The Gelfand-Naimark-Segal Construction

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In this document, we prove the following famous result from the theory of C^*-algebras. All C^*-algebras are assumed to have a unit, which we denote by $\mathbb{1}$.

Theorem (Gelfand-Naimark-Segal)**.** *Let $\mathfrak{A}$ be a C^*-algebra with unit $\mathbb{1}$ and suppose that ω is a state on $\mathfrak{A}$. There exists a cyclic representation $(\mathfrak{H}, \pi, \Omega)$ of $\mathfrak{A}$, where Ω is a unit vector, such that*

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle, \quad \forall A \in \mathfrak{A}.$$

Furthermore, such a representation of $\mathfrak{A}$ is unique (up to unitary equivalence).

The proof we give follows that given in Math 595 (Topics in Analysis) at McGill university during the winter of 2018.

We first provide an easy lemma.

Lemma 1. *Let $\mathfrak{A}$ be a C^*-algebra with unit $\mathbb{1}$ and assume $\omega : \mathfrak{A} \to \mathbb{C}$ is a positive linear functional. Then,*

$$\omega\left((AB)^* AB\right) \leq \|A\|^2 \omega(B^* B), \quad \forall (A, B) \in \mathfrak{A} \times \mathfrak{A}.$$

Proof. Fix $A, B \in \mathfrak{A}$; we begin by considering the linear functional η on $\mathfrak{A}$ given by

$$\eta(C) := \omega(B^* CB) \quad \text{for } C \in \mathfrak{A}.$$

First, we verify that f is linear. For $\alpha, \beta \in \mathbb{C}$ and $C, D \in \mathfrak{A}$ there holds

$$\begin{aligned} \eta(\alpha C + \beta D) = \omega\left(B^*(\alpha C + \beta D)B\right) &= \omega(\alpha B^* CB + \beta B^* DB) \\ &= \alpha\omega(B^* CB) + \beta\omega(B^* DB) \\ &= \alpha\eta(C) + \beta\eta(D). \end{aligned}$$

Now, η is positive since

$$\eta(C^* C) = \omega\left(B^* C^* CB\right) = \omega\left((CB)^*(CB)\right) \geq 0.$$

From these last two facts we see that η is a positive linear function whence it follows that η is bounded with norm $\|\eta\| = \eta(\mathbb{1})$. From this, we obtain

$$0 \leq \omega((AB)^* AB) = \omega\left(B^* A^* AB\right) = \eta(A^* A) \leq \|\eta\| \, \|A^* A\| = \eta(\mathbb{1}) \, \|A\|^2 .$$

The statement of the lemma follows at once since $\eta(\mathbb{1}) = \omega(B^* B)$. □

We are now ready to prove the main theorem of this note. Once again, we will decompose the proof of this result into multiple steps.

1.1 Building up the Hilbert space $\mathfrak{H}$

We will first construct a prototype inner-product space $\mathfrak{U}$, and we will then deem its completion (a Hilbert space) to be $\mathfrak{H}$. Let now $\mathfrak{I}$ be the subset of the algebra $\mathfrak{A}$ consisting of

$$\mathfrak{I} := \left\{A \in \mathfrak{A} : \omega(A^* A) = 0\right\} .$$

We now claim that $\mathfrak{I}$ is an ideal in $\mathfrak{A}$. To establish this, we begin by proving that $\mathfrak{I}$ is closed under left multiplication. Let $A \in \mathfrak{A}$ and $B \in \mathfrak{I}$, then from Lemma 1 we obtain

$$\omega((AB)^* (AB)) \leq \|A\|^2 \, \omega(B^* B) = 0.$$

Thus, $AB \in \mathfrak{I}$ whenever $B \in \mathfrak{I}$. Now, we prove that $\mathfrak{I}$ is a vector subspace of $\mathfrak{A}$. Suppose $A \in \mathfrak{I}$ or $B \in \mathfrak{I}$, then the Cauchy-Schwarz inequality implies that

$$\left|\omega(A^* B)\right|^2 \leq \omega(A^* A) \omega(B^* B) = 0 \tag{1}$$

so that $A^* B \in \mathfrak{I}$. Hence, if $A \in \mathfrak{I}$ and $B \in \mathfrak{I}$ one has

