DIRICHLET CHARACTERS AND THE METHOD OF HYPERBOLAS

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1. Dirichlet Convolution and Möbius Inversion

Recall that an arithmetic function is a mapping $\alpha : \mathbb{N} \to \mathbb{C}$. The family of such functions is denoted by $\mathcal{A}[\mathbb{N}]$ and is a vector space over $\mathbb{C}$ when endowed with our usual notions of scalar multiplication and vector addition. Two crucial arithmetic functions are the δ and μ functions, the latter of which is called the Möbius function. These are defined by

$$\delta(s) := \begin{cases} 1 & s = 1 \\ 0 & \text{else} \end{cases} \tag{1}$$

Now consider $s \in \mathbb{N}$. If s is not square-free, then we set $\mu(s) = 0$. Else, write $s = \prod_{j=1}^{k} p_j$ and let $\mu(s) := (-1)^k$. This defines the Möbius function. Now,

Definition. *Given two arithmetic functions α, β we define $\alpha * \beta$, called the Dirichlet convolution, as follows:*

$$(\alpha * \beta)(n) := \sum_{rs=n} \alpha(r)\beta(s) = \sum_{d|n} \alpha(d)\, \beta\left(\frac{n}{d}\right) = \sum_{d|n} \alpha\left(\frac{n}{d}\right) \beta(d) \tag{2}$$

From these definitions it is clear that $(\alpha * \beta)(n)$ is itself an arithmetic function and moreover we find that the operation $*$ is both commutative and associative in $\mathcal{A}[\mathbb{N}]$. We show now that it preserves multiplicativity of a mapping:

Proposition 1. *Let $\alpha, \beta \in \mathcal{A}[\mathbb{N}]$ and suppose that α, β are weakly multiplicative. Then, so is $(\alpha * \beta)(\cdot)$.*

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Proof. Let $n, m \in \mathbb{N}$ be co-prime and note that any divisor $d \mid nm$ may be written as rs where $r \mid n$ and $s \mid m$. Therefore,

$$\begin{aligned}(\alpha * \beta)(nm) &= \sum_{d|nm} \alpha(d)\,\beta\left(\frac{nm}{d}\right) = \sum_{rs|nm} \alpha(rs)\,\beta\left(\frac{nm}{d}\right) \\ &= \sum_{r|n, s|m} \alpha(r)\alpha(s)\beta\left(\frac{n}{r}\right)\beta\left(\frac{m}{s}\right) \\ &= \sum_{r|n}\sum_{s|m} \alpha(r)\alpha(s)\beta\left(\frac{n}{r}\right)\beta\left(\frac{m}{s}\right) \\ &= \left(\sum_{r|n} \alpha(r)\beta\left(\frac{n}{r}\right)\right) \cdot \left(\sum_{s|m} \alpha(s)\beta\left(\frac{m}{s}\right)\right)\end{aligned}$$

where this last line is precisely the quantity $(\alpha * \beta)(n) \cdot (\alpha * \beta)(m)$.

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The above is in of itself a useful property but is not quite what we seek. Recall the definitions of the delta and Möbius functions $\delta, \mu \in \mathcal{A}[\mathbb{N}]$ which we have defined above. Obviously, any constant mapping $\gamma : \mathbb{N} \to \mathbb{C}$ is an element of $\mathcal{A}[\mathbb{N}]$ and the same can be said for the map $\mathbb{1} : \mathbb{N} \to \mathbb{C}$ which is defined by $n \mapsto 1$. What we now claim is the following:

Lemma 1. *One has* $(\mu * \mathbb{1}) \equiv \delta$.

Proof. We may use the properties derived in the previous result as well as some of our introductory results. Obviously, $\mathbb{1}$ is *strongly multiplicative*. We claim now that μ is multiplicative. It is sufficient to show this for $p^a q^b$ where p, q are distinct primes.

If one of $a, b \geq 2$ then $\mu(p^a q^b) = 0$ and $\mu(p^a)\mu(q^b) = 0$ as well, and it is verified in this case. In the other-case we have the result by direct calculation on the number of such primes. By our previous proposition , it follows that $(\mu * \mathbb{1})$ is multiplicative. It is therefore sufficient to establish this for powers of primes (see the "behaviour" of the δ function!).

