The Identity & Holomorphic Equivalence Theorems

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Theorem 1. *Let $\Omega \subseteq \mathbb{C}$ be a connected set and let $f : \overline{\Omega} \to \mathbb{C}$ be holomorphic in the domain Ω and continuous up to the boundary (unless $\Omega = \mathbb{C}$). Further assume that there exists a sequence $(\omega_k)_{k\in\mathbb{N}} \subset \Omega$ with infinitely many distinct values converging to some interior point $\omega \in \Omega$ so that $f(\omega_n) = 0$ for all indices n. Then, $f \equiv 0$ in all of Ω.*

Proof. Without loss of generality, by extracting a subsequence, we may assume that all the ω_n are distinct and none are equal to ω. Now, since $\omega \in \Omega$ is an interior point we may find $\rho > 0$ so that the ball $B(\omega, \rho) \subseteq \Omega$. Since f is holomorphic in Ω, we may take this ρ so that

$$f(z) = \sum_{n\in\mathbb{N}} a_n(z-\omega)^n, \quad \forall z \in B(\omega,\rho) \tag{1}$$

converges absolutely in this same ball. We now show that $f \equiv 0$ in this ball $B(\omega, \rho)$. To see this, we argue by contradiction and suppose that $f \not\equiv 0$ in the above ball. In this case, we may find a *lowest* index $m \in \mathbb{N}$ so that the coefficient $a_m \neq 0$. Hence, for appropriate $z \in \Omega$

$$f(z) = a_m(z-\omega)^m \left(1+\varphi(z)\right), \quad \varphi(z) \xrightarrow{z\to\omega} 0$$

In the above, observe that the $\varphi(z)$ is given by a series locally and hence is holomorphic in this same radius of convergence. Thus, since $(\omega_n) \to \omega$ and $\varphi(\omega_n) \to 0$ in n we may find $N \in \mathbb{N}$ so large that $\omega_N \in B(\omega,\rho)$ and $|\varphi(\omega_N)| < \frac{1}{2}$. For such an index:

$$0 = |f(\omega_N)| = |a_m|\,|\omega_N - \omega|\,|1+\varphi(\omega_N)| \geq \frac{|a_m|}{2}\,|\omega_N - \omega| \tag{2}$$

Implying since $\omega_N \neq \omega$ that $a_m = 0$, which is a contradiction. Hence, we conclude that $f \equiv 0$ in a small ball contained in Ω.

We now show that $f \equiv 0$ inside all of Ω. Indeed, define the auxiliary set

$$\mathcal{S} := \{z \in \Omega \mid f \equiv 0 \text{ in some neighbourhood of } z\} \tag{3}$$

It is obvious that $\mathcal{S}$ is open. We claim that $\mathcal{S}$ is closed as well. Otherwise, there would be a sequence of elements (φ_n) in $\mathcal{S}$ converging to $\varphi \notin \mathcal{S}$. However, the previous argument would imply that $f \equiv 0$ in a small ball around φ and hence that $\varphi \in \mathcal{S}$ by definition of $\mathcal{S}$.

Thus, $\mathcal{S}$ is clopen. We may then write $\Omega = \mathcal{S} \cup \mathcal{S}^c$ as the disjoint union of two open sets. But Ω is connected, and hence one of these sets must be empty. By hypothesis we know $\mathcal{S} \neq \emptyset$, whence we conclude $\mathcal{S}^c = \emptyset$ or equivalently $\mathcal{S} = \Omega$. Especially, we have $f \equiv 0$ in Ω.

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Corollary 2. *Let $f, g : \overline{\Omega} \subseteq \mathbb{C} \to \mathbb{C}$ be holomorphic in a connected open set Ω and suppose that $f \equiv g$ in a neighbourhood $\mathcal{U} \subset \Omega$. Then $f \equiv g$ in all of Ω.*

Proof. In $\mathcal{U}$ we need have $f - g \equiv 0$, where the difference is holomorphic in all of Ω. Hence there are infinitely many distinct points in Ω where $f - g$ vanishes and by the previous theorem we have that $f - g \equiv 0$ in all of Ω.

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