HEAT EQUATION WITH INITIAL DATA IN ONE SPATIAL DIMENSION

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In these notes we are interested in twice continuously differentiable solutions to the following initial value problem:

$$\begin{cases} \partial_t u(x,t) - \gamma \partial_{xx} u(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = \phi(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

In the above, ϕ is a given smooth function and $\gamma > 0$ is a constant. We shall give an explicit form for a C^2 solution to this PDE. However, we first require a technical lemma.

Lemma 1. *Define* $f(\xi) = e^{-a\xi^2}$ *for* $a > 0$. *Then*

$$\mathcal{F}^{-1}[f](x) = \frac{1}{\sqrt{4\pi a}} \exp\left(-\frac{x^2}{4a}\right).$$

Proof. By definition of the Fourier inverse,

$$\mathcal{F}^{-1}[f](x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-a\xi^2 + ix\xi} \, \mathrm{d}\xi.$$

Now,

$$\begin{aligned} -a\xi^2 + ix\xi = a\left(\xi^2 - \frac{ix}{a}\xi\right) = -a\left(\xi^2 - \frac{ix}{a}\xi - \frac{x^2}{4a^2} + \frac{x^2}{4a^2}\right) &= -a\left[\left(\xi - \frac{ix}{2a}\right)^2 + \frac{x^2}{4a^2}\right] \\ &= -a\left(\xi - \frac{ix}{2a}\right)^2 - \frac{x^2}{4a}. \end{aligned}$$

This implies that

$$\begin{aligned} \mathcal{F}^{-1}[f](x) = \frac{1}{2\pi} \exp\left(-\frac{x^2}{4a}\right) \int_{\mathbb{R}} e^{-a\left(\xi - \frac{ix}{2a}\right)^2} \, \mathrm{d}\xi = \frac{1}{2\pi} \exp\left(-\frac{x^2}{4a}\right) \int_{\mathbb{R}} e^{-a\zeta^2} \, \mathrm{d}\zeta \\ = \frac{1}{2\pi\sqrt{a}} \exp\left(-\frac{x^2}{4a}\right) \int_{\mathbb{R}} e^{-\sigma^2} \, \mathrm{d}\sigma \end{aligned}$$

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where we have made the substitution $\sigma := \sqrt{a}\zeta$ in this last line. Recalling now that

$$\int_{-\infty}^{\infty} e^{-\sigma^2}\,\mathrm{d}\sigma = \sqrt{\pi}$$

we find that

$$\mathcal{F}^{-1}[f](x) = \frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{x^2}{4a}\right),$$

as was asserted. □

Solving the Heat Equation. Now let u solve (1). Taking the Fourier transform of both sides, we obtain

$$\partial_t\widehat{u}(\xi,t) = \widehat{u_t}(\xi,t) = \gamma\widehat{u_{xx}}(\xi,t) = \gamma i^2\xi^2\widehat{u}(\xi,t) = -\gamma\xi^2\widehat{u}(\xi,t).$$

For fixed ξ this is an ODE in t which evaluates to

$$\widehat{u}(\xi,t) = A(\xi)e^{-\gamma\xi^2 t}.$$

Note that $\widehat{u}(\xi,0) = \mathcal{F}(u(x,0)) = \mathcal{F}(\phi(x)) = \widehat{\phi}(\xi)$ whence

$$\widehat{u}(\xi,t) = \widehat{\phi}(\xi)e^{-\gamma\xi^2 t}.$$

Define now

$$g(x) := \frac{1}{\sqrt{4\pi\gamma t}}e^{-\frac{x^2}{4\gamma t}};$$

by virtue of the previous lemma, we know that $\widehat{g}(\xi) = e^{-\gamma\xi^2 t}$. Hence, for (x,t) in the given domain:

$$u(x,t) = \left(\widehat{\phi}\cdot\widehat{g}\right)^{\vee} = \left(\widehat{\phi * g}\right)^{\vee} = (\phi * g)(x,t).$$

Of course, the convolution is taken with respect to the spatial variable x. Thus,

$$u(x,t) = \frac{1}{\sqrt{4\pi\gamma t}}\int_{-\infty}^{\infty}\phi(y)e^{-\frac{(x-y)^2}{4\gamma t}}\,\mathrm{d}y.$$