We now proceed with the calculation where $\ell \in \mathbb{N}$:

$$(\mu * \mathbb{1})\,(p^\ell) = \sum_{d|p^\ell} \mu(d) = \sum_{j=0}^{\ell} \mu(p^j) = 1 + \mu(p) = 0$$

If, however, $\ell = 0$ (i.e. $n = 1$) then this simply yields 1. By the fundamental theorem of arithmetic, the proof is now complete.

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Equipped with this result we are now prepared to prove the rather useful *Möbius Inversion Formula*. Loosely speaking, this is a discrete analogue to the problem of the inverse Laplace transform. Suppose we are given the following relation:

$$\beta(n) = \sum_{d|n} \alpha(d), \quad \alpha, \beta \in \mathcal{A}[\mathbb{N}] \quad \text{and} \quad n \in \mathbb{N}$$

How would one recover α? The answer lies in the previous lemma. Compute now the convolution with μ, then for each n we have

$$\beta * \mu = (\alpha * \mathbb{1}) * \mu = \alpha * (\mu * \mathbb{1}) = \alpha * \delta$$

We must now compute $\alpha * \delta$ for some n. This is not difficult at all; write:

$$(\alpha * \delta)(n) = \sum_{rs|n} \alpha(r)\delta(s) = \alpha(n)$$

Putting these facts together we obtain the following:

Theorem 1 (Möbius Inversion Formula)**.** *Let $\alpha, \beta \in \mathcal{A}[\mathbb{N}]$ be given by $\beta(n) \equiv \sum_{d|n} \alpha(d)$. Then, for all such n:*

$$\alpha(n) = (\beta * \mu)(n) \tag{$\mathfrak{M}$}$$

2. Dirichlet Characters

In this document be briefly introduce and explore the concept of a Dirichlet character modulo some integer $n \geq 1$. Simply put, a Dirichlet character may be defined on the group $\mathbb{Z}_n^*$ for any n as above. More-precisely:

Definition. *Given an integer $n \geq 1$ a **Dirichlet character** modulo n is a group homomorphism: $\chi : \mathbb{Z}_n^* \to \mathbb{C}^*$, where the group operation is multiplication.*

Some notable properties are now in order. Note that $\chi \not\equiv 0$ on $\mathbb{Z}_n^*$. Indeed, this follows from the fact that a homomorphism fixes the identity, i.e. $\chi(1) = 1$. It is clear that it makes sense to set $\chi(\cdot) = 0$ whenever the argument is an integer not co-prime to n. Hence, there is a natural extension to all of $\mathbb{Z}$ achieved by setting

$$\chi(z) := \chi\left(z \pmod n\right)$$

which is equivalent to a periodic extension of χ. Moreover, χ is a multiplicative map, since homomorphisms preserve the group operations.

There is always the trivial Dirichlet character, called the **principal character** which is simply the constant-map $\chi_0 \equiv 1$. Our first useful property is the following one:

Proposition 2. *Let $n \in \mathbb{N}$ and $\chi : \mathbb{Z}_n^* \to \mathbb{C}^*$ a Dirichlet character modulo n. Then, for all $\chi^{\varphi(n)} \equiv 1$. Namely, $\chi(x)$ is an n^{th} root of unity for all $x \in \mathbb{Z}_n^*$.*

Proof. Fix $x \pmod n$, or rather, its equivalence class. Now, since χ is assumed to be a homomorphism one has of course $\chi(x)^{\varphi(n)} = \chi\left(x^{\varphi(n)}\right) = \chi(1) = 1$.

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We note that it follows from the above that χ must always be a map from $\mathbb{Z}_n^*$ to $\{|z| = 1\}$: the unit disc in $\mathbb{C}$.

Our next goal is to "turn" the set of all these characters into a group. We note that modulo any n taking two characters χ, ψ the product:

$$(\chi\psi)(\cdot) = \chi(\cdot)\psi(\cdot)$$

allows us to discuss the "product characters". Namely, let us define by G the set of all characters modulo some $n \geq 1$, which we fix. Under the product above we observe that G becomes a group in its own right. Indeed, it is clear that this operation commutes

and it is only left to show that each character has an inverse character with respect to this operation. Fix χ, some character modulo n. If $\chi = \chi_0$ there is nothing to show, as this is our identity element. Else, consider a character defined by:

$$\chi^{-1}(\cdot) := \overline{\chi(\cdot)}$$

clearly, $\chi \cdot \chi^{-1} \equiv \chi_0$ and it is only left to verify that χ^{-1} is a Dirichlet character. Clearly, it fixes the identity element and preserves the group operations by simple properties of the complex conjugate and hence we know $(G, \odot)$ is a group, as was asserted.

