A USEFUL CONSTRUCTION OF A HOLOMORPHIC COMPLEX LOGARITHM

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We begin by considering the following problem, given a non-zero complex number z does there exist a complex number w such that $e^w = z$? Yes, however the answer is not unique. Moreover, in general we cannot simply choose a single determination. We may write for $\arg z \in [0, 2\pi)$

$$z = |z|\, e^{i \arg z} = |z| \left(\cos \arg z + i \sin \arg z\right)$$

Thus, if we write $w = x + iy$

$$e^w = e^{x+iy} e^x e^{iy} = |z|\, e^{i \arg z}$$

Equating now real and imaginary parts yields $e^w = |z|$ and $e^{iy} = e^{i \arg z}$. The former has a former *unique* solution $x = \ln |z|$ and the latter implies

$$y = \arg z$$

The trouble here is that the argument of z **not** uniquely determined, indeed simply the complex number 1 may be expressed as

$$1 = 1 \cdot e^0 = 1 \cdot e^{2\pi i} = \ldots = 1 \cdot 1^{2\pi i n}, \quad n \in \mathbb{Z}$$

Nonetheless, we may say several nice things about the complex logarithm. In fact, we shall construct one here.

Theorem. *Let $\Gamma \subset \mathbb{C}$ be a simple closed curve and f be holomorphic on and inside Γ with $f \neq 0$ everywhere. There exists a holomorphic* single valued *function w so that*

$$f(z) = e^{w(z)} \quad \textit{inside and on } \Gamma \tag{1}$$

Proof. We proceed in 4 steps.

Step 1. Here we define the logarithm. Fix an interior point of the curve Γ and let $z_1, z_2 \in \Gamma$ be points such that the segments

$$\lambda : [0,1] \to \mathbb{C}, \quad z_1 \to z_0$$
$$\lambda' : [0,1] \to \mathbb{C}, \quad z_2 \to z_0$$

lie in inside the curve Γ, except the points $z_1, z_2 \in \Gamma$. We will begin by showing that

$$f(z_1) \exp \int_\lambda \frac{f'(\xi)}{f(\xi)} d\xi = f(z_2) \exp \int_{\lambda'} \frac{f'(\xi)}{f(\xi)} d\xi \tag{2}$$

Along the arc $\overline{z_1 z_2} \subset \Gamma$ with positive orientation consider the function

$$\frac{1}{f(\zeta)} \exp \int_{z_1}^{\zeta} \frac{f'(\xi)}{f(\xi)} d\xi$$

where the integral is taken along this same curve. This is defined as f is holomorphic and non vanishing on Γ. Taking the derivative,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{1}{f(\zeta)}\exp\int_{z_1}^{\zeta}\frac{f'(\xi)}{f(\xi)}d\xi\right) \\ = -\frac{f'(\zeta)}{f(\zeta)^2}\exp\int_{z_1}^{\zeta}\frac{f'(\xi)}{f(\xi)}d\xi + \frac{1}{f(\zeta)}\cdot\exp&\int_{z_1}^{\zeta}\frac{f'(\xi)}{f(\xi)}d\xi\cdot\frac{f'(\zeta)}{f(\zeta)} \equiv 0\end{aligned}$$

for any ζ chosen along this sub-path of Γ. Thus, it is constant in ζ, it takes value $\dfrac{1}{f(z_1)}$ at $\zeta = z_1$, and must also assume this value at $\zeta = z_2$, that is:

$$\frac{1}{f(z_1)} = \frac{1}{f(z_2)}\exp\int_{z_1}^{z_2}\frac{f'(\xi)}{f(\xi)}d\xi$$

and consequently

$$\frac{f(z_2)}{f(z_1)} = \exp\int_{z_1}^{z_2}\frac{f'(\xi)}{f(\xi)}d\xi =: \exp\int_{\overline{z_1z_2}}\frac{f'(\xi)}{f(\xi)}d\xi \tag{3}$$

Now consider the positive closed curve with positive orientation from by travelling along $\overline{z_1z_2}$, along λ' and λ^-, which is λ with reverse orientation. Denoting this closed curve by γ we see by Cauchy's Theorem that

$$\begin{aligned} 0 = \oint_{\gamma}\frac{f'(\xi)}{f(\xi)}d\xi &= \int_{\overline{z_1z_2}}\frac{f'(\xi)}{f(\xi)}d\xi + \int_{\lambda'}\frac{f'(\xi)}{f(\xi)}d\xi + \int_{\lambda^-}\frac{f'(\xi)}{f(\xi)}d\xi \\ &= \int_{\overline{z_1z_2}}\frac{f'(\xi)}{f(\xi)}d\xi + \int_{\lambda'}\frac{f'(\xi)}{f(\xi)}d\xi - \int_{\lambda}\frac{f'(\xi)}{f(\xi)}d\xi \end{aligned}$$

Whence, by the above

$$\frac{f(z_2)}{f(z_1)} = \exp\int_{\lambda}\frac{f'(\xi)}{f(\xi)}d\xi - \int_{\lambda'}\frac{f'(\xi)}{f(\xi)}d\xi$$

Implying

$$f(z_2)\exp\int_{\lambda'}\frac{f'(\xi)}{f(\xi)}d\xi = f(z_1)\exp\int_{\lambda}\frac{f'(\xi)}{f(\xi)}d\xi \tag{4}$$

STEP 2. Here we show uniqueness of representation, so to speak. Namely, that the above is equal to $f(z_0)$ and hence the function is independent of chosen path. This is a reprise of the argument above, where we may again show that

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{1}{f(\zeta)}\exp\left(-\int_{z_1}^{\zeta}\frac{f'(\xi)}{f(\xi)}d\xi\right)\right] = 0$$

as ζ moves from z_1 to z_0 on the curve λ. Then, taking $\zeta = z_0$ the result follows; i.e

$$f(z_0) = f(z_1)\exp\int_{\lambda}\frac{f'(\xi)}{f(\xi)}d\xi \tag{5}$$

the second equality follows from step 1.

