NOTES ON DISTRIBUTIONS AND SOLVED EXERCISES

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In a previous set of notes (http://cs.mcgill.ca/~echern2/repo/475dist.pdf, for more information) we introduced distributions on the real line and considered a notion of differentiation for these "generalized functions". In this text, we will continue this topic and study analogous mappings in higher dimensions and see what it means to satisfy a partial differential equation in the sense of distributions. Along the way, we provide some examples with detailed solutions.

1. Distributions in Higher Dimensions

We begin by fixing a space dimension $n \geq 1$ and letting $\mathcal{D}(\mathbb{R}^n)$ denote $C_c^\infty(\mathbb{R}^n)$: the space of all smooth functions of compact support $\phi : \mathbb{R}^n \to \mathbb{C}$. Note that $\mathcal{D}(\mathbb{R}^n)$ is a vector space (of infinite dimension) over the complex numbers. We "topologize" $\mathcal{D}(\mathbb{R}^n)$ by attaching the sup-norm

$$\|\cdot\|_\infty := \sup_{x \in \mathbb{R}^n} |\cdot(x)| .$$

This makes sense as every $\phi \in \mathcal{D}(\mathbb{R}^n)$ is continuous with compact support whence $\|\phi\|_\infty < \infty$. It is left as a simple exercise to check that this indeed defines a norm on $\mathcal{D}(\mathbb{R}^n)$.

1.1. Multi-indices.

A multi-index in $\mathbb{R}^n$ is a n-tuple $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where each coordinate α_j is a non-negative integer (i.e. $\alpha_j \in \mathbb{N}_0$). We also define

$$|\alpha| := \sum_{j=1}^{n} \alpha_j.$$

Now, suppose we are given a function $\phi \in \mathcal{D}(\mathbb{R}^n)$, we set

$$\partial^\alpha \phi(x) := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n} \phi(x).$$

Note that since ϕ is smooth, it does not matter in which order we write the differential operators, since they will commute. This notion of multi-indices will be of use when speaking about the derivatives of distribution in $\mathbb{R}^n$.

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1.2. **Distributions.** If $\{\phi_k\}_{k=1}^{\infty}$ is a sequence in $\mathcal{D}(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ we shall say that ϕ_k converges to ϕ (as $k \to \infty$) provided

(1) For every multi-index α,

$$\|\partial^\alpha \phi_k - \partial^\alpha \phi\| \xrightarrow{k\to\infty} 0;$$

(2) There exists a compact set K in $\mathbb{R}^n$ such that $\operatorname{supp}(\phi_k) \subseteq K$ for each $k \in \mathbb{N}$.

This notion of convergence in $\mathcal{D}(\mathbb{R}^n)$ allows us to consider continuous linear functionals on $\mathcal{D}(\mathbb{R}^n)$.

Definition 1. A distribution (or generalized function) is a continuous linear functional

$$\langle F, \cdot \rangle : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}.$$

By continuous, we mean that for every sequence $\{\phi_k\}_{k=1}^{\infty}$ in $\mathcal{D}(\mathbb{R}^n)$ converging to ϕ, we have

$$\lim_{k\to\infty} \langle F, \phi_k \rangle = \langle F, \phi \rangle .$$

We should also point out that "many" functions that we are familiar with induce a distribution in a canonical way. Certainly, let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and define a linear functional $\langle F_f, \cdot \rangle$ by setting

$$\langle F_f, \phi \rangle := \int_{\mathbb{R}^n} f(x)\phi(x)\,\mathrm{d}x, \quad \forall \phi \in \mathcal{D}(\Omega).$$

Proposition 1.1. *If $f : \mathbb{R}^n \to \mathbb{C}$ is locally integrable, then $\langle F_f, \cdot \rangle$ is a distribution.*

Proof. The fact that this map is a linear functional is immediate from familiar properties of the Lebesgue integral. It this remains only to check that this functional is continuous. Suppose that $\{\phi_k\}$ converges to ϕ in $\mathcal{D}(\mathbb{R}^n)$ and note then that

$$\langle F_f, \phi_k \rangle = \int_{\mathbb{R}^n} f(x)\phi_k(x)\,\mathrm{d}x.$$

