SOME FACTS ABOUT CONVERGENCE IN PROBABILITY AND DISTRIBUTION

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Lemma 1. *Let* $(\Omega, \Sigma, \mathbb{P})$ *be a probability space and* $X : \Omega \to \mathbb{R}$ *a random variable with distribution function* $F_X : \mathbb{R} \to [0,1]$. *Suppose that* x *is a continuity point of* $F_X(x)$. *Then,*

$$\mathbb{P}(X \leq x) = \mathbb{P}(X < x), \quad \mathbb{P}(X > x) = \mathbb{P}(X \geq x)$$

Proof. By simply observing that $\{X < x\} \subseteq \{X \leq x\}$ we have $\mathbb{P}(X < x) \leq \mathbb{P}(X \leq x)$. Now let $\varepsilon > 0$ be given and fix a continuity point $x \in \mathbb{R}$. Note that $\{X \leq x - \varepsilon\} \subseteq \{X < x\}$ and thus

$$F_X(x - \varepsilon) \leq \mathbb{P}(X < x), \quad \forall \varepsilon > 0$$

Since x is a continuity point, and $\varepsilon > 0$ was arbitrarily given, we may pass to the limit to discover

$$\mathbb{P}(X \leq x) = F_X(x) = \lim_{\varepsilon \to 0} F_X(x - \varepsilon) \leq \mathbb{P}(X < x)$$

We shall now show the second equality. This is a reprise of the above argument. We again have by monotonicity that $\mathbb{P}(X > x) \leq \mathbb{P}(X \geq x)$. For the reverse inclusion, let $\varepsilon > 0$ and note again that $\{X \geq x + \varepsilon\} \subseteq \{X > x\}$ so that

$$\mathbb{P}(X \geq x + \varepsilon) \leq \mathbb{P}(X > x)$$

Since x is a continuity point, we again pass to the limit as $\varepsilon \to 0$ to conclude that

$$\mathbb{P}(X \geq x) = F_X(x) = \lim_{\varepsilon \to 0} F_X(x + \varepsilon) \leq \mathbb{P}(X > x)$$

which completes the proof of this lemma. □

Lemma 2. *Let* $(\Omega, \Sigma, \mathbb{P})$ *be a probability space and* $X : \Omega \to \mathbb{R}$ *a random variable and fix* $a, b \in \mathbb{R}$ *with* $a \neq 0$. *If* x *is a continuity point of the distribution for* $aX + b$ *then* $\frac{x-b}{a}$ *is a continuity point of* X.

Proof. First suppose $a > 0$ and let x be as given and suppose (γ_n) is a sequence of real numbers converging to $\gamma := \frac{x-b}{a}$. Consider now

$$\lim_{n \to \infty} \mathbb{P}(X \leq \gamma_n) = \lim_{n \to \infty} \mathbb{P}(aX + b \leq a\gamma_n + b)$$

Now, $a\gamma_n + b$ is a sequence of real numbers converging to x, a continuity point of $aX + b$. Thence,

$$\begin{aligned}\lim_{n \to \infty} \mathbb{P}(X \leq \gamma_n) = \lim_{n \to \infty} \mathbb{P}(aX + b \leq a\gamma_n + b) &= \mathbb{P}(aX + b \leq x) \\ &= \mathbb{P}\left(X \leq \frac{x-b}{a}\right)\end{aligned}$$

Since (γ_n) was an arbitrary sequence, we have completed this step. If instead $a < 0$, then by the previous lemma and the very same argument as above

$$\lim_{n \to \infty} \mathbb{P}(X \leq \gamma_n) = \mathbb{P}(aX + b \geq x)$$

However,

$$\mathbb{P}\left(aX+b \leq x\right) = \mathbb{P}\left(X \leq \frac{x-b}{a}\right)$$

which completes the proof since (γ_n) was taken without restriction. □

Theorem 3. *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and suppose $a, b \in \mathbb{R}$ and $X : \Omega \to \mathbb{R}$ a random variable on the space. If there exists a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables on the space so that*

$$X_n \xrightarrow{d} X, \quad n \to \infty$$

then

$$aX_n + b \xrightarrow{d} aX + b, \quad n \to \infty$$

Proof. The case $a = 0$ is trivial. Fix $x \in \mathbb{R}$ at the distribution of $aX + b$ is continuous. Note then that by the previous lemma X is continuous at $\frac{x-b}{a}$.

