THE FOURIER TRANSFORM

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CONTENTS

In this text we shall consider the development of the Fourier transform for square-integrable functions $f : \mathbb{R} \to \mathbb{C}$. We shall develop most of the theory for the Schwartz space $\mathcal{S}$, and extend the Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ to a (unique) mapping

$$\widetilde{\mathcal{F}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

by showing that $\mathcal{S}$ is a dense subspace of $L^2(\mathbb{R})$.

1. THE FOURIER TRANSFORM ON $\mathcal{S}$

Here we develop the theory of Fourier analysis for the "smaller" space $\mathcal{S}$. Recall that the **Schwartz Space**, denoted $\mathcal{S} = \mathcal{S}(\mathbb{R})$, consists of all smooth functions vanishing faster than any inverse power of x at infinity; along with all of its derivatives. That is,

$$\mathcal{S} := \left\{ f \in C^\infty(\mathbb{R}) : \forall \alpha, \beta \in \mathbb{N}_0,\ \exists c_{\alpha,\beta} \geq 0 \text{ such that } \left\| x^\alpha \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^\beta f \right\|_\infty \leq c_{\alpha,\beta} \right\}^1 \tag{1}$$

Clearly, any smooth function of compact support $f : \mathbb{R} \to \mathbb{R}$ belongs to the Schwartz space, and so it is easy to come up with examples of such functions. Our first observation is that $\mathcal{S}$ is a vector space when endowed with the obvious notions of scalar multiplication and addition. Certainly, this is ultimately a consequence of the triangle inequality. Moreover, this same observation tells us that if $f \in \mathcal{S}$ then for each $\beta \in \mathbb{N}_0$ and any polynomial $p(x) \in \mathbb{C}[x]$ there is an associated constant $C \geq 0$ such that

$$\left\| p(x) \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^\beta f \right\|_\infty \leq C$$

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[1] Here we write $\mathbb{N}_0$ to denote the set $\{0, 1, 2, \ldots\}$.

as is immediate from the triangle inequality. This allows us to deduce the following:

Proposition 1.1. *The Schwartz space is a subspace of* $L^1(\mathbb{R})$.

Proof. Let $f \in \mathcal{S}$ be given. Then, for $N \in \mathbb{N}$ large we consider the polynomial $p(x) = (1+x^2)^N$. By our remarks above, we may find a constant C independent of x such that

$$\left\| (1+x^2)^N f \right\|_\infty \leq C$$

whence we obtain $(1+x^2)^N |f(x)| \leq C$ for all $x \in \mathbb{R}$. Especially, this yields

$$|f(x)| \leq \frac{C}{(1+x^2)^N}, \quad \forall x \in \mathbb{R}$$

implying that $\int_{\mathbb{R}} |f(x)|\,\mathrm{d}x < \infty$ and that $f \in L^1(\mathbb{R})$.

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Corollary 1.2. $\mathcal{S}(\mathbb{R})$ *is a subspace of* $L^p(\mathbb{R})$ *for each* $1 \leq p < \infty$.

It is now time to determine the Fourier transform on $\mathcal{S}(\mathbb{R})$.

Definition 1. *For any* $f \in \mathcal{S} = \mathcal{S}(\mathbb{R})$ *we define the Fourier transform of* f*, denoted* $\widehat{f}$ *or* $\mathcal{F}(f)$*, to be the function* $\widehat{f} : \mathbb{R} \to \mathbb{C}$ *defined by*

$$\widehat{f}(y) := \int_{\mathbb{R}} f(x) e^{-ixy} \mathrm{d}x \tag{2}$$

This is well defined. Certainly, by the previous proposition we know that $f \in L^1(\mathbb{R})$. Therefore, for all $y \in \mathbb{R}$ one has easily:

$$\left|\widehat{f}(y)\right| \leq \int_{\mathbb{R}} \left| f(x) e^{-ixy} \right| \mathrm{d}x = \int_{\mathbb{R}} |f(x)| \left| e^{-ixy} \right| \mathrm{d}x = \int_{\mathbb{R}} |f(x)|\,\mathrm{d}x$$

where this last term is precisely the quantity $\|f\|_{L^1(\mathbb{R})} < \infty$.

We first claim that if $f \in \mathcal{S}$ then $\widehat{f} \in \mathcal{S}$. This is more-or-less a two-step process.

