BRIEF NOTE ON COMPLEX HILBERT SPACES WITH HERMITIAN INNER-PRODUCTS

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1. Definitions and Orthogonality

We shall begin by giving the definition of a complex Hilbert space. Recall that a complex vector space is a non-empty set V endowed with two operations:

$$+ : \mathrm{V} \times \mathrm{V} \to \mathrm{V}, \quad (u, v) \mapsto u + v$$
$$\odot : \mathrm{V} \times \mathbb{C} \to \mathrm{V}, \quad (u, \zeta) \mapsto \zeta \cdot u$$

so that V is an Abelian group with respect to both operations. We may extend this further to a "nicer" class of spaces: inner-product space.

Definition 1. *A complex inner-product space is a complex vector space* V *equipped with an extra mapping:*

$$\langle \cdot, * \rangle : \mathrm{V} \times \mathrm{V} \to \mathbb{C}, \quad (u, v) \mapsto \langle u, v \rangle \tag{1}$$

that satisfies each of the following for all $u, v, w \in \mathrm{V}$ *and* $\zeta \in \mathbb{C}$*:*

(i) Conjugate symmetry: $\langle u, v \rangle = \overline{\langle v, u \rangle}$.
(ii) Linearity in the first argument: $\langle u + \zeta w, v \rangle = \langle u, v \rangle + \zeta \langle w, v \rangle$.
(iii) Positive definiteness: $\langle u, u \rangle \geq 0$ *and* $\langle u, u \rangle = 0$ *if and only if* $u = 0$.

Remark. Note that above in (3) we require $\langle u, u \rangle$ to be *real* for all $u \in \mathrm{V}$.

Observe that any inner-product induces a norm on the complex space V. Certainly, this norm arises by letting $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$. It is an easy consequence that any inner-product as above satisfies the Cauchy-Schwarz inequality:

$$|\langle u, v \rangle| \leq \|u\| \, \|v\| , \quad \forall u, v \in \mathrm{V} \tag{2}$$

and that the triangle inequality holds: $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in \mathrm{V}$. This tells us that any complex inner product space is a complex, normed vector space. Note that the inner-product may map to $\Im z \neq 0$ but that the norm maps into the reals.

Definition 2. *A complex Hilbert space* $\mathcal{H}$ *is a complex inner-product space which is a separable Banach space with respect to the induced norm.*

Date: March 19, 2017.

It should be evident from the very definition of the inner-product that $\langle \cdot, * \rangle$ is anti-linear in the second argument. Recall that a metric space (X, d) is separable provided it has a countable dense subset. Given two vectors $f, g \in \mathcal{H}$ we shall say that f and g are **orthogonal** (or perpendicular), written $f \perp g$, whenever $\langle f, g \rangle = 0$. Our first result is a generalization of the Pythagorean theorem to these spaces:

Lemma 1.1. *Let $\mathcal{H}$ be a complex Hilbert space and $f, g \in \mathcal{H}$ with $f \perp g$. Then $\|f+g\|^2 = \|f\|^2 + \|g\|^2$.*

Proof. To see this we write:

$$\begin{aligned} \|f+g\|^2 = \langle f+g, f+g \rangle &= \langle f, f+g \rangle + \langle g, f+g \rangle \\ &= \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle \\ &= \|f\|^2 + \langle f, g \rangle + \langle g, f \rangle + \|g\|^2 \end{aligned}$$

Now, since $f \perp g$ we get $\langle f, g \rangle = \overline{\langle g, f \rangle} = 0$ which concludes the proof.