It follows that

$$\begin{aligned}\left| \int_{\mathbb{R}^n} f(x)\phi_k(x)\,\mathrm{d}x - \int_{\mathbb{R}^n} f(x)\phi(x)\,\mathrm{d}x \right| &\le \int_{\mathbb{R}^n} |f(x)|\,|\phi_k(x) - \phi(x)|\,\mathrm{d}x \\ &= \int_K |f(x)|\,|\phi_k(x) - \phi(x)|\,,\end{aligned}$$

where K is some compact set containing the support of every ϕ_k. From the above, we obtain

$$|\langle F_f, \phi_k \rangle - \langle F_f, \phi \rangle| \le \|\phi_k - \phi\|_\infty \cdot \|f\|_{L^1(K)} .$$

Letting $k \to \infty$ proves the continuity of $\langle F_f, \cdot \rangle$. □

We are now in a position to discuss the differentiation of distributions. Let $\langle F, \cdot \rangle$ be a distribution, for any multi-index α we define (as a natural extension of the one dimensional case) a new distribution called the α-**derivative** of $\langle F, \cdot \rangle$, by

$$\langle \partial^\alpha F, \phi \rangle := (-1)^{|\alpha|} \langle F, \partial^\alpha \phi \rangle, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

It is easy to check that $\langle \partial^\alpha F, \cdot \rangle$ is itself a distribution. This brings us to what it means to be a "weak solution" to a partial differential equation.

Definition 2. Let L be a differential endomorphism of the space $\mathcal{D}(\mathbb{R}^n)$. We say a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a weak solution to the PDE $L[f] = g$ provided

$$\langle L[f], \phi \rangle = \langle g, \phi \rangle, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

That is, if $L[f] = g$ in the sense of distributions.

Remark 1. This is particularly convenient when considering solutions to homogeneous equations. A locally integrable function f is a weak solution to $L[f] = 0$ provided

$$\langle L[f], \phi \rangle = 0, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

1.3. **Solved Exercises.** This subsection consists of examples (with solutions) relating to the section above. These exercises are mostly taken from assignments in Math 475 (Honours PDE).

Exercise 1.1. *Let $g : \mathbb{R}^n \to \mathbb{C}$ be a smooth function and suppose $\langle F, \cdot \rangle$ is a distribution. We can define a distribution $\langle gF, \cdot \rangle$ by letting*

$$\langle gF, \phi \rangle := \langle F, g\phi \rangle, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

Show that $|\mathbf{x}|^2 \Delta\delta_0 = 6\delta_0$ in the sense of distributions (in $\mathbb{R}^3$).

Solution. Fix $\phi \in \mathcal{D}(\mathbb{R}^3)$ and note that

$$\langle |\mathbf{x}| \Delta\delta_0, \phi \rangle = \langle \Delta\delta_0, |\mathbf{x}| \phi \rangle = \frac{\partial^2}{\partial x_1^2} [\mathbf{x}\phi(\mathbf{x})]_{\mathbf{x}=0} + \cdots + \frac{\partial^2}{\partial x_3^2} [\mathbf{x}\phi(\mathbf{x})]_{\mathbf{x}=0} .$$

Fix an index j and compute

$$\frac{\partial}{\partial x_j} \left[\left(x_1^2 + x_2^2 + x_3^2 \right) \phi(\mathbf{x}) \right] = 2x_j \phi(\mathbf{x}) + |\mathbf{x}|^2 \phi_{x_j}(\mathbf{x}).$$

Thus,

$$\frac{\partial^2}{\partial x_j^2} \left[\left(x_1^2 + x_2^2 + x_3^2 \right) \phi(\mathbf{x}) \right] = 2\phi(\mathbf{x}) + 4x_j \phi_{x_j}(\mathbf{x}) + |\mathbf{x}|^2 \phi_{x_j x_j}(\mathbf{x}).$$

This means that

$$\frac{\partial^2}{\partial x_1^2}\left[\mathbf{x}\phi(\mathbf{x})\right]_{\mathbf{x}=0} + \cdots + \frac{\partial^2}{\partial x_3^2}\left[\mathbf{x}\phi(\mathbf{x})\right]_{\mathbf{x}=0} = 6\phi(\mathbf{x}).$$