Theorem 1.3 (Riemann-Lebesgue Lemma)**.** *Let* $f \in L^1(\mathbb{R})$. *Then,*

$$\lim_{y \to \infty} \int_{\mathbb{R}} f(x) e^{-ixy} \mathrm{d}x = 0$$

Proof. We first establish it for the characteristic function of a compact interval $[a,b]$ in $\mathbb{R}$. To see that this claim holds, we observe now that for each $y \neq 0$:

$$\left| \int_{\mathbb{R}} \mathbf{1}_{[a,b]}(x) e^{-ixy} \mathrm{d}x \right| = \left| \int_a^b e^{-iyx} \mathrm{d}x \right| = \left| \frac{-1}{iy} e^{-iyx} \bigg]_a^b \right| = \frac{1}{y} \left| e^{-iyb} - e^{-iya} \right|$$

whence $\left| \int_{\mathbb{R}} \mathbf{1}_{[a,b]}(x) e^{-ixy} \mathrm{d}x \right| \leq \frac{1}{y}$ which vanishes at infinity.

Now, we recall that step functions are dense in $L^1(\mathbb{R})$. Let $\varepsilon > 0$ be given. Then, there exists a step function $\varphi_\varepsilon = \sum_{n=1}^{N} \alpha_n \mathbf{1}_{[a_n,b_n]}(x)$ in $L^1(\mathbb{R})$ such that $\|f - \varphi_\varepsilon\|_{L^1(\mathbb{R})} < \varepsilon$. Now, we may now compute:

$$\begin{aligned} \left| \int_{\mathbb{R}} f(x)e^{-iyx}\mathrm{d}x \right| &= \left| \int_{\mathbb{R}} \left(f(x) - \varphi_\varepsilon(x)\right) e^{-iyx}\mathrm{d}x + \int_{\mathbb{R}} \varphi_\varepsilon(x)e^{-iyx}\mathrm{d}x \right| \\ &\leq \int_{\mathbb{R}} |f(x) - \varphi_\varepsilon(x)|\,\mathrm{d}x + \sum_{n=1}^{N} |\alpha_n| \left| \int_{\mathbb{R}} \mathbf{1}_{[a_n,b_n]}(x)e^{-iyx}\mathrm{d}x \right| \end{aligned}$$

However, we have that $\|f - \varphi_\varepsilon\| \leq \varepsilon$ and taking y large-enough we can make each term of the summation arbitrarily small. Now, it follows from this that for y sufficiently large we have $\left|\int_{\mathbb{R}} f(x)e^{-iyx}\mathrm{d}x\right| \leq \varepsilon$ and the proof is complete.

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We now prove one more lemma:

Lemma 1.4. *Let $f \in \mathcal{S}(\mathbb{R})$. Then, the following are satisfied:*

(1) $\frac{\mathrm{d}}{\mathrm{d}y}\widehat{f}(y) = -i\widehat{xf}(y)$.

(2) $\widehat{\frac{\mathrm{d}f}{\mathrm{d}x}}(y) = iy\widehat{f}(y)$

Proof.

(1) We use that f is smooth here to differentiate under the integral. Here, the integrand is most certainly smooth and therefore:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y}\widehat{f}(y) = \frac{\mathrm{d}}{\mathrm{d}y}\int_{\mathbb{R}} f(x)e^{iyx}\mathrm{d}x = \int_{\mathbb{R}} f(x)\frac{\mathrm{d}}{\mathrm{d}y}e^{-iyx}\mathrm{d}x &= \int_{\mathbb{R}} f(x)(-ix)e^{-iyx}\mathrm{d}x \\ &= -i\int_{\mathbb{R}} xf(x)e^{-iyx}\mathrm{d}x \\ &= -i\widehat{xf}(y) \end{aligned}$$

(2) Here we integrate by parts and use the property that $f(x)$ vanishes at infinity to deduce the following:

$$\begin{aligned} \widehat{\frac{\mathrm{d}f}{\mathrm{d}x}}(y) = \int_{\mathbb{R}} f'(x)e^{-iyx}\mathrm{d}x &= f(x)e^{-iyx}\Big]_{x=-\infty}^{\infty} - \int_{\mathbb{R}} f(x)\frac{\mathrm{d}}{\mathrm{d}x}e^{-iyx}\mathrm{d}x \\ &= -\int_{\mathbb{R}} f(x)(-iy)e^{-iyx}\mathrm{d}x = (iy)\widehat{f}(y) \end{aligned}$$

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Corollary 1.5. *The Fourier map $\widehat{\cdot}$ takes elements of $\mathcal{S}$ into $\mathcal{S}$. Namely, $\mathcal{F} : \mathcal{S} \to \mathcal{S}$.*