(1) $a > 0$. In this case the event $\{aX + b \leq x\} = \{x \leq \frac{x-b}{a}\}$. Therefore,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(aX_n + b \leq x) = \lim_{n\to\infty} \mathbb{P}\left(X_n \leq \frac{x-b}{a}\right) &= \mathbb{P}\left(X_n \leq \frac{x-b}{a}\right)\\ &= \mathbb{P}(aX + b \leq x)\end{aligned}$$

(2) Suppose $a < 0$. Then,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(aX_n + b \leq x) = \lim_{n\to\infty} \mathbb{P}\left(X_n \geq \frac{x-b}{a}\right) &= \lim_{n\to\infty}\left[1 - \mathbb{P}\left(X_n < \frac{x-b}{a}\right)\right]\\ &= 1 - \lim_{n\to\infty} \mathbb{P}\left(X_n \leq \frac{x-b}{a}\right)\\ &= 1 - \mathbb{P}\left(X \leq \frac{x-b}{a}\right)\\ &= 1 - \mathbb{P}\left(aX + b \geq x\right)\\ &= 1 - \mathbb{P}\left(aX + b > x\right)\\ &= \mathbb{P}\left(aX + b \leq x\right)\end{aligned}$$

This is what we had to show. □

Theorem 4. *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $X, Y : \Omega \to \mathbb{R}$ non-trivial random variables on the sample subspace (Ω, Σ). Further assume there exist two auxiliary sequences $\{X_\ell\}_{\ell\in\mathbb{N}}, \{Y_\ell\}_{\ell\in\mathbb{N}}$ so that*

$$X_\ell \xrightarrow{\mathbb{P},\ \ell\to\infty} X, \quad Y_\ell \xrightarrow{\mathbb{P},\ \ell\to\infty} Y$$

Then,

(1) $X_\ell + Y_\ell \xrightarrow{\mathbb{P},\ \ell\to\infty} X + Y$
(2) $X_\ell Y_\ell \xrightarrow{\mathbb{P},\ \ell\to\infty} XY$

Proof. We shall distinguish both results.

(1) Note that $|X_\ell + Y_\ell - (X+Y)| \leq |X_\ell - X| + |Y - \ell - Y|$ and thus whenever $\varepsilon > 0$ is given:

$$\mathbb{P}(|X_\ell + Y_\ell - (X+Y)| \geq 2\varepsilon) \leq \mathbb{P}(|X_\ell - X| + |Y_\ell - Y| \geq 2\varepsilon)$$

Or,

$$\mathbb{P}(|X_\ell + Y_\ell - (X+Y)| \geq 2\varepsilon) \leq \mathbb{P}(|X_\ell - X| \geq \varepsilon) + \mathbb{P}(|Y_\ell - Y| \geq \varepsilon)$$

taking $n \to \infty$ yields the desired result by hypothesis on $\{X_\ell\}, \{Y_\ell\}$.