What is of greater interest is the following:

Theorem 2. *Let $q = p^\alpha$ for some integer α and a prime p. Then, where G denotes the group as above modulo q, we have $\#G = \varphi(q)$. Moreover, G is a cyclic group.*

Proof. We may take a primitive root g for $\mathbb{Z}_q^*$. Now, since any character χ is a homomorphism it will be completely determined on $\mathbb{Z}_q^*$ by its value on g, i.e. $\chi(g)$. Now, we know from a previous proposition that $\chi(g)$ is a root of unity with $\varphi(q)$. That is,

$$\chi(g) = \exp\left(2\pi i \cdot \frac{a}{\varphi(q)}\right), \quad 0 \leq a < \varphi(q)$$

showing that they are distinct for different values of a.

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We now show some orthogonality results, which are in practice quite useful.

Theorem 3 (Orthogonality Theorems)**.** *Let $n \geq 1$ be an integer. Then;*

(1) *For each Dirichlet character modulo n one has:*

$$\sum_{a=1}^{n} \chi(a) = \begin{cases} \varphi(q) & \chi = \chi_0 \\ 0 & \textit{else} \end{cases} \tag{3}$$

(2) *For all $a \in \mathbb{N}$ one has:*

$$\sum_{\chi \bmod n} \chi(a) = \begin{cases} \varphi(q) & a \equiv 1 \pmod{n} \\ 0 & \textit{else} \end{cases} \tag{4}$$

(3) *For any two Dirichlet characters χ, ψ modulo n one has:*

$$\sum_{a=1}^{n} \chi(a)\overline{\psi(a)} = \begin{cases} \varphi(q) & \chi = \psi \\ 0 & \textit{else} \end{cases} \tag{5}$$

(4) *For all $a, b \in \mathbb{N}$ one has:*

$$\sum_{\chi \bmod n} \chi(a)\overline{\chi(b)} = \begin{cases} \varphi(q) & a \equiv b \pmod{n} \textit{ and co-prime to } n \\ 0 & \textit{else} \end{cases} \tag{6}$$

Proof. We begin by proving (1). In the case $\chi = \chi_0$ the result is trivial since $\sum_{a=1}^{n} \chi(a) = \sum_{a \in \mathbb{Z}_n^*} \chi(a)$ and there are precisely $\varphi(n)$ such elements. Otherwise, we still need only

consider $\sum_{a\in\mathbb{Z}_n^*}\chi(a)$ but there is some $b\in\mathbb{Z}_n^*$ such that $\chi(b)\neq 1$. Since b is invertible modulo n, it is not difficult to see that $b\mathbb{Z}_n^*=\mathbb{Z}_n^*$ and therefore;

$$\chi(b)\sum_{a\in\mathbb{Z}_n^*}\chi(a)=\sum_{a\in\mathbb{Z}_n^*}\chi(ab)=\sum_{\alpha\in\mathbb{Z}_n^*}\chi(\alpha)$$

proving that this sum vanishes, since $\chi(b)\neq 1$.

To prove (2), we argue in a very similar fashion. Consider now the case $a\equiv 1 \pmod n$, since the χ always fix the identity and we sum over a set of cardinality $\#\mathbb{Z}_n^*=\varphi(n)$ the result follows. Now, assume that $a\not\equiv 1 \pmod n$. Then, there is some Dirichlet character ψ such that $\psi(a)\neq 1$ (we know this from the previous theorem). Now, let us calculate:

$$\psi(a)\sum_{\chi \bmod n}\chi(a)=\sum_{\tau \bmod n}\tau(a)$$

since if $h\in G$ where G is Abelian then $hG=G$.
Define now $\tau:=\chi\cdot\overline{\psi}$. We need only appeal to (1) here with τ and noting that $\tau=\chi_0$ if and only if $\chi=\psi$.
Much like in (3) we shall reduce this to the case established in (2). Note here that the result is trivial if one of a,b shares a divisor with n, as it vanishes. Therefore we may presume without loss of generality that $a,b\in\mathbb{Z}_n^*$. Note that it is a general property of a homomorphism σ of groups that $\sigma(g^{-1})=\sigma(g)^{-1}$. Therefore,