This is $6\delta_0$ in the sense of distributions. □

Exercise 1.2. *We know from multivariate calculus that for a smooth function*

$$f : \mathbb{R}^2 \to \mathbb{C}$$

it is always the case that $f_{xy} = f_{yx}$, i.e. differential operators commute for smooth functions (but it is not necessarily the case if f is not $C^2(\mathbb{R}^2)$). Let $\langle F, \cdot \rangle$ be a distribution, show that for all $\phi \in \mathcal{D}(\mathbb{R}^2)$

$$\left\langle \partial_x \partial_y F, \phi \right\rangle = \left\langle \partial_y \partial_x F, \phi \right\rangle.$$

Solution. By definition,

$$\begin{aligned}\left\langle \partial_x \partial_y F, \phi \right\rangle = -\left\langle \partial_y F, \phi_x \right\rangle = \left\langle F, \phi_{xy} \right\rangle = \left\langle F, \phi_{yx} \right\rangle &= -\left\langle \partial_x F, \phi_y \right\rangle \\ &= \left\langle \partial_y \partial_x F, \phi \right\rangle.\end{aligned}$$

□

Exercise 1.3. *Let $\mathbf{u}(x, y) = (u_1(x, y), u_2(x, y))$ be a 2D vector field. What does is mean for $\mathbf{u}$ to be a solution to $\operatorname{div} \mathbf{u} = 0$, in the sense of distributions?*

Solution. This is precisely the statement that, for all $\phi \in \mathcal{D}(\mathbb{R}^n)$,

$$\begin{aligned}0 = \langle \operatorname{div} \mathbf{u}, \phi \rangle &= \left\langle \partial_x u_1(x, y) + \partial_y u_2(x, y), \phi \right\rangle \\ &= -\langle u_1, \phi_x \rangle - \left\langle u_2, \phi_y \right\rangle \\ &= -\int_{\mathbb{R}^2} u_1(x, y)\phi_x(x, y)\, \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^2} u_2(x, y)\phi_y(x, y)\, \mathrm{d}\mathbf{x} \\ &= -\int_{\mathbb{R}^2} \mathbf{u} \cdot \nabla\phi\, \mathrm{d}\mathbf{x}.\end{aligned}$$

□

Exercise 1.4. *Consider the Lebesgue measurable function*

$$f(x, y) := \begin{cases} 1, & \text{if } y \le x^3, \\ 0, & \text{if } y > x^3. \end{cases}$$

What is f_{xy} in the sense of distributions?

Solution. Let $\phi \in \mathcal{D}(\mathbb{R}^2)$ and calculate

$$\left\langle \partial_y \partial_x F_f, \phi \right\rangle = \left\langle F_f, \phi_{xy} \right\rangle = \iint_{\mathbb{R}^2} f(x,y)\phi_{xy}(x,y)\,\mathrm{d}A$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x^3} \phi_{xy}(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

The fundamental theorem of calculus then yields (since ϕ has compact support)

$$\int_{-\infty}^{x^3} \phi_{xy}(x,y)\,\mathrm{d}y = \phi_x(x,x^3).$$

This gives

$$\left\langle \partial_y \partial_x F_f, \phi \right\rangle = \int_{\mathbb{R}} \phi_x(x,x^3)\,\mathrm{d}x.$$

□

Exercise 1.5. *Let $f \in L^1_{loc}(\mathbb{R})$. Prove that the function*

$$u(x,y) := f(bx - ay)$$

is a solution to $au_x + b_y = 0$, in the sense of distributions.