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Corollary 1.2 (Pythagorean Theorem)**.** *Let $\mathcal{H}$ be a complex Hilbert space and assume $\{f_j\}_{j=1}^N$ is a family of pairwise orthogonal vectors in $\mathcal{H}$. Then,*

$$\left\| \sum\nolimits_{j=1}^N f_j \right\|^2 = \sum_{j=1}^N \|f_j\|^2 \tag{3}$$

Proof. We argue by induction on N. The case $N = 2$ is clear from the previous lemma, now assume (3) holds for N, we show the case $N+1$ follows. Certainly, if $\{f_j\}_{j=1}^{N+1}$ is pairwise orthogonal then so is $\{f_j\}_{j=1}^N$. Moreover, it is obvious that $\langle \sum_{j=1}^N f_j, f_{N+1} \rangle = 0$ by linearity. Whence we find:

$$\begin{aligned} \left\| \sum\nolimits_{j=1}^{N+1} f_j \right\|^2 &= \left\langle \sum\nolimits_{j=1}^{N+1} f_j, \sum\nolimits_{j=1}^{N+1} f_j \right\rangle \\ &= \left\langle \sum\nolimits_{j=1}^{N} f_j, \sum\nolimits_{j=1}^{N+1} f_j \right\rangle + \left\langle f_{N+1}, \sum\nolimits_{j=1}^{N+1} f_j \right\rangle \\ &= \left\langle \sum\nolimits_{j=1}^{N} f_j, \sum\nolimits_{j=1}^{N} f_j \right\rangle + \left\langle \sum_{j=1}^{N} f_j, f_{N+1} \right\rangle + \left\langle f_{N+1}, \sum\nolimits_{j=1}^{N+1} f_j \right\rangle \\ &= \left\langle \sum\nolimits_{j=1}^{N} f_j, \sum\nolimits_{j=1}^{N} f_j \right\rangle + \left\langle f_{N+1}, \sum\nolimits_{j=1}^{N+1} f_j \right\rangle \\ &= \left\| \sum\nolimits_{j=1}^{N} f_j \right\|^2 + \langle f_{N+1}, f_{N+1} \rangle + \left\langle f_{N+1}, \sum\nolimits_{j=1}^{N} f_j \right\rangle \\ &= \left\| \sum\nolimits_{j=1}^{N} f_j \right\|^2 + \|f_{N+1}\|^2 \end{aligned}$$

as was to be shown.

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Let $\mathcal{H}$ be a complex Hilbert space and $\{e_k\}_{k \in \mathbb{N}}$ be a countable subset of $\mathcal{H}$. This set is said to be **orthonormal** provided for all indices $(k, \ell) \in \mathbb{N}^2$ one has:

$$\langle e_k, e_\ell \rangle = \begin{cases} 1 & k = \ell \\ 0 & \text{else} \end{cases}$$

This same set is called a **Hilbert basis** for $\mathcal{H}$ if their linear combinations are dense in $\mathcal{H}$. These are sometimes called orthonormal bases. We shall now give a complete characterization of these bases for a Hilbert space $\mathcal{H}$ (over $\mathbb{C}$). First, we introduce notation. In the next theorem we shall write $\xi_k := \langle f, e_k\rangle \in \mathbb{C}$ and set $S_N(f) := \sum_{k=1}^N \xi_k e_k$ for $f \in \mathcal{H}$.

Theorem 1.3 (Characterization of Hilbert Bases). *Let $\mathcal{H}$ be a complex Hilbert space and $\{e_k\}_{k\in\mathbb{N}}$ an orthonormal subset of $\mathcal{H}$. The following statements are equivalent:*

(i) $\{e_k\}_{k\in\mathbb{N}}$ is a Hilbert basis for $\mathcal{H}$.
(ii) If $f \in \mathcal{H}$ satisfies $\langle f, e_j\rangle = 0$ for all $j \in \mathbb{N}$ then $f = 0$.
(iii) For all $f \in \mathcal{H}$ the combination $S_N(f) \to f$ as $N \to \infty$.
(iv) Parseval's identity holds true for all $f \in \mathcal{H}$:

$$\|f\|^2 = \sum_{k\in\mathbb{N}} |\xi_k|^2 \tag{$\mathfrak{P}$}$$

Proof of Theorem. (i $\implies$ ii) Assume that $\langle f, e_j\rangle = 0$ for all j. Let ε positive be given; select a vector $g_\varepsilon \in \mathcal{H}$ where $g_\varepsilon = \sum_{k=1}^N \zeta_k e_k$ with $\|f - g_\varepsilon\| \le \varepsilon$. Observe that by assumption on f one has by linearity of the Hermitian inner-product: $\langle f, g_\varepsilon\rangle = 0$. Therefore,

$$\|f\|^2 = \langle f, f, \rangle = \langle f, f - g_\varepsilon + g_\varepsilon\rangle = \underbrace{\langle f, g_\varepsilon\rangle}_{=0} + \langle f, f - g_\varepsilon\rangle = \langle f, f - g_\varepsilon\rangle$$