(2) This is somewhat more difficult. Let $\varepsilon > 0$, and note that

$$|X_\ell Y_\ell - XY| = |X_\ell Y_\ell - X_\ell Y + X_\ell Y - XY| \leq |X_\ell|\,|Y_\ell - Y| + |Y|\,|X_\ell - X|$$

implying

$$\mathbb{P}(|X_\ell Y_\ell - XY| \geq 2\varepsilon) \leq \mathbb{P}\left(|X_\ell|\,|Y_\ell - Y| \geq \varepsilon\right) + \mathbb{P}(|Y|\,|X_\ell - X| \geq \varepsilon)$$

We shall only estimate the first term on the right, as the same argument applies to the second. Now, we claim that for all $M > 0$:

$$\{|X_\ell|\,|Y_\ell - Y| \geq \varepsilon\} \subseteq \{|X_\ell - X| \geq 1\} \cup \{|X + 1| \geq M\} \tag{1}$$

$$\cup \left\{|Y_n - Y| \geq \frac{\varepsilon}{M+2}\right\} \tag{2}$$

Indeed, if $\omega \in \{|X_\ell|\,|Y_\ell - Y|$ but $\omega \notin \{|X_\ell - X| \geq 1\} \cup \{|X + 1| \geq M\}$ then we have

$$|X_\ell - X| < 1 \implies |X_\ell| < 1 + |X|$$

Thus,

$$|Y_\ell - Y| \geq \frac{\varepsilon}{|X_n|} \geq \frac{\varepsilon}{M+2}$$

Which proves the inclusion. Two of the above tend to 0 as $n \to \infty$ by convergence in probability. For the middle, take M very large to make this term as small as you'd like. Thus, we have established what was required.

□

Theorem 5. *In the context of the previous problem, suppose that* $X_n \xrightarrow{d} X$ *and* $Y_n \xrightarrow{\mathbb{P}} c$*, then*

$$X_n + Y_n \xrightarrow{d} X + c$$

Proof. Let us first observe that we may assume wlog that $c = 0$. Certainly, if not then write

$$Z_n = X_n + c, \quad W_n = Y_n - c$$

Then $Y_n \to 0$ in probability and $X_n \to X - c$ in distribution. Having noted this, let x be a continuity point of $X = X + c$. Consider for given $\varepsilon > 0$:

$$\begin{aligned}
\mathbb{P}\left(X_n + Y_n \leq x\right) &= \mathbb{P}\left(X_n + Y_n \leq x,\ |Y_n| < \varepsilon\right) + \mathbb{P}\left(X_n + Y_n \leq x,\ |Y_n| \geq \varepsilon\right) \\
&\leq \mathbb{P}\left(X_n + Y_n \leq x,\ |Y_n| < \varepsilon\right) + \mathbb{P}(|Y_n| \geq \varepsilon) \\
&\leq \mathbb{P}\left(X_n \leq x + \varepsilon\right) + \mathbb{P}(|Y_n| \geq \varepsilon)
\end{aligned}$$

In a similar vein,

$$\begin{aligned}
\mathbb{P}(X_n \leq x - \varepsilon) &= \mathbb{P}\left(X_n \leq x - \varepsilon,\ |Y_n| < \varepsilon\right) + \mathbb{P}\left(X_n - \varepsilon \leq x,\ |Y_n| \geq \varepsilon\right) \\
&\leq \mathbb{P}\left(X_n \leq x - \varepsilon,\ |Y_n| < \varepsilon\right) + \mathbb{P}\left(|Y_n| \geq \varepsilon\right) \\
&\leq \mathbb{P}\left(X_n + Y_n \leq x\right) + \mathbb{P}\left(|Y_n| \geq \varepsilon\right)
\end{aligned}$$

Putting all this together yields,

$$\mathbb{P}(X_n \leq x - \varepsilon) - \mathbb{P}\left(|Y_n| \geq \varepsilon\right) \leq \mathbb{P}\left(X_n + Y_n \leq x\right) \leq \mathbb{P}\left(X_n \leq x + \varepsilon\right) + \mathbb{P}(|Y_n| \geq \varepsilon)$$

Wlog suppose that $x \pm \varepsilon$ are continuity points of the DF of X (continuous almost everywhere), then we may let $n \to \infty$ for each of these to discover

$$\mathbb{P}(X \leq x - \varepsilon) \leq \lim_{n \to \infty} \mathbb{P}(X_n + Y_n \leq x) \leq \mathbb{P}(X \leq x + \varepsilon)$$