Solution. We fix a function $\phi \in \mathcal{D}(\mathbb{R}^2)$ and let L denote the operator $a\partial_x + b\partial_y$. Then,

$$\langle LF_u, \phi \rangle = -\langle u, a\phi_x \rangle - \left\langle u, b\phi_y \right\rangle.$$

Now,

$$\langle u, a\phi_x \rangle = a \iint_{\mathbb{R}^2} u(x,y)\phi_x(x,y)\,\mathrm{d}A = a \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(bx-ay)\phi_x(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= -ab \iint_{\mathbb{R}^2} f'(bx-ay)\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

On the other hand,

$$\left\langle u, b\phi_y \right\rangle = b \iint_{\mathbb{R}^2} u(x,y)\phi_y(x,y)\,\mathrm{d}A = b \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(bx-ay)\phi_y(x,y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= ab \iint_{\mathbb{R}^2} f(bx-ay)\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

The claim follows. □

2. Harmonic Functions

This section comprises of some results regarding harmonic functions in $\mathbb{R}^n$. We state some powerful results, such as the mean value property and maximum principles. However, we shall not prove these results. The proofs can be found here. Recall that in $\mathbb{R}^n$ the Laplacian is the linear differential (elliptic) operator given by

$$\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

A $C^2(\mathbb{R}^n)$ function u is then called **harmonic** if $\Delta u \equiv 0$. Unless stated otherwise, Ω will denote a domain in $\mathbb{R}^n$. Recall that a domain is a non-empty, bounded, open and connected set. We say u is harmonic in Ω if $\Delta u \equiv 0$ in Ω.

Theorem 2.1 (Mean Value Property)**.** *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. The following statements are equivalent.*

(1) u is harmonic in Ω;
(2) For all open balls $B(\mathbf{x}, \rho)$ whose closure is contained in Ω, there holds

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},\rho)} u(\mathbf{y}) \, \mathrm{d}\mathbf{S} = \fint_{B(\mathbf{x},\rho)} u(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

We recall the averaging notation '$\fint$'. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $E \in \mathfrak{M}$ a set of positive (but finite) measure. If $f : X \to \mathbb{C}$ is measurable, we define

$$\fint_E f(x) \, \mathrm{d}\mu := \frac{1}{\mu(E)} \int_E f(x) \, \mathrm{d}\mu.$$

2.1. The Maximum Principles.

We continue to state fundamental properties regarding harmonic functions. Our last main results are the strong and weak maximum principles.

Theorem 2.2 (Strong Maximum Principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω. If u achieves a local extremum in Ω, then u is constant on $\overline{\Omega}$.*

As a corollary we obtain the weak maximum principle.

Corollary 2.3 (Weak Maximum Principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω. Then*

$$\max_{\mathbf{x} \in \overline{\Omega}} u(\mathbf{x}) = \max_{\mathbf{x} \in \partial\Omega} u(\mathbf{x}).$$

In what follows, we will solve some classic exercises related to these properties.

2.2. **Solved Exercises.** The following problems are in some sense "classical", and are often given as exercises to students.

Exercise 2.1. *Assuming the mean value property over the boundary of balls in* $\mathbb{R}^3$*, prove the mean value property over solid balls in* $\mathbb{R}^3$*.*

Solution. Let $\Omega \subset \mathbb{R}^3$ be a domain and suppose $B(\mathbf{x}, \rho)$ is compactly supported in Ω. In addition, suppose u is continuous on $\overline{\Omega}$ and that u satisfies the mean value property for the boundary of all such balls. Then,

$$\int_{B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_0^\rho \left(\int_{\partial B(\mathbf{x},\varepsilon)} u(\mathbf{y})\,\mathrm{d}S \right) \mathrm{d}\varepsilon. \tag{2.1}$$

However, by assumption on u:

$$\int_{\partial B(\mathbf{x},\varepsilon)} u(\mathbf{y})\,\mathrm{d}S = 4\pi\varepsilon^2 ⨍_{\partial B(\mathbf{x},\varepsilon)} u(\mathbf{y})\,\mathrm{d}S = 4\pi\varepsilon^2 u(\mathbf{x}).$$

Using this in (2.1) gives

$$\int_{B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}\mathbf{y} = 4\pi u(\mathbf{x}) \int_0^\rho \varepsilon^2\,\mathrm{d}\varepsilon = \frac{4\pi\rho^3}{3} u(\mathbf{x}).$$

It follows that

$$⨍_{B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}\mathbf{y} = u(\mathbf{x}).$$

□

Exercise 2.2. *Suppose* $u(r,\theta)$ *is harmonic inside the open disk* $B(0,2)$ *and continuous on* $\overline{B(0,2)}$*. Suppose in addition that we have* $u(r,\theta) = 3\sin(2\theta) + 1$ *on the circle*

$$\gamma := \left\{ (x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} = 2 \right\}.$$

Determine each of the following:

(1) The maximum value of u *on the closed disk* $\overline{B(0,2)}$*,*
(2) The value of $u(0)$*.*

Solution. We handle each part separately.