Now, using Cauchy-Schwarz we find that $\|f\|^2 \le \|f\|\,\|f - g_\varepsilon\|$. Suppose now that $\|f\| \ne 0$, then we get $\|f\| \le \|f - g_\varepsilon\| \le \varepsilon$. Since $\varepsilon > 0$ was arbitrary we find that $\|f\| = 0$.

(ii $\implies$ iii) There are some preliminary "calculations" to be made. Fix a vector $f \in \mathcal{H}$ and define $S_N(f)$ as in (iii). We claim first that $f - S_N(f) \perp S_N(f)$ for all N sufficiently large. Indeed, to see this we write by the Pythagorean theorem (the $\{e_k\}_{k\in\mathbb{N}}$ are orthonormal)

$$\langle f - S_N(f), S_N(f)\rangle = \langle f, S_N(f)\rangle - \|S_N(f)\|^2 = \langle f, S_N(f)\rangle - \sum_{k=1}^N |\xi_k|^2$$

But now

$$\langle f, S_N(f)\rangle = \left\langle f, \sum\nolimits_{k=1}^N \xi_k e_K \right\rangle = \sum_{k=1}^N \overline{\xi_k}\langle f, e_k\rangle = \sum_{k=1}^N |\xi_k|^2$$

which proves that $f - S_N(f) \perp S_N(f)$ as was asserted. Therefore, applying the Pythagorean theorem proven in Corollary 3 we obtain that

$$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{k=1}^N |\xi_k|^2 \ge \sum_{k=1}^N |\xi_k|^2 \tag{4}$$

Now, passing to the limit in $N \to \infty$ in the righthand side of the equation above gives *Bessel's Identity*:

$$\sum_{k\in\mathbb{N}} |\xi_k|^2 \le \|f\|^2 \tag{5}$$

We now make the bold claim that $\{S_N(f)\}_{N\in\mathbb{N}}$ is Cauchy in $\mathcal{H}$. Certainly, from (5) we know that $\sum_k |\xi_k|^2 < \infty$ and so for all $N, M \in \mathbb{N}$, taking without harm $M > N$:

$$\|S_N(f) - S_M(f)\| \le \sum_{k=N+1}^M |\xi_k|^2 \xrightarrow{N,M\to\infty} 0$$

Since $\mathcal{H}$ is also a Banach space, there is a point, say, $g \in \mathcal{H}$ so that $S_N(f) \to g$ in norm as $N \to \infty$. We claim now that $f = g$. Indeed, it suffices by our assumption in (ii) to prove that $\langle f - g, e_j \rangle = 0$ for arbitrary j. Fix j and let $N \gg j$ be an integer. We note that

$$|\langle f - g, e_j \rangle| \leq |\langle f - S_N(f), e_j \rangle| + |\langle S_N(f) - g, e_j \rangle| \leq |\langle f - S_N(f), e_j \rangle| + \|S_N - g\|$$

Therefore this boils down to showing that $|\langle f - S_N(f), e_j \rangle| \to 0$ as $N \to \infty$. Certainly, for N is large

$$\langle f - S_N(f), e_j \rangle = \langle f, e_j \rangle - \left\langle \sum_{k=1}^{N} \xi_k e_k, e_j \right\rangle = \langle f, e_j \rangle - \langle f, e_j \rangle = 0$$

Thus passing to the limit in $|\langle f - g, e_j \rangle| \leq \|S_N - g\|$ we find that $f - g \perp e_j$ for all j whence $f = g$ as vectors.

