ON THE EXISTENCE OF AN UNCOUNTABLE DENSE SET OF LEBESGUE MEASURE ZERO IN $\mathbb{R}^n$

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The purpose of this note is to establish the existence of an uncountable, yet dense, subset of $\mathbb{R}^n$ having Lebesgue measure 0. Despite being a well known fact, it is remarkable and counter intuitive. Throughout this text $m(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^n$.

Proposition. *There exists an uncountable set of Lebesgue measure zero dense in* $\mathbb{R}^n$.

Proof. Let $\mathcal{C}$ denote the Cantor dust in $\mathbb{R}^n$ constructed by taking the cartesian product of n Cantor sets; for two sets $A, B \subseteq \mathbb{R}^n$ introduce the notation

$$A + B \stackrel{\text{def}}{=} \{a + b : a \in A,\, b \in B\}\,.$$

Put

$$K \stackrel{\text{def}}{=} \mathbb{Q}^n + \mathcal{C},$$

we claim that this is the desired subset of $\mathbb{R}^n$. We must show that K has Lebesgue measure zero. Indeed, to see that this is true we need only observe that

$$K = \bigcup_{r \in \mathbb{Q}^n} \{r + c : c \in \mathcal{C}\}$$

whence

$$m(K) \leq \sum_{r \in \mathbb{Q}^n} m\left(\{r + c : c \in \mathcal{C}\}\right) = \sum_{r \in \mathbb{Q}^n} m\left(\mathcal{C}\right) = 0$$

by translation invariance of $m(\cdot)$. Since $0 \in \mathcal{C}$ it is clear that $\mathbb{Q}^n \subset K$, and hence K is dense in $\mathbb{R}^n$. Finally, we see that K is uncountable since $\mathcal{C} \subset K$. $\square$

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