Proof. Clearly, $\widehat{f}$ is smooth by the simple property that $f(x)e^{-iyx}$ is smooth in y. Now, we let $\alpha, \beta \geq 0$ be given. By the Riemann-Lebesgue lemma we need only consider the cases where $\alpha, \beta > 0$. Then, we write:

$$y^{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^{\beta}\widehat{f}(y) = y^{\alpha}(-i)^{\beta}\widehat{x^{\beta}f}(y) = (-i)^{\alpha}y^{\alpha}\widehat{x^{\beta}f}(y) = (-i)^{\alpha+\beta}\widehat{D_x^{\alpha}x^{\beta}f}(y)$$

However, this last line is:

$$C\int_{\mathbb{R}} D_x(x^{\beta}f(x))e^{-iyx}\mathrm{d}x$$

Now, by the product integral together with the fact that $f \in \mathcal{S}(\mathbb{R})$ the result follows again from the Riemann-Lebesgue lemma.

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We now show that we may "exchange" the transform operator under the integral sign:

Proposition 1.6. *Let $f, g \in \mathcal{S}$. Then, $\int_{\mathbb{R}}\widehat{f}(y)g(y)\mathrm{d}y = \int_{\mathbb{R}} f(x)\widehat{g}(x)\mathrm{d}x$.*

Proof. By Fubini's theorem we may write:

$$\begin{aligned}\int_{\mathbb{R}}\widehat{f}(y)g(y)\mathrm{d}y &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x)e^{-iyx}\mathrm{d}x\right)g(y)\mathrm{d}y = \int_{\mathbb{R}\times\mathbb{R}} f(x)g(y)e^{-iyx}\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(y)e^{-iyx}\mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}} f(x)\left(\int_{\mathbb{R}} g(y)e^{-iyx}\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x)\widehat{g}(x)\mathrm{d}x\end{aligned}$$

as was asserted.

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We shall use the notation $f_a(x) = f(a+x)$ for any $a \in \mathbb{R}$. In this case, we have instead that $\widehat{f_a}(y) = e^{iay}\widehat{f}(y)$. To see this, we simply calculate:

$$\begin{aligned}\widehat{f_a}(y) = \int_{\mathbb{R}} f_a(x)e^{-ixy}\mathrm{d}x = \int_{\mathbb{R}} f(x+a)e^{-iyx}\mathrm{d}x &\xrightarrow{z\to x+a} \int_{\mathbb{R}} f(z)e^{-iy(z-a)}\mathrm{d}x \\ &= \int_{\mathbb{R}} f(z)e^{iya}e^{-iyz}\mathrm{d}z = e^{iya}\widehat{f}(y) \qquad (\oplus)\end{aligned}$$

If instead we let $f_a(x) := f(x/a)$ then we find that:

$$\widehat{f_a}(y) = \int_{\mathbb{R}} f\left(\frac{x}{a}\right)e^{-iyx}\mathrm{d}x \xrightarrow{z\to x/a} a\int_{\mathbb{R}} f(z)e^{-iyaz}\mathrm{d}x = a\widehat{f}(ay)$$

2. THE FOURIER INVERSION THEOREM

The main result of this section (the *Fourier Inversion Formula*) makes use of a special calculation.

2.0.1. *Simple Differential Equation.* We consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by the mapping $f(x) = e^{-\frac{x^2}{2}}$. Clearly, this is a smooth function on $\mathbb{R}$ that is of Schwarz class. We wish to determine the Fourier transform $\mathcal{F}(f) = \widehat{f}$. To do this, we first observe that we have the relation

$$f'(x) = -xe^{-\frac{x^2}{2}} = -xf(x)$$

This yields $\widehat{f'} = -\widehat{xf}$. We now employ Lemma 1.4 to see that $-\widehat{xf} = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}y}\widehat{f}(y)$. That is, we obtain

$$\widehat{f'} = -\widehat{xf} = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}y}\widehat{f}(y)$$

However, another application of Lemma 1.4 yields $\widehat{f'} = \widehat{\frac{\mathrm{d}}{\mathrm{d}x}f} = iy\widehat{f}$. Putting all this together gives us an ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}y}\widehat{f} = -y\widehat{f} \tag{3}$$

Namely, integration gives us:

$$\ln \widehat{f} = -\frac{y^2}{2} + C, \quad C \in \mathbb{R}$$

Especially, upon exponentiation we find that $\widehat{f}(y) = \gamma e^{-\frac{y^2}{2}}$ for some constant $\gamma \in \mathbb{R}$. To determine this γ, we note finally that $\widehat{f}(0) = \gamma$ and that this may be computed directly via the Gaussian integral:

$$\widehat{f}(0) = \int_{\mathbb{R}} f(x)e^{-iy\cdot 0}\mathrm{d}x = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\mathrm{d}x = \sqrt{2\pi}$$