(iii $\implies$ iv) We refer again to (4). As per our assumption we know that $S_N(f) \to f$ in norm as $N \to \infty$. Thus, taking the limit in (4) we get Parseval's identity in ($\mathfrak{P}$).

(iv $\implies$ i). To see this, we assume that Parseval's identity holds true. Now, referring to (4) we find that $\|f - S_N\| \to 0$ as $N \to \infty$. Since S_N are linear combinations we have (i).

The theorem has now been proven.

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We conclude this section with the observation that any Hilbert space $\mathcal{H}$ over $\mathbb{C}$ has a Hilbert basis. Indeed, since $\mathcal{H}$ is a vector space we may select a basis, say, $\mathcal{B}$. Now, to construct an orthonormal subset one needs only follow Gram-Schmidt.

2. Orthogonal Subspaces and Linear Operators

For this section we fix a Hilbert space $\mathcal{H}$ over $\mathbb{C}$. A subspace $\mathcal{S}$ of $\mathcal{H}$ (written $\mathcal{S} < \mathcal{H}$) is a vector subspace of $\mathcal{H}$, viewed as a vector space. We shall say that $\mathcal{S}$ is a **closed** subspace, denoted $\mathcal{S} \leq \mathcal{H}$ provided it is topologically closed in $\mathcal{H}$.

Proposition 2.1. *Let $\mathcal{S} < \mathcal{H}$. Then $\mathcal{S} \leq \mathcal{H}$ if and only if for every sequence (f_n) in $\mathcal{S}$ converging to $f \in \mathcal{H}$ one has $f \in \mathcal{S}$*

The above proposition is a consequence of the *characterization of closed sets*[1] in metric spaces (X, d).

Corollary 2.2. *If $\mathcal{S} \leq \mathcal{H}$ then $\mathcal{S}$ is a Hilbert space.*

Proposition 2.3 (Parallelogram Law)**.** *For all vectors $A, B \in \mathcal{H}$ the parallelogram law holds:*

$$\|A + B\|^2 + \|A - B\|^2 = 2\left(\|A\|^2 + \|B\|^2\right) \tag{6}$$

Proof. This is a straightforward calculation. We write $\|A + B\|^2 = \langle A + B, A + B \rangle$ and similarly for $\|A - B\|^2$. This gives us

$$\|A + B\|^2 + \|A - B\|^2 = \langle A + B, A + B \rangle + \langle A - B, A - B \rangle = 2\langle A, A \rangle + 2\langle B, B \rangle$$

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Theorem 2.4 (Existence and Uniqueness of Perpendicular Minimizers)**.** *Let $\mathcal{H}$ be a complex Hilbert space with $\mathcal{S} \leq \mathcal{H}$. For all $f \in \mathcal{H}$ there exists a unique $g_0 \in \mathcal{S}$ such that $\|f - g_0\| = \inf_{g \in \mathcal{S}} \|f - g\|$ and $(f - g_0) \perp \mathcal{S}$* [2].

[1] In a metric space a subspace $\mathcal{S}$ is closed if and only if it contains all of its limit points.

[2] By this we mean that $(f - g_0) \perp g$ for all $g \in \mathcal{S}$

Proof of Theorem. For existence we shall use an argument that is frequently used in the derivation of maximum principles for elliptic differential operators. Note that if $f \in \mathcal{S}$ then we may simply let $g_0 := f$ and this obviously satisfies the claim. Hence, we may presume without harm to the proof that $f \notin \mathcal{S}$. Then, since $\mathcal{S} \leq \mathcal{H}$ one has

$$\inf_{g \in \mathcal{S}} \|f - g\| = d > 0$$

for otherwise we would have $f \in \mathcal{S}$. By definition of the infimum we may select a sequence (g_n) living in $\mathcal{S}$ so that $\lim \|f - g_n\| = d > 0$. We set $A = f - g_n$ and $B = f - g_m$ in (6) and glean

$$\|2f - (g_n + g_m)\|^2 + \|g_m - g_n\|^2 = 2\left(\|f - g_n\|^2 + \|f - g_m\|^2\right)$$

Since $\mathcal{S} \leq \mathcal{H}$ is also a vector subspace we note that $\frac{g_n + g_m}{2} \in \mathcal{S}$ and therefore we get

$$\|g_m - g_n\|^2 \leq 2\left(\|f - g_n\|^2 + \|f - g_m\|^2\right) - 4d^2 \xrightarrow{n,m\to\infty} 0$$

This proves that (g_n) as constructed above is Cauchy in $\mathcal{S}$. Hence, there is a limit point, say, $g_0 \in \mathcal{S}$ of (g_n). Passing to the limit and using the continuity of norms we get $\|f - g_0\| = d$.