$$\begin{aligned} \omega((A+B)^*(A+B)) &= \omega\left((A^* + B^*)(A+B)\right) \\ &= \omega(A^* A) + \omega(B^* A) + \omega(A^* B) + \omega(B^* B) \\ &= \omega(B^* A) + \omega(A^* B) \end{aligned}$$

where we have used that $A, B \in \mathfrak{I}$ in this last step. But, we see from (1) that,

$$\omega(B^* A) = \omega(A^* B) = 0$$

and it follows that $A + B \in \mathfrak{I}$. Finally, let $\zeta \in \mathbb{C}$ and $A \in \mathfrak{I}$; it is clear that

$$\omega((\zeta A)^* (\zeta A)) = |\zeta|^2 \, \omega(A^* A) = 0$$

whence $\mathfrak{J}$ is an ideal and vector subspace of $\mathfrak{A}$. This last property allows us to define a quotient vector space over $\mathbb{C}$:

$$\mathfrak{U} := \mathfrak{A}/\mathfrak{J} = \{\phi_A : A \in \mathfrak{A}\}$$

where, of course, ϕ_A denotes the equivalence class containing A:

$$\phi_A = A + \mathfrak{J} = \{B \in \mathfrak{A} : B - A \in \mathfrak{J}\}\,.$$

It is known that $\mathfrak{U}$ inherits operations

$$\phi_A + \phi_B := \phi_{A+B}, \quad \zeta\phi_A := \phi_{\zeta A}$$

which make $\mathfrak{U}$ into a $\mathbb{C}$ vector space. We now turn towards making $\mathfrak{U}$ into an inner-product space. To this end, consider the "inner-product" given by

$$\langle \phi_A, \phi_B \rangle := \omega(A^* B), \quad \phi_A, \phi_B \in \mathfrak{U}.$$

We must show that this is well defined, i.e. independent of representation. Suppose that $\phi_{A'} = \phi_A$ and $\phi_{B'} = \phi_B$. This is to say that

$$A' = A + J_1 \quad \text{and} \quad B' = B + J_2$$

for $J_1, J_2 \in \mathfrak{J}$. Then,

$$\begin{aligned}\langle \phi_{A'}, \phi_{B'} \rangle = \omega\left((A')^*(B')\right) &= \omega\left(\left(A^* + J_1^*\right)(B + J_2)\right) \\ &= \omega(A^*B) + \omega(J_1^*B) + \omega(A^*J_2) + \omega(J_1^*J_2).\end{aligned}$$

Using what we have proven in (1), it follows that

$$\omega(J_1^*B) + \omega(A^*J_2) + \omega(J_1^*J_2) = 0$$

which implies that

$$\langle \phi_{A'}, \phi_{B'} \rangle = \omega(A^*B) = \langle \phi_A, \phi_B \rangle\,.$$

Being positive, we see that $0 \le \omega(A^*A) = \langle \phi_A, \phi_A \rangle$ for all $A \in \mathfrak{A}$. By definition of $\mathfrak{J}$, we then see that

$$\langle \phi_A, \phi_A \rangle = 0 \iff \omega(A^*A) = 0 \iff A \in \mathfrak{J} \iff \phi_A = 0 \text{ in } \mathfrak{U}.$$

From class, we know that

$$\langle \phi_A, \phi_B \rangle = \omega(A^*B) = \overline{\omega(B^*A)} = \overline{\langle \phi_B, \phi_A \rangle}, \quad \forall A, B \in \mathfrak{A}.$$

Finally, fix $\phi_A, \phi_B, \phi_C \in \mathfrak{U}$ and let $\zeta, \xi \in \mathbb{C}$ be given. We see that

$$\begin{aligned}\langle \phi_A, \zeta\phi_B + \xi\phi_C \rangle = \omega\left(A^*\left[\zeta B + \xi C\right]\right) &= \zeta\omega(A^*B) + \xi\omega(A^*C)\\ &= \zeta\left\langle \phi_A, \phi_B \right\rangle + \xi\left\langle \phi_A, \phi_C \right\rangle.\end{aligned}$$

This verifies that $\langle \cdot, * \rangle$ is an inner product on $\mathfrak{U}$. Let now $\mathfrak{H}$ be the Hilbert space obtained through the completion of $\mathfrak{U}$. We know then that $\mathfrak{U}$ will be a dense subspace of this Hilbert space $\mathfrak{H}$.