That is,

$$\boxed{\mathcal{F}\left[e^{-\frac{x^2}{2}}\right](y) = \sqrt{2\pi}e^{-\frac{y^2}{2}}} \tag{$\mathfrak{E}$}$$

2.0.2. *Proof of the Fourier Inversion Theorem.* We state and prove the Fourier inversion formula here:

Theorem 2.1 (Fourier Inversion Theorem). *The operator $\mathcal{F}[\cdot] : \mathcal{S} \to \mathcal{S}$ is a bijection, and for all $f \in \mathcal{S}$ such that $g = \widehat{f}$ we may recover f via*

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(y)e^{ixy}\mathrm{d}y \tag{$\mathfrak{F}$}$$

Proof. We begin by considering the function $g(y) = e^{-\frac{y^2}{2a^2}}$ for $a \gg 0$. By our earlier calculations together with our earlier remarks on $g_a(\cdot)$, we may deduce that

$$\widehat{g}(x) = a\sqrt{2\pi}e^{-\frac{(ax)^2}{2}}$$

Therefore, by our result in Proposition 1.6 we find that

$$\int_{\mathbb{R}} \widehat{f}(y)g(y)\mathrm{d}y = \int_{\mathbb{R}} f(x)\widehat{g}(x)\mathrm{d}x$$

Whence,

$$\int_{\mathbb{R}} \widehat{f}(y)e^{-\frac{y^2}{2a^2}}\mathrm{d}y = a\sqrt{2\pi}\int_{\mathbb{R}} f(x)e^{-\frac{(ax)^2}{2}}\mathrm{d}x$$

Now, we make the substitution $x \mapsto ax$ to recover:

$$a\sqrt{2\pi}\int_{\mathbb{R}} f(x)e^{-\frac{(ax)^2}{2}}\mathrm{d}x \xrightarrow{u=ax} \sqrt{2\pi}\int_{-\infty}^{\infty} f\left(\frac{u}{a}\right)e^{-\frac{u^2}{2}}\mathrm{d}u$$

All this put together yields:

$$\int_{\mathbb{R}} \widehat{f}(y)e^{-\frac{y^2}{2a^2}}\mathrm{d}y = \sqrt{2\pi}\int_{-\infty}^{\infty} f\left(\frac{u}{a}\right)e^{-\frac{u^2}{2}}\mathrm{d}u \tag{4}$$

We now wish to pass to the limit as $a \to \infty$. We first consider the left-hand-side of equation (4). We observe that by integrability of $\widehat{f}$ we may apply the dominated convergence theorem to deduce

$$\lim_{a\to\infty}\int_{\mathbb{R}} \widehat{f}(y)e^{-\frac{y^2}{2a^2}}\mathrm{d}y = \int_{\mathbb{R}} f(y)\lim_{a\to\infty} e^{-\frac{y^2}{2a^2}}\mathrm{d}y = \int_{\mathbb{R}} \widehat{f}(y)\mathrm{d}y$$

The same argument implies moreover that in the limit of the right-hand-side of (4) one has by continuity of f:

$$\begin{aligned}\lim_{a\to\infty}\sqrt{2\pi}\int_{-\infty}^{\infty} f\left(\frac{u}{a}\right)e^{-\frac{u^2}{2}}\mathrm{d}u = \sqrt{2\pi}\int_{-\infty}^{\infty} f(0)e^{-\frac{u^2}{2}}\mathrm{d}u &= \sqrt{2\pi}f(0)\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\mathrm{d}u \\ &= 2\pi f(0)\end{aligned}$$

This with the equality in (4) implies that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(y)\mathrm{d}y = f(0)$$

We are now almost done. For $z \in \mathbb{R}$ we consider the mapping $h(x) := f(x+z) = f_z(x)$. Now, from our result directly above, we know that

$$f(z) = h(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{h}(y)\mathrm{d}y = \int_{\mathbb{R}} \widehat{f}(y)e^{izy}\mathrm{d}y$$

where in the last equality we have used ($\oplus$). This concludes the proof of the *Fourier Inversion Formula.*

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The final result of this section is of course the following:

Theorem 2.2 (Plancherel's Theorem)**.** *Let $f \in \mathcal{S}$. Then, $2\pi\left\|f\right\|^2 = \left\|\mathcal{F}(f)\right\|^2$ where $\mathcal{F}$ denotes the Fourier transform and $\|\cdot\|$ is the norm in $L^2(\mathbb{R})$.*

