The Essentials of Harmonic Functions in $\mathbb{R}^n$

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1 Definition and Examples

We will first get some notation out of the way. Recall that $\mathbb{R}^n$ denotes the n-Dimensional Euclidean space. That is, it is the set

$$\mathbb{R}^n := \{(x_1, x_2, \ldots, x_n), \quad x_i \in \mathbb{R},\ \forall 1 \leq i \leq n\} \tag{1}$$

When endowed with the usual vector addition and scalar multiplication (over any field $\mathbb{F}$ actually) this forms the vector space you have all (hopefully) worked with in linear algebra. We will use the symbol Ω (said omega) to denote any arbitrary set $\Omega \subset \mathbb{R}^n$ that is *bounded, open and connected*. Such sets are often called **domains**.

In like, we will study functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ that are assumed to be of class $C^2(\Omega) \cap C(\overline{\Omega})$ where $\overline{\Omega}$ is the **closure** of Ω: the smallest closed set containing Ω. $C^2(\Omega)$ simply means that all second order partials of u exist and are continuous in Ω and $C(\overline{\Omega})$ means that u is continuous on $\overline{\Omega}$.

Definition 1 (Harmonic). *Given a domain Ω, we say a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is **harmonic** in Ω if*

$$\sum_{d=1}^{n} \frac{\partial^2}{\partial x_d^2} u(x) = 0, \quad \textit{for all } x \in \Omega \tag{2}$$

The operator

$$\Delta := \sum_{d=1}^{n} \frac{\partial^2}{\partial x_d^2} \tag{Δ}$$

is called the Laplacian *operator. We will often use the shorthand $\Delta u \equiv 0$ in Ω to say that u is harmonic in the domain Ω.*

We now wish to give some examples. This equation (a partial differential equation or PDE) arises in many physical contexts: particularly in physics and fluid mechanics. It is an essential part of the heat and wave equations and may be generalized to *elliptic operators*, of which the Laplacian is the simplest. Intuitively, it should be notes that the Laplacian is the divergence of the gradient operator, i.e $\Delta = \nabla \cdot \nabla$. In one dimension the operator becomes $\frac{d^2}{dx^2}$ which is the curvature of the curve. In more general setting one could say that the Laplacian quantifies the notion of curvature or stress. More precisely, it gives the rate at which the function changes from it's average (it is the gradient of the divergence after all).

With that being said, it should come to no surprise that we will see a lot of very powerful averaging results in the short future. Some of you may have studied some complex variable theory or possibly some complex analysis. One should also note that the restriction of our domains to $\mathbb{R}^n$ doesn't prevent us from considering functions defined on $\mathbb{C}$, as the complex plane may be identified with $\mathbb{R}^2$.

In fact, if f is a holomorphism over a domain $\Omega \subset \mathbb{C}$ writing $f = u(x,y) + iv(x,y)$ with $z = x + iy \in \mathbb{C}$ the *Cauchy-Riemann* equations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{3}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4}$$

in all of Ω. Then we need only write

$$\begin{aligned}\Delta \Re f = u_{xx} + u_{yy} = \frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y = \frac{\partial}{\partial x}v_y - \frac{\partial}{\partial y}v_x \\ = \frac{\partial v}{\partial yx} - \frac{\partial v}{\partial xy} \equiv 0\end{aligned}$$

by equality of mixed partials showing that any such holomorphism is harmonic. The same argument works just as well for $\Delta \Im f$.

By trying to find a function $v(r) = u(x)$ for $r := \|x\|$ we can find a *fundamental solution* to $\Delta u = 0$ in a domain. We give this equation below, and the actual derivation is straightforward; one need only substitute $v(r)$ into the PDE and simplify, which will yield an ODE in terms of v which is easily solved. Those of you who have taken an ODE course will be able to do this, and the rest of you can even easily find this online. The fundamental solution:

$$\Phi(x) := \begin{cases} \frac{1}{2\pi}\ln\|x\| & n = 2 \\ \frac{-1}{\omega_n}\cdot\frac{1}{\|x\|^{n-2}} & n \geq 3 \end{cases} \tag{5}$$

where ω_n is the surface area of the unit sphere in n dimensional space ($\omega_3 = 4\pi$).

