbert. Introduction to Real Analysis. Hoboken, NJ: Wiley, 2011. Print.