$$\sum_{\chi \bmod n}\chi(a)\overline{\chi(b)}=\sum_{\chi \bmod n}\chi(a)\chi(b)^{-1}=\sum_{\chi \bmod n}\chi(ab^{-1})$$

passing to (2) we have the result since $ab^{-1}\equiv 1 \pmod n$ if and only if $b\equiv a \pmod n$.

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3. Abel Summation

In this section we derive an identity used in the computation of certain series which is of practical use in many situation. Consider now a function $f:I\subseteq\mathbb{R}\to\mathbb{C}$ that is of class C^1 and fix an associated series $\sum_{A<n\leq B}a_nf(n)$ where $A,B\in\mathbb{R}$ and $a_n\in\mathbb{C}$ for all indices. Of course, we shall presume that $[A,B]\subsetneq I$ and that I is open in $\mathbb{R}$.

We begin by defining:

$$S(x):=\sum_{n=1}^{x}a_n,\quad n\in\mathbb{Z}$$

Whereby it is easy to calculate:

$$\begin{aligned}\sum_{A<n\leq B}a_nf(n)&=\sum_{A<n\leq b}\left(S(n)-S(n-1)\right)f(n)\\&=\sum_{A<n\leq B}S(n)f(n)-\sum_{A<n\leq B}S(n-1)f(n)\\&=\sum_{A<n\leq B}S(n)f(n)-\sum_{A-1<n\leq B-1}S(n)f(n+1)\end{aligned}$$

However, this last line is precisely the quantity given below by [1]:

$$S(B)f(B) - S(A)f(A) + \sum_{A\leq n<B} S(n)f(n) - \sum_{A-1<n\leq B-1} S(n)f(n+1)$$

which is, of course, merely

$$S(B)f(B) - S(A)f(A) - \sum_{A\leq n<B} S(n)\left(f(n+1) - f(n)\right) \tag{♭}$$

We turn our attention to the tail term $\sum_{A\leq n<B} S(n)\left(f(n+1) - f(n)\right)$. This is, by the Fundamental Theorem of Calculus:

$$\sum_{A\leq n<B} S(n)\left(f(n+1) - f(n)\right) = \sum_{A\leq n<B} S(n)\int_n^{n+1} f'(x)\,\mathrm{d}x$$

where we have used that f' is continuous and therefore Riemann integrable. Summing these integrals, we obtain now that:

$$\begin{aligned}\sum_{A\leq n<B} S(n)\int_n^{n+1} f'(x)\,\mathrm{d}x &= \sum_{A\leq n<B} S(n)\int_n^{n+1} f'(x)\,\mathrm{d}x\\ &= \sum_{A\leq n<B}\int_n^{n+1} S(x)f'(x)\,\mathrm{d}x\\ &= \int_A^B S(x)f'(x)\,\mathrm{d}x\end{aligned}$$

Putting this last line together with (♭) we obtain Abel's Summation Formula:

$$\boxed{\sum_{A<n\leq B} a_n f(n) = f(B)S(B) - f(A)S(A) - \int_A^B S(x)f'(x)\,\mathrm{d}x} \tag{Abel}$$

In the above we have $\sum_{n=1}^{x} a_n = S(x)$. Now, an example is in order.

Example. Evaluate $\sum_{n=1}^{N}\log n$ for N fixed. Here the coefficients are fixed (i.e. the a_n) and so it is quite easy to see that for all $x\in\mathbb{R}_+$ one has:

$$S(x) = \sum_{n=1}^{x} 1 = \lfloor x\rfloor$$

Therefore, we employ the formula in (Abel) to calculate:

$$\sum_{n=1}^{N}\log x = N\log N - \log 0 - \int_1^N \frac{\lfloor x\rfloor}{x}\,\mathrm{d}x = N\log N - \int_1^N \frac{\lfloor x\rfloor}{x}\,\mathrm{d}x$$

Recall the notation $\{x\} = x - \lfloor x\rfloor$. We then have by the above

$$\sum_{n=1}^{N}\log x = N\log N - \int_1^N \frac{x-\{x\}}{x}\,\mathrm{d}x = N\log N - N + \int_1^N \frac{\{x\}}{x}\,\mathrm{d}x$$

[1] This may be seen by "playing around" with the inequalities of the summations.