We shall now prove that $(f - g_0) \perp g$ for all $g \in \mathcal{S}$. Let $\varepsilon \in \mathbb{R}$ be small in absolute value and consider a perturbation set by $g \mapsto g_0 - \varepsilon g$. Again, since $\mathcal{S}$ is a vector subspace of $\mathcal{H}$ we know that $g_0 - \varepsilon g \in \mathcal{S}$ so that

$$\|f - g_0\|^2 + \varepsilon^2 \|g\|^2 + 2\varepsilon\Re\langle f - g_0, g\rangle = \|f - (g_0 - g\varepsilon)\|^2 \geq \|f - g_0\|^2$$

Whence $2\varepsilon\Re\langle f - g_0, g\rangle + \varepsilon^2 \|g\|^2 \geq 0$. If $\Re\cdot > 0$ we take $\varepsilon < 0$ small in norm and vice-versa for a contradiction. Similarly one shows the imaginary part of this inner-product is 0 and hence obtain that $(f - g_0) \perp g$ for all $g \in \mathcal{S}$.

As for uniqueness, suppose g, h are minimizers in $\mathcal{S}$. Then, we know that $g - h \in \mathcal{S}$ since $\mathcal{S}$ is closed and thence $(f - g) \perp h$ so that by the Pythagorean theorem:

$$\|f - h\|^2 = \|f - g + g - h\|^2 = \|f - g\|^2 + \|g - h\|^2$$

which gives $\|g - h\|^2 = 0$.

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An important consequence of this theorem is that we may decompose a Hilbert space $\mathcal{H}$ into the direct-sum[3] of two of its subspaces. Certainly, suppose $\mathcal{S} \leq \mathcal{H}$ and let $\mathcal{S}^\perp$ consist of the subspace perpendicular to $\mathcal{S}$:

$$\mathcal{S}^\perp := \{f \in \mathcal{H} : \langle f, g\rangle = 0,\ \forall g \in \mathcal{S}\}$$

We make the claim that $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$. First, however, we note some properties regarding the space $\mathcal{S}^\perp$. We wish to show that $\mathcal{S}^\perp \leq \mathcal{H}$. Of course, by the (anti)-linearity of the Hermitian inner-product that $\mathcal{H}$ is equipped with it follows that $\mathcal{S}^\perp < \mathcal{H}$ (i.e. $\mathcal{S}^\perp$ is a vector subspace). To see that $\mathcal{S}^\perp$ is topologically closed, pick a sequence (f_n) in $\mathcal{S}^\perp$ with limit point $f \in \mathcal{H}$. For all indices $n \in \mathbb{N}$ we know that $\langle f_n, g\rangle = 0$, where $g \in \mathcal{S}$ is fixed. Then observe that by Cauchy-Schwarz

$$|\langle f, g\rangle| = |\langle f, g\rangle - \langle f_n, g\rangle| = |\langle f - f_n, g\rangle| \leq \|f - f_n\|\,\|g\| \xrightarrow{n\to\infty} 0$$

Proving that $\langle f, g\rangle = 0$ for all $g \in \mathcal{S}$ whence $f \in \mathcal{S}^\perp$. Moreover, note that $\mathcal{S} \cap \mathcal{S}^\perp = \{0\}$. Certainly, let $f \in \mathcal{S} \cap \mathcal{S}^\perp$, so that $\langle f, f\rangle = \|f\|^2 = 0$ since $f \perp f$.

We may now prove the aforementioned claim.