2 Theory of Harmonic Functions

Here we discuss the more interesting results involving harmonic functions. Recall that a couple dozen lines before, we mentioned the averaging properties. It turns out that harmonic functions behave quite nicely. We will see that if a function u is harmonic in a domain Ω (and of class $C^2(\Omega) \cap C(\overline{\Omega})$), then for any ball $B(x,\rho) \subset \Omega$ one has both

$$⨍_{B(x,\rho)} u(y)\mathrm{d}y = u(x) = ⨍_{\partial B(x,\rho)} u(y)\mathrm{d}S \tag{6}$$

Here the $⨍$ symbol denotes the *mean integral*. That is, we take the integral and divide by the volume of the ball $B(x,\rho)$ (or $\partial B(x,\rho)$ in the other case):

$$⨍_{B(x,\rho)} u(y)\mathrm{d}y = \frac{1}{\mathrm{Vol}(B(x,\rho))}\int_{B(x,\rho)} u(y)\mathrm{d}y \tag{7}$$

In general, for a domain $\Sigma \subseteq \mathbb{R}^n$ and an integrable function $f : \Sigma \to \mathbb{R}$ we define

$$⨍_\Sigma f := \frac{1}{\mu(\Sigma)}\int_\Sigma f$$

Where μ denotes the *Lebesgue measure* in $\mathbb{R}^n$.

The converse is also true, if equation (6) is satisfied in a domain Ω for **any** ball then the function is also harmonic in this domain. We state this below as a theorem:

Theorem 2.1 (Mean Value Theorem). *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. The following are equivalent:*

1. *u is harmonic in Ω (i.e $\Delta u \equiv 0$ in Ω)*
2. *For any ball $B(x,\rho) \subset \Omega$:*

$$\fint_{B(x,\rho)} u(y)\mathrm{d}y = u(x) = \fint_{\partial B(x,\rho)} u(y)\mathrm{d}S$$

Gauss's Proof. ($1 \implies 2$). Let u be harmonic and pick an arbitrary ball $B(x,\rho) \subset \Omega$ (hence $x \in \Omega$). We will first consider the integral over $\partial B(x,\rho)$. Recall now the *Divergence Theorem* which we will require:

$$\int_S \Delta f(y)\mathrm{d}y = \int_{\partial S} \nabla f(y) \cdot \mathrm{d}\mathbf{S} \tag{δ}$$

where $\mathrm{d}\mathbf{S} = \nu dS$ and ν is the outwards normal of the ball at a point. Then, since $\Delta u \equiv 0$ in Ω and hence in $B(x,\rho)$:

$$\begin{aligned} 0 = \int_{B(x,\rho)} \Delta u(y)\mathrm{d}y &= \int_{\partial B(x,\rho)} \nabla u(y) \cdot \mathrm{d}\mathbf{S} = \int_{\partial B(x,\rho)} \nabla u(y) \cdot \nu \mathrm{d}S \\ &= \int_{\partial B(0,1)} \nabla u(x+\rho z) \cdot \nu \rho^{n-1}\mathrm{d}S = \rho^{n-1} \int_{\partial B(0,1)} \nabla u(x+\rho z) \cdot \nu \mathrm{d}S(z) \\ &= \rho^{n-1} \int_{\partial B(0,1)} \nabla u(x+\rho z) \cdot z\mathrm{d}S(z) = \rho^{n-1} \int_{\partial B(0,1)} \frac{\partial}{\partial \rho}\left[u(x+\rho z)\right]\mathrm{d}S(z) \\ &= \rho^{n-1}\frac{\partial}{\partial \rho}\int_{\partial B(0,1)} u(x+\rho z)\mathrm{d}S(z) = \rho^{n-1}\frac{\partial}{\partial \rho}\left[\frac{1}{\omega_{n-1}\rho^{n-1}}\int_{\partial B(0,1)} u(x+\rho z)\rho^{n-1}\mathrm{d}S(z)\right] \\ &= \rho^{n-1}\frac{\partial}{\partial \rho}\left[\frac{1}{\omega_{n-1}\rho^{n-1}}\int_{\partial B(x,\rho)} u(y)\mathrm{d}S\right] \end{aligned}$$

Since $\rho^{n-1} \neq 0$ everywhere we conclude that

$$\frac{\partial}{\partial \rho}\fint_{\partial B(x,\rho)} u(y)\mathrm{d}S = \frac{\partial}{\partial \rho}\left[\frac{1}{\omega_{n-1}\rho^{n-1}}\int_{\partial B(x,\rho)} u(y)\mathrm{d}S\right] = 0 \tag{8}$$

for all $\rho > 0$ and is hence a constant function of ρ. Taking the limit as $\rho > 0$ implies

$$\fint_{\partial B(x,\rho)} u(y)\mathrm{d}S = \lim_{\rho \to 0^+} \fint_{\partial B(x,\rho)} u(y)\mathrm{d}S = u(x)$$