1.2 Construction of cyclic representation for $\mathfrak{A}$

We now construct the tuple $(\mathfrak{H}, \pi, \Omega)$ at hand. We will begin by defining a "prototype" function τ using $\mathfrak{U}$ and we will *extend* this to $\mathfrak{H}$. Fix now $A \in \mathfrak{A}$ and consider the map

$$\tau(A)(\cdot) : \mathfrak{U} \to \mathfrak{U}, \quad \phi_B \mapsto \phi_{AB}.$$

Once again, we must show this to be well defined. Assume that we have two representatives $B, B' \in \mathfrak{A}$ for the same equivalence class, i.e. $\phi_B = \phi_{B'}$. Then,

$$B' = B + J_1$$

for a choice of $J_1 \in \mathfrak{I}$. Now, since $\mathfrak{I}$ is a left-ideal of $\mathfrak{A}$, there holds

$$AB' = AB + \underbrace{AJ_1}_{\in\mathfrak{I}}$$

whence $\phi_{AB} = \phi_{AB'}$. This means that $\tau(A)(\cdot)$ is well defined for each A and therefore we may consider a function τ defined on $\mathfrak{A}$. Notice also that, by definition,

$$\begin{aligned}\left\|\tau(A)\phi_B\right\|^2 = \langle \phi_{AB}, \phi_{AB} \rangle = \omega\left((AB)^*AB\right) &\leq \|A\|^2\,\omega(B^*B)\\ &= \|A\|^2\left\langle \phi_B, \phi_B \right\rangle\\ &= \|A\|^2\left\|\phi_B\right\|^2\end{aligned}$$

where we have made use of Lemma 1. This means that $\tau(A)$ is a bounded operator $\mathfrak{U} \to \mathfrak{U}$ for every fixed $A \in \mathfrak{A}$. Furthermore, for $\xi, \zeta \in \mathbb{C}$

$$\begin{aligned}\tau(A)\left(\zeta\phi_B + \xi\phi_C\right) = \tau(A)\left(\phi_{\zeta B+\xi C}\right) = \phi_{A(\zeta B)+A(\xi C)} &= \phi_{\zeta AB+\xi AC}\\ &= \zeta\phi_{AB} + \xi\phi_{AC}\\ &= \zeta\tau(A)(\phi_B) + \xi\tau(A)(\phi_C).\end{aligned}$$

It follows that $\tau(A) \in \mathcal{B}(\mathfrak{U})$ for every $A \in \mathfrak{A}$. Thus, τ is a map $\mathfrak{A} \to \mathcal{B}(\mathfrak{U})$. Now, if $A, B, C \in \mathfrak{A}$ and $\zeta, \xi \in \mathbb{C}$

$$\tau(\zeta A + \xi B)(\phi_C) = \phi_{(\zeta A + \xi B)C} = \zeta\phi_{AC} + \xi\phi_{BC} = \zeta\tau(A)(\phi_C) + \xi\tau(B)(\phi_C).$$

Since ϕ_C was arbitrary, we conclude that

$$\tau(\zeta A + \xi B) \equiv \zeta\tau(A) + \xi\tau(B)$$

and hence that τ is linear over $\mathfrak{A}$. Similarly,

$$\tau(AB)(\phi_C) = \phi_{ABC} = \tau(A)(\phi_{BC}) = (\tau(A)\tau(B))(\phi_C)$$

whence τ is multiplicative. An interesting fact is that $\tau(A)$ will always have an adjoint in $\mathcal{B}(\mathfrak{U})$ (remember, $\mathfrak{U}$ is not always a Hilbert space). Certainly, fix ϕ_B and ϕ_C in $\mathfrak{U}$ and consider

$$\begin{aligned}\langle\tau(A)\phi_B, \phi_C\rangle = \langle\phi_{AB}, \phi_C\rangle = \omega\left((AB)^*C\right) &= \omega\left(B^*A^*C\right) \\ &= \langle\phi_B, \phi_{A^*C}\rangle \\ &= \langle\phi_B, \tau(A^*)\phi_C\rangle.\end{aligned}$$