(1) An application of the weak maximum principle shows that

$$\max_{\mathbf{x}\in\gamma} u(\mathbf{x}) = \max_{\mathbf{x}\in\overline{B(0,2)}} u(\mathbf{x}).$$

Since $u(r,\theta)$ clearly achieves a maximum of 4, we obtain that the maximum of u on $\overline{B(0,2)}$ is 4.

(2) Let $n \in \mathbb{N}$ be given and consider the circle

$$\Gamma_n := \left\{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} = 2 - \frac{1}{n}\right\} \subset B(0,2).$$

By the mean value property,

$$u(\mathbf{0}) = \fint_{\Gamma_n} u(\mathbf{y})\,\mathrm{d}\mathbf{S}.$$

By continuity of u on $\overline{B(0,2)}$, we may apply dominated convergence to find that

$$\fint_{\gamma} u(\mathbf{y})\,\mathrm{d}\mathbf{S} = \lim_{n\to\infty} \fint_{\Gamma_n} u(\mathbf{y})\,\mathrm{d}\mathbf{S} = \lim_{n\to\infty} u(\mathbf{0}) = u(\mathbf{0}).$$

We may now directly compute this left hand integral:

$$\begin{aligned}\fint_{\gamma} u(\mathbf{y})\,\mathrm{d}\mathbf{S} &= \frac{1}{2\pi\cdot 2}\int_0^{2\pi} (3\sin(2\theta)+1)\ \mathrm{d}\theta \\ &= \frac{1}{4\pi}\left[-\frac{3}{2}\sin(2\theta)\big|_0^{2\pi} + 2\pi\right] \\ &= \frac{1}{2}.\end{aligned}$$

This yields $u(\mathbf{0}) = 1/2$.

□

ADDENDUM

In this section we explore some additional topics, at a much quicker pace. These range from fundamental solutions and Green's functions (for the Laplacian Δ) to random walks.

A fundamental solution to the Laplace equation in $\mathbb{R}^n$ is a function $\Phi(\mathbf{x})$ such that $\Delta\Phi(\mathbf{x}) = \delta_0$, in the sense of distributions. We will mostly focus on the fundamental solution for Δ in $\mathbb{R}^3$, as is simpler and not much different to those in higher dimensions.

Proposition 2.4. *Define*

$$\Phi(\mathbf{x}) := \frac{1}{4\pi\,|\mathrm{x}|}.$$

Then $\Delta\Phi(\mathbf{x}) = \delta_0$ *in the sense of distributions.*

We shall not prove this statement, but we will proceed with the concept. Henceforth, Ω denotes a domain in $\mathbb{R}^3$. Given a function G in two variables, we define

$$H(\mathbf{x}) := G(\mathbf{x}, \mathbf{x}_0) - \Phi(\mathbf{x} - \mathbf{x}_0), \quad \mathbf{x} \in \Omega \tag{2.2}$$

where $\mathbf{x}_0 \in \Omega$ is fixed.

Definition 3. For fixed $\mathbf{x}_0 \in \Omega$ the Green's function for Δ sourced at x_0 is a function $G(\mathbf{x}, \mathbf{x}_0)$ such that

(1) The function H defined by (2.2) is smooth and harmonic in Ω;
(2) $G(\mathbf{x}, \mathbf{x}_0) \equiv 0$ for $\mathbf{x} \in \partial\Omega$.

The typical algorithm for finding a Green's function for a domain Ω goes as follows (we fix $\mathbf{x}_0 \in \Omega$)

(i) Find a harmonic smooth function $H(\mathbf{x})$ equal to $-\Phi(\mathbf{x} - \mathbf{x}_0)$ for $x \in \partial\Omega$.
(ii) Take $G(\mathbf{x}, \mathbf{x}_0) := H(\mathbf{x}) + \Phi(\mathbf{x} - \mathbf{x}_0)$.