For the remainder of (2) write

$$\begin{aligned} \fint_{B(x,\rho)} u(y)\mathrm{d}y = \frac{1}{\omega_n \rho^n}\int_0^\rho \left(\int_{\partial B(x,\varepsilon)} u(y)dS(y)\right)\mathrm{d}\varepsilon &= \frac{1}{\omega_n \rho^n}\int_0^\rho n\omega_n u(x)\varepsilon^{n-1}\mathrm{d}\varepsilon \\ &= \frac{u(x)}{\rho^n}\int_0^\rho n\varepsilon^{n-1} \\ &= u(x) \end{aligned}$$

($2 \implies 1$). Suppose that (2) holds but that (1) is false: $\Delta u \not\equiv 0$ in Ω. So there exists a point $x_0 \in \Omega$ so that $\Delta u(x_0) \neq 0$. We may assume without loss of generality that

$\Delta u(x_0) = \delta > 0$. Then, we may take a small ball $B(x_0, \rho) \subset \Omega$ over which $\Delta u > 0$ since u is of class C^2 and consequently we have that Δu is continuous inside Ω. Then, we see by our assumption in (2) that for all sufficiently small ρ (i.e whenever $B(x_0, \rho) \subset \Omega$):

$$\frac{\partial}{\partial \rho} \fint_{\partial B(x_0,\rho)} u(y)\mathrm{d}S = 0$$

On the other hand,

$$\begin{aligned}
\fint_{\partial B(x_0,\rho)} u(y)\mathrm{d}S(y) &= \frac{1}{n\omega_{n-1}\rho^{n-1}} \int_{\partial B(x_0,\rho)} u(y)\mathrm{d}S(y) \\
&= \frac{1}{n\omega_{n-1}\rho^{n-1}} \int_{\partial B(0,1)} u(x_0 + \rho z)\rho^{n-1}\mathrm{d}S(z) \\
&= \frac{1}{n\omega_{n-1}} \int_{\partial B(0,1)} u(x_0 + \rho z)\mathrm{d}S(z)
\end{aligned}$$

Whence,

$$\begin{aligned}
\frac{\partial}{\partial \rho} \fint_{\partial B(x_0,\rho)} u(y)\mathrm{d}S(y) &= \frac{1}{n\omega_{n-1}} \int_{\partial B(0,1)} \nabla u(x_0 + \rho z) \cdot z\mathrm{d}S(z) \\
&= \frac{1}{n\omega_{n-1}} \int_{\partial B(0,1)} \nabla u(x_0 + \rho z) \cdot \nu\mathrm{d}S(z) \\
&= \fint_{\partial B(x_0,\rho)} \nabla u(y) \cdot \nu\mathrm{d}S(y)
\end{aligned}$$

Hence, by multiplying both sides by appropriate constants we deduce that

$$\fint_{B(x_0,\rho)} \Delta u(y)\mathrm{d}S(y) = 0, \quad \forall \rho \text{ small} \tag{9}$$

This contradicts that $\Delta u > 0$ in a small neighbourhood of x_0. The theorem is then proven.

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With the help of this theorem we may now deduce the so called *Strong Maximum Principle*. This is a very powerful theorem, which is greatly used in classical and complex analysis (remember all that the real and imaginary parts of holomorphic functions are harmonic!). The idea is that if a function u is harmonic in a domain Ω, it can only achieve it's maximum on the boundary, unless it is constant.

Theorem 2.2 (Strong Maximum Principle). *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω. Then, if u achieves it's maximum in the interior of Ω it must be constant.*

Proof. Suppose that u achieves it's maximum M at an interior point $x_0 \in \Omega$. Define the set

$$X := \{x \in \Omega \mid u(x) = M\} \tag{10}$$

Note that by continuity of u this set is closed. Indeed, if $(x_n) \subset \Omega$ is a sequence converging to some $x \in \mathbb{R}^n$ then for all indices n one has easily that $u(x_n) = 0$. Thus, we have $\lim_{n\to\infty} u(x_n) = M = u(x_n)$ by continuity of u. But then, we have $u(x) = M$ and hence $x \in X$.