We may now define for an interior point z_0 and such a boundary point z_1:

$$w(z_0) := w(z_1) + \int_{\lambda}\frac{f'(\xi)}{f(\xi)}\mathrm{d}\xi = w(z_2) + \int_{\lambda'}\frac{f'(\xi)}{f(\xi)}\mathrm{d}xi \tag{6}$$

Ultimately, it is this *uniqueness* of representation that will yield continuity of this function, which is defined on and inside Γ.

Step 3. The function $w(z)$ is continuous where it is defined. Fix a point z_0 inside Γ and consider now a small disc centred about it contained inside the interior of Γ except for one point where it intersects a point z_1 on the boundary of Γ. Consider again an arc λ from z_1 to z_0 with $(z_1, z_0]$ contained in the interior of the curve. If we take z_0' near z_0 then it lies inside this same disc, which is convex. Consequently for all such z_0' we may choose a path λ' from $(z_1, z_0'])$ contained inside the interior of the disc and consequently inside the region enclosed by Γ.

We now employ a usual strategy. Consider the closed curve γ formed by traveling along $[z_0, z_1] \to [z_1, z_0'] \to [z_0', z_0]$ where this last path is taken along the radius of the disc. This is a closed curve (a triangle) on and inside which f is holomorphic and thus

$$0 = \oint_\gamma \frac{f'(\xi)}{f(\xi)} d\xi = \int_{\lambda-} \frac{f'(\xi)}{f(\xi)} d\xi + \int_{\lambda'} \frac{f'(\xi)}{f(\xi)} d\xi + \int_{[z_0', z_0]} \frac{f'(\xi)}{f(\xi)} d\xi \tag{7}$$

Therefore,

$$w(z_0) - w(z_0') = \int_\lambda \frac{f'(\xi)}{f(\xi)} d\xi - \int_{\lambda'} \frac{f'(\xi)}{f(\xi)} d\xi = \int_{[z_0', z_0]} \frac{f'(\xi)}{f(\xi)} d\xi$$

Now observe that since $f'(\zeta)$ and $f(\zeta)$ are jointly holomorphic with $f(\zeta) \neq 0$, their ratio is continuous (even holomorphic) inside and on Γ and must achieve a maximum, say, $M \geq 0$. In any case it is bounded and we may make the following estimate for all z_0' near z_0:

$$\left|w(z_0) - w(z_0')\right| \leq M \int_{z_0'}^{z_0} d\ell = M \left|z_0 - z_0'\right| \xrightarrow{z_0' \to z_0} 0$$

Hence, w is continuous.

Step 4. We now show the function $w(z)$ is holomorphic. This is really a consequence of the introductory result, as we have already shown continuity, it is sufficient to establish the following:

Lemma. *If f is holomorphic and non vanishing inside a domain $\Omega \subset \mathbb{C}$ and we have a continuous function w there such that*

$$f(z) = e^{w(z)}$$

then w is also holomorphic with $w'(z) = \dfrac{f'(z)}{f(z)}$.

Proof of Lemma. Employing the holomorphic property of f and the exponential we may expand

$$f(z + \Delta z) - f(z) = e^{w(z+\Delta z)} - e^{w(z)}$$

so that

$$\begin{aligned} f(z + \Delta z) - f(z) &= e^{w(z)} \left(e^{w(z+\Delta z) - w(z)} - 1\right) \\ &= f(z) \left(e^{\Delta w} - 1\right), \quad \Delta w = w(z + \Delta z) - w(z) \end{aligned}$$

Dividing through by Δz

$$\begin{aligned}\frac{f(z+\Delta z)-f(z)}{\Delta z} &= \frac{f(z)}{\Delta z}\left(\Delta w + \frac{(\Delta w)^2}{2!} + \dots\right)\\ &= f(z)\cdot\frac{w(z+\Delta z)-w(z)}{\Delta z}\left(1+\frac{(\Delta w)}{2!}+\dots\right)\end{aligned}$$

We see that by continuity of $w(z)$ the power series tends to 1 as $\Delta z \to 0$. Hence, letting $z \to z_0$ we see

$$w'(z) = \frac{f'(z)}{f(z)}$$

and hence that w is holomorphic.

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Having established this, we note that this holds in particular for the logarithm we constructed in the previous steps. Thus, there exists a function $w(z)$ holomorphic on and inside $\Gamma \subset \mathbb{C}$ where

$$w'(z) = \frac{f'(z)}{f(z)}, \quad e^{w(z)} = f(z) \tag{8}$$

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