Bounding this last integral using $\int_1^N \frac{\{x\}}{x}\,\mathrm{d}x = \mathcal{O}(\log N)$ we deduce that:

$$\sum_{n=1}^{N} \log n = N\log N - N + \mathcal{O}(\log N)$$

4. The Method of Hyperbolas

This section is dedicated to *Dirichlet's Hyperbola Method*, which we may sum up in the following theorem:

Theorem 4. *Let $x \geq 1$, then*

$$D(x) := \sum_{n\leq x} d(n) = x\log x + 2(\gamma-1)x + \mathcal{O}\left(\sqrt{x}\right)$$

Where $d(n) = \sum_{r|n} 1$ is the divisor function and γ is a constant defined by:

$$\gamma := 1 - \int_1^{\infty} \frac{\{t\}}{t^2}\,\mathrm{d}t$$

Our proof begins with a lemma/calculation. This lemma is "well-placed", as it provides us with another example of Abel's summation formula. Consider now the summation (for $x \geq 3$):

$$\sum_{n\leq x} \frac{1}{n}$$

we may relate this to the equation in (Abel) by taking $a_n = 1$ for all $n \in \mathbb{N}$ and $S(x) := \lfloor x \rfloor$. Then, using Abel's summation formula we recover

$$\sum_{n\leq x} \frac{1}{n} = \frac{1}{x}S(x)^{\,2} + \int_1^x \frac{S(t)}{t^2}\,\mathrm{d}t = \frac{x-\{x\}}{x} + \int_1^x \frac{t-\{t\}}{t^2}\,\mathrm{d}t$$

This is precisely, $1 + \mathcal{O}\left(\frac{1}{x}\right) + \log x - \int_1^{\infty} \frac{\{t\}}{t^2}\,\mathrm{d}t$, i.e.

$$\log x + \mathcal{O}\left(\frac{1}{x}\right) + \gamma \qquad (\blacktriangle)$$

The time has now come to estimate our main function $D(x) := \sum_{n\leq x} d(n)$. We note that we may re-write this as follows:

$$D(x) = \sum_{n\leq x} d(n) = \sum_{n\leq x}\sum_{r|n} 1 = \sum_{ab\leq x} 1$$

Indeed, note that if $n \leq x$ and $r \mid n$ then $n = \alpha\cdot r \leq x$ and conversely if $ab \leq x$ then taking $n = ab$ we sum over $a \mid n$ in the LHS. Therefore, we need only study the summation $\sum_{ab\leq x} 1$. Note that if $ab \leq x$ for integers a, b then we must have one of $a \leq \sqrt{x}$ or $b \leq \sqrt{x}$ for otherwise one has

$$ab > \sqrt{x}\cdot\sqrt{x} = x$$

[2]Here we use the fact that Abel's formula has a strict inequality in the series indices, and hence here our $A = 0$.

Now, we may write $\sum_{ab\leq x} 1 = \sum_1 + \sum_2 - \sum_3$ where:

$$\Sigma_1 := \sum_{a\leq\sqrt{x}} \sum_{b\leq\frac{x}{a}} 1, \quad \Sigma_2 := \sum_{b\leq\sqrt{x}} \sum_{a\leq\frac{x}{b}} 1, \quad \Sigma_3 = \sum_{\substack{a\leq\sqrt{x}\\ b\leq\sqrt{x}}} 1$$

To see that $\sum_{ab\leq x} 1$ may indeed be expressed in this way, we need only note that the pairs (a,b) for which $a \leq \sqrt{x}$ **and** $b \leq \sqrt{x}$ appear twice: once in $\sum_1$ and again in $\sum_2$. To correct this, we remove one "copy" in $\sum_3$. Let us now observe that it is obvious by symmetry that $\sum_1 = \sum_2$ and thence it follows that

$$\begin{aligned}\sum_{ab\leq x} 1 = 2\sum_{a\leq\sqrt{x}} \sum_{b\leq\frac{x}{a}} 1 - \sum_{\substack{a\leq\sqrt{x}\\ b\leq\sqrt{x}}} 1 &= 2\sum_{a\leq\sqrt{x}} \sum_{b\leq\frac{x}{a}} 1 - \left\lfloor\sqrt{x}\right\rfloor^2 \\ &= 2\sum_{a\leq\sqrt{x}} \left(\frac{x}{a} + \mathcal{O}\left(1\right)\right) - \left\lfloor\sqrt{x}\right\rfloor^2\end{aligned}$$