Moreover, X is open. This follows from the mean value property. Indeed, if $x \in X$ then we have $u(x) = M$. By the mean value property, we have some small ball $B(x,\rho) \subset \Omega$ so that

$$\fint_{B(x,\rho)} u(y)\mathrm{d}y = u(x)$$

which is possible only if $u \equiv M$ in $B(x,\rho)$. In other words, $B(x,\rho) \subset X$ and hence X is open. Consequently, X is clopen and we may then write Ω as the disjoint union of two open sets. One must be empty (because Ω is a domain and thus connected), but by hypothesis we know $X \neq \emptyset$. Thus, $X^c = \emptyset$ and we see that $u \equiv M$ in Ω; proving our claim.

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Corollary 2.3. *Under the assumptions of the strong maximum principle.*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

The reader should observe that all of these results are equally valid for holomorphic functions, as $\mathbb{C} \sim \mathbb{R}^2$ and the real and imaginary parts of holomorphisms are harmonic by a previous observation.

Theorem 2.4 (Liouville). *If a function $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic in $\mathbb{R}^n$ and bounded, it is constant.*

Edward Nelson's Proof. We have some constant $M > 0$ so that $|u| \leq M$ for all of $\mathbb{R}^n$. Fix two points $x_1, x_2 \in \mathbb{R}^n$. Suppose without harm that $x_1 \neq x_2$, we will show $u(x_1) = u(x_2)$. Take any balls $B(x_1, R)$ and $B(X_2, R)$ for $R \gg 0$. No matter how large we make the balls, the integrals

$$\fint_{B(x_1,R)} u(y)\mathrm{d}dy \leq M, \quad \fint_{B(x_2,R)} u(y)\mathrm{d}dy \leq M \tag{11}$$

Now, by making R arbitrarily large we may make symmetric difference $B(x_1,R)\Delta B(x_2,R)$ arbitrarily small and hence make these integrals arbitrarily close. However, these are equal to $u(x_1)$ and $u(x_2)$ respectively. Thus, $u(x_1) = u(x_2)$ and we are done as this pair was arbitrary.

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The reader should note some consequences of this beautiful theorem. It states loosely that non-trivial (constant) harmonic function (and holomorphic functions) must behave erratically at infinity, so to speak, in the sense that they grow. With this result, we may now prove the elusive *Fundamental Theorem of Algebra.* It is actually quite surprising that we required only analysis to prove a fundamental result of algebra. Now, many of you ought to be familiar with the statement of this theorem, but have likely never seen a rigorous proof (as it requires a fair bit of difficult work, even in our case). Personally, I find this to be the most elegant proof of the theorem.

Theorem 2.5 (Fundamental Theorem of Algebra). *Let $P : \mathbb{C} \to \mathbb{C}$ be a non constant polynomial of degree n, i.e*

$$P(z) := a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0, \quad a_n \neq 0 \tag{12}$$

where all the coefficients are complex numbers. $P(z)$ has a root in $\mathbb{C}$. That is, there exists a point $z_0 \in \mathbb{C}$ so that $P(z_0) = 0$.

Proof of the Fundamental Theorem of Algebra. We argue by contradiction and assume $P(z) \neq 0$ in all of $\mathbb{C}$. Then, $|P(z)| > 0$ for all $z \in \mathbb{C}$. We now consider the quotient

$$\frac{P(z)}{z^n} = a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \xrightarrow{z \to \infty} a_n > 0$$

Taking $\varepsilon := \frac{a_n}{2} > 0$ we may find $R > 0$ so that for $|z| \geq R$ one has

$$|P(z)| \geq C\,|z|^n\,, \quad C > 0 \text{ a constant}$$

Therefore,

$$\frac{1}{|P(z)|} \leq \frac{C'}{|z^n|}, \quad \forall\,|z| \geq R \gg 0 \tag{13}$$

However, when $|z| \leq R$ we use that $\frac{1}{P(z)}$ is continuous in z to find that it is bounded above in absolute value for $\{|z| \leq R\}$. Consequently, putting these facts together shows that $\frac{1}{P(z)}$ is bounded in all of $\mathbb{C}$. Since $P(z)$ is holomorphic and non-zero in all of $\mathbb{C}$ by hypothesis it's inverse is also holomorphic in $\mathbb{C}$. By Liouville's theorem this function $\frac{1}{P(z)}$ must be constant and thus $P(z)$ is constant in $\mathbb{C}$. This is absurd, given that we assumed $P(z)$ is a non-constant polynomial.

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