Now we may let ε shrink, which gives us the desired result, since x is a continuity point of F_X. □

Problem. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $\{X_\ell\}_\ell$ a sequence of i.i.d random variables with $\mathbb{E}[X_1] = m$ and $\mathrm{Var}(X_1) = \sigma^2 > 0$. Let $\{Y_\ell\}_\ell$ be another i.i.d sequence of random variables with $\mathbb{E}[Y_1] = \mu > 0$ and $\mathrm{Var}(Y_1) = \gamma^2 > 0$. Define

$$\overline{S_n} := \frac{1}{n}\sum_{\jmath=1}^{n} X_\jmath, \quad \overline{T_n} := \frac{1}{n}\sum_{\jmath=1}^{n} Y_\jmath$$

and

$$Z_n := \frac{\sqrt{n}\left(\overline{S_n} - m\right)}{\overline{T_n}}$$

show that Z_n converges in distribution to some random variable Y and determine the limiting distribution.

Solution. First note that by the weak law of large numbers we may assert that

$$\overline{T_n} \xrightarrow{\mathbb{P}} \mu > 0 \implies \frac{1}{\overline{T_n}} \xrightarrow{\mathbb{P}} \frac{1}{\mu}$$

Moreover, the Central Limit theorem states that

$$\frac{\sqrt{n}\left(\overline{S_n} - m\right)}{\sigma} = \frac{S_n - nm}{\sqrt{n}\sigma} \xrightarrow{\mathrm{d}} Y \sim \mathcal{N}(0,1)$$

We shall now use the previous theorem to deduce that:

$$Z_n = \frac{\sqrt{n}\left(\overline{S_n} - m\right)}{\overline{T_n}} = \frac{\sqrt{n}\left(\overline{S_n} - m\right)}{\sigma} \cdot \frac{\sigma}{\overline{T_n}} \xrightarrow{\mathrm{d}} \frac{\sigma}{\mu} Y$$

where $Y \sim \mathcal{N}(0,1)$ and thus $\mathcal{W} := \dfrac{\sigma}{\mu} Y \sim \mathcal{N}\left(0, \dfrac{\sigma^2}{\mu^2}\right)$.

Problem. Prove that

$$\lim_{n\to\infty} e^{-nt}\sum_{k=0}^{n}\frac{(nt)^k}{k!} = \begin{cases} 0 & t > 1 \\ \frac{1}{2} & t = 1 \\ 1 & 0 < t < 1 \end{cases} \tag{3}$$

Solution. Fix any admissible t. Let $X_1, X_2, \cdots, X_n$ be i.i.d random variables with $X_i \sim \mathcal{P}(t)$. Then, adopting the notation above, $S_n \sim \mathcal{P}(nt)$.

$$e^{-nt}\sum_{k=0}^{n}\frac{(nt)^k}{k!} = \mathbb{P}\left(S_n \leq n\right)$$

Now, $\mathbb{E}[X_i]$ for any such i is given by t. Moreover, $\mathrm{Var}(X_i) = t$. Thus, the Central Limit Theorem dictates the following:

$$\frac{S_n - nt}{\sqrt{nt}} \xrightarrow{\mathrm{d}} \Phi \sim \mathcal{N}(0,1) \tag{4}$$

Thus,

$$\mathbb{P}(S_n \leq n) = \mathbb{P}\left(\frac{S_n - nt}{\sqrt{nt}} \leq \frac{n(1-t)}{\sqrt{nt}}\right) \xrightarrow{n\to\infty} \Phi\left(\frac{n(1-t)}{\sqrt{nt}}\right)$$

$$= \begin{cases} \Phi(-\infty) = 0 & t > 1 \\ \Phi(0) = \frac{1}{2} & t = 1 \\ \Phi(\infty) = 1 & 0 < t < 1 \end{cases}$$