This is precisely,

$$2x\sum_{a\leq\sqrt{x}} \left(\frac{1}{a} + \mathcal{O}\left(\frac{1}{x}\right)\right) - x + \mathcal{O}\left(1\right) = 2x\left(\sum_{a\leq\sqrt{x}} \frac{1}{a} + \mathcal{O}\left(\frac{1}{\sqrt{x}}\right)\right) - x + \mathcal{O}\left(1\right)$$

Which is

$$\begin{aligned}2x\sum_{a\leq\sqrt{x}} \frac{1}{a} + \mathcal{O}\left(\sqrt{x}\right) - x + \mathcal{O}\left(1\right) &= 2x\sum_{a\leq\sqrt{x}} \frac{1}{a} + \mathcal{O}\left(\sqrt{x}\right) - x \\ &= 2x\left(\log\sqrt{x} + \gamma + \mathcal{O}\left(\frac{1}{\sqrt{x}}\right)\right) - x + \mathcal{O}\left(\sqrt{x}\right) \\ &= x\log x + 2x\gamma - x + \mathcal{O}\left(\sqrt{x}\right)\end{aligned}$$

this last line is precisely $x\log x + (2\gamma - 1) + \mathcal{O}\left(\sqrt{x}\right)$; having used the identity in (▲), thereby proving the theorem stated at the beginning of this section.

5. Topics and Solved Problems

In this less formal section we present some results from number theory and demonstrate some "simple" problems that may solved by applying these results. We recall the following definition:

Definition (Liouville)**.** *A Liouville number is an irrational number x if there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ we may find a rational number $\frac{p}{q}$ such that:*

$$\left|x - \frac{p}{q}\right| \leq \frac{c}{q^n}$$

where $q > 1$.

We claim the following, in the hopes of obtaining a method for proving that certain numbers are transcendental. Note first of all that requiring x to be irrational in the above definition is of no great harm, since we will show below that all Liouville numbers are transcendental, and all transcendental numbers are irrational, in any case.

Theorem 5. *Any Liouville number is transcendental.*

Proof. Here we argue by contradiction. Let η be a Liouville number. We construct a sequence of integers $(p,q)_n$ as follows: for $n \in \mathbb{N}$ pick a pair $(p,q) \in \mathbb{Z}^2$ with $q > 1$ such that

$$\left|\eta - \frac{p}{q}\right| \leq \frac{c}{q^n}$$

Assume now that there is some $f \in \mathbb{Z}[x]$ with $m = \deg f$ and $f(\eta) = 0$. Note that f has at-most m roots in $\mathbb{R}$, and hence at most finitely many of these (p,q) correspond to rational roots of this polynomial f, for otherwise in passing to a constant subsequence $\{(p,q)\}$ one would have $\eta = \frac{p}{q} \in \mathbb{Q}$. Hence, we only need handle the case for all large n.

Now, since f is a polynomial it is $C^\infty(\mathbb{R})$ and therefore Lipschitz continuous with constant $L > 0$ on any large compact interval about η. Observe that since $f \in \mathbb{Z}[x]$ we trivially have for all (p,q) associated to n large:

$$\left|f\left(\frac{p}{q}\right)\right| \geq \frac{1}{q^m}$$

Therefore,

$$\frac{1}{q^m} \leq \left|f\left(\frac{p}{q}\right)\right| = \left|f(\eta) - f\left(\frac{p}{q}\right)\right| \leq L\left|\eta - \frac{p}{q}\right| \leq \frac{Lc}{q^n}$$

and taking n large gives a contradiction.