Proof. We first derive an expression for $\overline{f(x)}$ with $x \in \mathbb{R}$. This is ultimately a consequence of the Fourier inversion formula:

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}[f](y) e^{ixy} \mathrm{d}y$$

Thence, $\overline{f(x)} = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\mathcal{F}[f](y)} e^{-ixy} \mathrm{d}y$. In other-words;

$$\overline{f(x)} = \frac{1}{2\pi} \mathcal{F}\left[\overline{\mathcal{F}[f]}\right] \tag{5}$$

Using this, we obtain

$$\|f\|^2 = \int_{\mathbb{R}} f(x)\overline{f(x)} \mathrm{d}x = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) \mathcal{F}\left[\overline{\mathcal{F}[f]}\right](x) \mathrm{d}x \overset{(\star)}{=} \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}[f](x)\overline{\mathcal{F}[f](x)} \mathrm{d}x$$

where the equality denoted $(\star)$ follows from Proposition 1.6. This last equation is clearly $\frac{1}{2\pi} \|\mathcal{F}(f)\|^2$ and the proof is complete.

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3. THE FOURIER TRANSFORM ON $L^2(\mathbb{R})$

Of course, the Schwartz Space $\mathcal{S}(\mathbb{R})$ is far too restrictive: we wish to extend the notion of the Fourier transform so that it utilizes the generality of the Lebesgue integral. Of course, the natural setting for this is the space $L^2(\mathbb{R})$.

We begin with an observation that will serve us well:

Theorem 3.1 (Dense-Subspace Extension Theorem)**.** *Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces over some field $\mathbb{K}$ and $\mathcal{S}$ is some dense subspace of $\mathcal{B}_1$. If $T : \mathcal{S} \to \mathcal{B}_2$ is a bounded linear transformation, there exists a unique linear transformation $T_0 : \mathcal{B}_1 \to \mathcal{B}_2$ agreeing with T on $\mathcal{S}$.*

Proof. We shall first assume existence and show uniqueness. This is a consequence of the density of $\mathcal{S}$ in $\mathcal{B}_1$. Let $f \in \mathcal{B}_1$ and let $(f_n) \to f$ be a sequence in $\mathcal{S}$. Clearly, by continuity of T_0, T_1 (recall that a linear operator is continuous iff bounded) we have

$$T_0(f) = \lim_{n\to\infty} T_0(f_n) = \lim_{n\to\infty} T_1(f_n) = T_1(f)$$

establishing uniqueness. We now show existence. Let $f \in \mathcal{B}_1$ and pick a sequence (f_n) in $\mathcal{S}$ converging to f. Then, we claim that the sequence $(T(f_n))$ is Cauchy in $\mathcal{B}_2$. To see this, we simply use the boundedness of T to discover

$$\|T(b_n) - T(b_m)\|_{\mathcal{B}_2} \le M \|b_n - b_m\|_{\mathcal{B}_1}, \quad M > 0$$

Since (f_n) is a convergent sequence, it must Cauchy and therefore we have found that $(T(b_n))_n$ is again Cauchy in $\mathcal{B}_2$. Since $\mathcal{B}_2$ is a Banach space, there is a limit to this sequence in $\mathcal{B}_2$ and the limit above makes sense. It remains to show that this is *well-defined*. Suppose that (g_n) is another sequence in $\mathcal{S}$ converging to f in $\mathcal{B}_1$. We must show that $\lim T(g_n) = \lim T(f_n)$.

To see this, write for n large:

$$\|T(f_n) - T(g_n)\|_{\mathcal{B}_2} \le M \|f_n - g_n\|_{\mathcal{B}_1} \xrightarrow{n\to\infty} 0$$

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This theorem tells us that we simply need to show that the Schwartz Space $\mathcal{S} = \mathcal{S}(\mathbb{R})$ is *dense* in $L^2(\mathbb{R})$ to ensure the existence of a well defined extension to $L^2(\mathbb{R})$ of the map $\mathcal{F}$. We shall devote this next theorem to this result.

As mentioned previously, we wish to establish the density of $\mathcal{S} = \mathcal{S}(\mathbb{R})$. The first observation is that $C_0^\infty(\mathbb{R})$ is a subspace of $\mathcal{S}$. Certainly, at infinity these functions are identical to zero and therefore the derivatives, as well as the function, vanish entirely. It follows from this observation that it suffices to show that $C_0^\infty(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$.