Hence, $\tau(A)$ has an adjoint operator, and it is given by

$$(\tau(A))^* = \tau(A^*).$$

Let us now fix $A \in \mathfrak{A}$; we have shown that $\tau(A)$ is a bounded linear operator $\mathfrak{U} \to \mathfrak{U}$. Since $\mathfrak{U}$ is a dense subspace of $\mathfrak{H}$, we are free to (uniquely) extend $\tau(A)$ to a *bounded linear operator*

$$\pi(A) : \mathfrak{H} \to \mathfrak{H}.$$

This association grants us a map $\pi : \mathfrak{A} \to \mathcal{B}(\mathfrak{H})$ via $A \mapsto \pi(A)$. We now claim that $(\mathfrak{H}, \pi)$ is a representation of $\mathfrak{A}$. The first step is to check that π is linear. Let $A, B \in \mathfrak{A}$ and $\psi \in \mathfrak{H}$. Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{U}$ converging to ψ in $\mathfrak{H}$. Then, for all $\zeta, \xi \in \mathbb{C}$ one has

$$\begin{aligned}\pi(\zeta A + \xi B)(\psi) = \lim_{n\to\infty} \pi(\zeta A + \xi B)(\phi_n) &= \lim_{n\to\infty} \tau(\zeta A + \xi B)(\phi_n) \\ &= \zeta \lim_{n\to\infty} \tau(A)(\phi_n) + \xi \lim_{n\to\infty} \tau(B)(\phi_n) \\ &= \zeta\pi(A)(\psi) + \xi\pi(B)(\psi).\end{aligned}$$

This shows that π respects linearity. Similarly,

$$\begin{aligned}\pi(AB)(\psi) = \lim_{n\to\infty} \pi(AB)(\phi_n) = \lim_{n\to\infty} \tau(AB)(\phi_n) &= \lim_{n\to\infty} \tau(A)\tau(B)(\phi_n) \\ &= \lim_{n\to\infty} \pi(A)\pi(B)(\phi_n).\end{aligned}$$

Now, we know that

$$\pi(B)(\phi_n) \xrightarrow{n\to\infty} \pi(B)(\psi)$$

and so, by composition, we will have

$$\lim_{n\to\infty} \pi(A)\pi(B)(\phi_n) = \pi(A)\pi(B)(\psi).$$

This shows that π respects the multiplicative structure of $\mathfrak{A}$. Let now $\phi, \psi \in \mathfrak{H}$ and choose sequences $(\phi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}}$ in $\mathfrak{U}$ such that

$$\phi_n \to \phi \quad \text{and} \quad \psi_n \to \psi.$$

Then, notice that

$$\begin{aligned} \langle \pi(A^*)\phi, \psi \rangle = \lim_{n\to\infty} \langle \pi(A^*)\phi_n, \psi_n \rangle = \lim_{n\to\infty} \langle \tau(A^*)\phi_n, \psi_n \rangle &= \lim_{n\to\infty} \langle \phi_n, \tau(A)\psi_n \rangle \\ &= \lim_{n\to\infty} \langle \phi_n, \pi(A)\psi_n \rangle \\ &= \langle \phi, \pi(A)\psi \rangle . \end{aligned}$$

Since ϕ and ψ were arbitrary, we conclude that $\pi(A^*) = \pi(A)^*$. It follows that π is a $*$-morphism and therefore $(\mathfrak{H}, \pi)$ must be a representation. Now, let us construct the candidate Ω. This is simply done via

$$\Omega := \phi_{\mathbb{1}} \in \mathfrak{U} \subseteq \mathfrak{H}.$$

This is a unit vector since

$$\|\Omega\|^2 = \langle \Omega, \Omega \rangle = \langle \phi_{\mathbb{1}}, \phi_{\mathbb{1}} \rangle = \omega(\mathbb{1}^*\mathbb{1}) = \omega(\mathbb{1}) = 1.$$

Now,

$$\{\pi(A)\Omega : A \in \mathfrak{A}\} = \{\pi(A)(\phi_{\mathbb{1}}) : A \in \mathfrak{A}\} = \{\phi_A : A \in \mathfrak{A}\}$$

is precisely the subspace $\mathfrak{U}$, which is dense in $\mathfrak{H}$ by construction. Notice also that for $A \in \mathfrak{A}$ one has

$$\langle \Omega, \pi(A)\Omega \rangle = \langle \phi_{\mathbb{1}}, \pi(A)\phi_{\mathbb{1}} \rangle = \langle \phi_{\mathbb{1}}, \phi_A \rangle = \omega(\mathbb{1}^*A) = \omega(A).$$

We have therefore proven the existence of a cyclic representation $(\mathfrak{H}, \pi, \Omega)$.