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Of course to properly illustrate this result it is best to give an example. Primarily, we wish to show a particular number is transcendental. Numbers such as π and e are themselves transcendental, but this is a result beyond this text. This example is the subsequent one:

Example. Define $\eta := \sum_{n\in\mathbb{N}} \frac{(-1)^n}{10^{n!}}$. We claim that η is transcendental. Of course, this is an exercise in Liouville's theorem. To see that η is irrational, it suffices to see that it will have a non-terminating and non-periodic decimal expansion. Clearly, it cannot terminate and moreover cannot be repeating due to the growth-rate of $n!$.

By Theorem 5 we may show that η is a Liouville number. Let $N \in \mathbb{N}$ be given. Note first off that

$$\left|\eta - \sum_{n\leq N} \frac{(-1)^n}{10^{n!}}\right| \leq \sum_{n>N} \frac{1}{10^{n!}}$$

Take now the two integers $p := \sum_{n=1}^{N} \frac{(-1)10^{N!}}{10^{n!}}$ and $q = 10^{N!}$. Clearly, $q > 1$ and $\frac{p}{q} = \sum_{n \leq N} \frac{(-1)^n}{10^{n!}}$. From the above we then have;

$$\left| \eta - \frac{p}{q} \right| \leq \sum_{n>N} \frac{1}{10^{n!}} = \frac{1}{10^{(N+1)!}} \cdot \sum_{n>N} \frac{10^{(N+1)!}}{10^{n!}} \leq \frac{1}{10^{(N+1)!}} \cdot \frac{10}{9}$$

Hence,

$$\left| \eta - \frac{p}{q} \right| \leq \frac{c}{10^{(N+1)!}} = \frac{c}{q^{N+1}}$$

Solving Pell Equations. This is best illustrated with an example. We seek integral solutions to $x^2 - 4y^2 = 1$. Since $2^2 = 4$ this is reduced to solving

$$(x - 2y)(x + 2y) = 1$$

whence we have the system $\begin{cases} x - 2y = \pm 1 \\ x + 2y = \pm 1 \end{cases}$ Which is easy to solve.

Let us now consider some case where the constant is not a perfect square. For instance, we seek solutions to $x^2 - 7y^2 = 1$. To illustrate this procedure, we shall begin by computing the continued fraction representation of $\sqrt{7}$. Of course, this entails the computation of the *continued fraction representation* of $\sqrt{7}$. Note that

$$4 < 7 < 9 \implies 2 < \sqrt{7} < 3 \implies \sqrt{7} = a_0 + (\sqrt{7} - 2), \quad a_0 := 2$$

$$\frac{1}{\sqrt{7} - 2} = \frac{1}{\sqrt{7} - 2} \cdot \frac{\sqrt{7} + 2}{\sqrt{7} + 2} = \frac{\sqrt{7} + 3 - 1}{3} = 1 + \frac{\sqrt{7} - 1}{3}, \quad a_1 := 1,\ t_1 := \frac{\sqrt{7} - 1}{3}$$

$$\frac{3}{\sqrt{7} - 1} \cdot \frac{\sqrt{7} + 1}{\sqrt{7} + 1} = \frac{\sqrt{7} + 2 - 1}{2} = 1 + \frac{\sqrt{7} - 1}{2}, \quad a_2 := 1,\ t_2 := \frac{\sqrt{7} - 1}{2}$$

$$\frac{2}{\sqrt{7} - 1} \cdot \frac{\sqrt{7} + 1}{\sqrt{7} + 1} = \frac{\sqrt{7} + 1}{3} = 1 + \frac{\sqrt{7} - 2}{3}, \quad a_3 := 1,\ t_3 := \frac{\sqrt{7} - 2}{3}$$

$$\frac{3}{\sqrt{7} - 2} \cdot \frac{\sqrt{7} + 2}{\sqrt{7} + 2} = \sqrt{7} + 2 = 4 + (\sqrt{7} - 2), \quad a_4 := 4,\ t_4 := (\sqrt{7} - 2)$$

Therefore, $\sqrt{7} = \left[2, \overline{1, 1, 1, 4}\right]$. Clearly, this has period $\rho := 4$. In the general case, we have $\rho = 2k$ and set

$$(x_1, y_1) = \begin{cases} (p_{\rho-1}, q_{\rho-1}) & \text{if } k \text{ is even} \\ (p_{2\rho-1}, q_{2\rho-1}) & \text{if } k \text{ is odd} \end{cases}$$

Up above we have k even, and hence we set $(x_1, y_1) = (p_3, q_3)$.