1.3 Establishing the Uniqueness of this Representation

We have proven that a cyclic representation exists, and we now prove uniqueness up to unitary maps. Suppose now that $(\mathfrak{H}', \pi', \Omega')$ is another cyclic representation as in the statement of the theorem:

$$\langle \Omega', \pi'(A)\Omega' \rangle = \omega(A).$$

In the previous step, we have shown that

$$\mathfrak{U} = \{\pi(A)\Omega : A \in \mathfrak{A}\} ;$$

let now $\mathfrak{V}$ denote the set

$$\{\pi'(A)\Omega' : A \in \mathfrak{A}\} =: \mathfrak{V} \subseteq \mathfrak{H}'.$$

We will once again study a prototype map $\mathfrak{U} \to \mathfrak{V}$, and from there we will use a density argument to extend it to a (unitary) map $\mathfrak{H} \to \mathfrak{H}'$.

Define

$$U : \mathfrak{U} \to \mathfrak{V}, \quad \pi(A)(\Omega) \mapsto \pi'(A)(\Omega').$$

As a first step, we must show that U is well defined. Assume that $\pi(A)\Omega = \pi(B)\Omega$; we must show that $\pi'(A)\Omega' = \pi'(B)\Omega'$. Certainly, using that π' is a $*$-morphism:

$$\begin{aligned} \langle \pi'(A-B)\Omega', \pi'(A-B)\Omega' \rangle &= \left\langle \Omega', \left[\pi'(A-B)\right]^* \pi'(A-B)\Omega' \right\rangle && (2) \\ &= \langle \Omega', \pi'\left((A-B)^*(A-B)\right)\Omega' \rangle && (3) \\ &= \omega\left((A-B)^*(A-B)\right) && (4) \\ &= \langle \Omega, \pi\left((A-B)^*(A-B)\right)\Omega \rangle && (5) \\ &= \langle \Omega, \pi(A-B)^*\pi(A-B)\Omega \rangle && (6) \\ &= \langle \pi(A-B)\Omega, \pi(A-B)\Omega \rangle && (7) \\ &= \|\pi(A-B)\Omega\|^2 . && (8) \end{aligned}$$

But, $\pi(A)\Omega = \pi(B)\Omega$ implies that $\pi(A-B)\Omega = 0$ in $\mathfrak{H}$. Therefore, it follows from this chain of equations that $\pi'(A-B)\Omega' = 0$, i.e.

$$\pi'(A)\Omega' = \pi'(B)\Omega'.$$

We see then that the mapping U is well defined. Assume that $\pi'(A)\Omega' = \pi'(B)\Omega'$ for some $A, B \in \mathfrak{A}$. Then,

$$\pi'(A-B)\Omega' = 0$$

whence $\langle \pi'(A-B)\Omega', \pi'(A-B)\Omega' \rangle = 0$. But then, we see from (2)-(8) that

$$\pi(A-B)\Omega = 0$$

whence $\pi(A)\Omega = \pi(B)\Omega$. This means that U is injective. By simple inspection, U is also seen to be surjective. Hence, we have constructed a bijection $U : \mathfrak{U} \to \mathfrak{V}$. To see that U "preserves the inner-product", notice that for $A, B \in \mathfrak{A}$ there holds

$$\begin{aligned}\langle U(\pi(A)\Omega), U(\pi(B)\Omega) \rangle = \langle \pi'(A)\Omega', \pi'(B)\Omega' \rangle &= \langle \Omega', \pi'(A^*)\pi'(B)\Omega' \rangle \\ &= \langle \Omega', \pi'(A^*B)\Omega' \rangle \\ &= \omega(A^*B),\end{aligned}$$

and a verbatim argument shows that $\omega(A^*B) = \langle \pi(A)\Omega, \pi(B)\Omega \rangle$. We conclude that

$$\langle U(\pi(A)\Omega), U(\pi(B)\Omega) \rangle = \langle \pi(A)\Omega, \pi(B)\Omega \rangle .$$

Thus, U preserves the inner-product. Notice that U is also *bounded* (since U preserves the inner product, and hence the norm). Luckily, the linearity of U is an easy consequence of the definition: for $\pi(A)\Omega, \pi(B)\Omega \in \mathfrak{U}$ and $\zeta, \xi \in \mathbb{C}$ one has

$$\begin{aligned}U\left(\zeta\pi(A)\Omega + \xi\pi(B)\Omega\right) = U\left(\pi(\zeta A + \xi B)\Omega\right) &= \pi'(\zeta A + \xi B)\Omega' \\ &= \zeta\pi'(A)\Omega' + \xi\pi'(B)\Omega' \\ &= \zeta U(\pi(A)\Omega) + \xi U(\pi(B)\Omega).\end{aligned}$$

From all these facts, we see that U is "unitary" from $\mathfrak{U} \to \mathfrak{V}$. Using the density of $\mathfrak{U}$ in $\mathfrak{H}$ (and that of $\mathfrak{V}$ in $\mathfrak{H}'$), we may uniquely extend U to a bounded linear operator $L : \mathfrak{H} \to \mathfrak{H}'$. We argue that L is a unitary equivalence.

We begin by checking that L is a surjection. Fix now $\Psi \in \mathfrak{H}'$ and select a sequence $(\Phi_n)_{n\in\mathbb{N}}$ in $\mathfrak{V} \subseteq \mathfrak{H}'$ such that

$$\Phi_n \to \Psi \text{ in } \mathfrak{H}'.$$

For every $n \in \mathbb{N}$, there exists unique $\phi_n \in \mathfrak{U}$ such that

$$U(\phi_n) = L(\phi_n) = \Phi_n.$$

We then have that $\lim_{n\to\infty} U(\phi_n) = \Phi_n$. This means that the sequence $\{U(\phi_n)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathfrak{H}'$. But, U is linear and preserves inner-product, and thus is an isometry. It follows that

$$\|\phi_n - \phi_m\| \xrightarrow{n,m\to\infty} 0.$$

This means that $(\phi_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathfrak{H}$ (which is complete) and thus converges to some $\psi \in \mathfrak{H}$. But then,

$$L(\psi) = \lim_{n\to\infty} U(\phi_n) = \lim_{n\to\infty} \Phi_n = \Psi.$$

We then conclude that L is surjective. Let $\phi, \psi \in \mathfrak{H}$ be given and select sequences $(\phi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}}$ in $\mathfrak{U}$ such that

$$\lim_{n\to\infty} \phi_n = \phi \quad \text{and} \quad \lim_{n\to\infty} \psi_n = \psi.$$

Then, we see that

$$\begin{aligned}\langle L(\phi), L(\psi)\rangle = \lim_{n\to\infty} \langle L(\phi_n), L(\psi_n)\rangle &= \lim_{n\to\infty} \langle U(\phi_n), U(\psi_n)\rangle \\ &= \lim_{n\to\infty} \langle \phi_n, \psi_n\rangle \\ &= \langle \phi, \psi\rangle .\end{aligned}$$

Hence, L also preserves the inner product. Assume now that $L(\phi) = 0$. Then,

$$0 = \langle L(\phi), L(\phi)\rangle = \langle \phi, \phi\rangle = \|\phi\|^2$$

whence $\phi = 0$. Since L is linear, we see that L is also injective. This makes it clear that L is a unitary extension of U. Let now $\pi'(B)\Omega'$ be any element of $\mathfrak{V}$ and $A \in \mathfrak{A}$ be given. We directly "evaluate"

$$\begin{aligned}\left[L\pi(A)L^{-1}\right](\pi'(B)\Omega') = [L\pi(A)]\,(\pi(B)\Omega) &= L\,(\pi(AB)(\Omega) \\ &= \pi'(AB)(\Omega') \\ &= \pi'(A)\,[\pi'(B)\Omega'] .\end{aligned}$$

This series of equations shows that $L\pi(A)L^{-1}$ and $\pi'(A)$ are equal on $\mathfrak{V} \subseteq \mathfrak{H}'$, for any $A \in \mathfrak{A}$. The next step is clear: let $\Psi \in \mathfrak{H}'$ and choose a sequence $(\Phi_n)_{n\in\mathbb{N}}$ in $\mathfrak{V}$ with $\Phi_n \to \Psi$. Then,

$$\left[L\pi(A)L^{-1}\right](\Phi) = \lim_{n\to\infty} \left[L\pi(A)L^{-1}\right]\Phi_n = \lim_{n\to\infty} \pi'(A)\Phi_n = \pi'(A)\Phi.$$

All this together shows that, in fact,

$$L\pi(A)L^{-1} \equiv \pi'(A), \quad \forall A \in \mathfrak{A}.$$

This completes the proof.