Proposition 2.5. *Let $\mathcal{S} \leq \mathcal{H}$. Then, $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$.*

[3]By this we mean that every vector in $\mathcal{H}$ has a unique representation $g + h$ where $g \in \mathcal{S}$ and $h \in \mathcal{S}^\perp$.

Proof. We shall first prove existence of representation. Fix $f \in \mathcal{H}$. By the previous theorem we may find $g_0 \in \mathcal{S}$ so that $(f - g_0) \perp g$ for all $g \in \mathcal{S}$. Especially, $(f - g_0) \perp g_0$. However, $f = (f - g_0) + g_0$ and so $(f - g_0) \in \mathcal{S}^{\perp}$ but $g_0 \in \mathcal{S}$. This gives us our representation.

To see uniqueness, suppose that $f = g + h = g' + h'$ where $g, g' \in \mathcal{S}$ and $h, h' \in \mathcal{S}^{\perp}$. Then we have that $(g - g') = (h' - h)$. Since $\mathcal{S} \cap \mathcal{S}^{\perp}$ consists only of the 0-vector we must have $g - g' = 0$ and $h' - h = 0$.

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Of course there is a natural projection from $\mathcal{H} \to \mathcal{S}$ defined by

$$P_{\mathcal{S}} : \mathcal{H} \to \mathcal{S}, \quad \mathcal{S} \oplus \mathcal{S}^{\perp} \ni (f, g) \mapsto f \tag{7}$$

This mapping is clearly linear and $\|P_{\mathcal{S}}(f)\| \leq \|f\|$ for all $f \in \mathcal{H}$.

Definition 3. *For complex Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ a linear mapping $L : \mathcal{H}_1 \to \mathcal{H}_2$ is called a **linear operator**. This operator Ł is said to be bounded if there is $M > 0$ so that*

$$\|L(f)\|_{\mathcal{H}_2} \leq M \|f\|_{\mathcal{H}_1}, \quad \forall f \in \mathcal{H}_1$$

Define $\|T\|$ to be the infimum of all such M.

Observe that by linearity all linear operators fix the origin: $Ł(0) = 0$. We shall call Ł continuous provided for all convergent sequences in $\mathcal{H}$: $f_n \to f$ one has $Ł(f_n) \to Ł(f)$. A surprising characterization of continuity follows:

Theorem 2.6 (Characterization of Continuity)**.** *Let $Ł : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear operator. Ł is continuous if and only if it is bounded.*

Proof of Theorem. ($\Longleftarrow$) Suppose that Ł is bounded, and let $M > 0$ be so that $\|Ł(f)\| \leq M \|f\|$ for all $f \in \mathcal{H}$. If we take a sequence (f_n) in $\mathcal{H}$ so that $f_n \to f$ then we consider

$$\|Ł(f_n) - Ł(f)\| = \|Ł(f_n - f)\| \leq M \|f_n - f\|$$

proving that Ł is continuous.

($\Longrightarrow$) If Ł is continuous, then Ł is continuous at $0 \in \mathcal{H}$. Assume for a contradiction that f is not-bounded, hence for all $n \in \mathbb{N}$ we may find $f_n \in \mathcal{H}$ so that $\|Ł(f_n)\| \geq n \|f_n\|$. Now we define a point

$$\mathcal{H} \ni g_n := \frac{f_n}{n \|f_n\|}$$

so that $\|g_n\| = 1/n$ and so it is clear that $\lim g_n = 0$. By continuity of Ł at 0, we must have $\lim Ł(g_n) = 0$ and so

$$\|Ł(g_n)\| \to 0, \quad n \to \infty$$

On the other-hand, we note that

$$\|Ł(g_n)\| = \frac{1}{n \|f_n\|} \|Ł(f_n)\| \geq \frac{n \|f_n\|}{n \|f_n\|} = 1$$

which is a contradiction.

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3. Functionals and The Riesz Representation Theorem

Given a complex Hilbert space $\mathcal{H}$ a linear functional ℓ is an element of the dual space $\mathcal{H}^*$. In other-words, a linear functional is a linear mapping $\ell : \mathcal{H} \to \mathbb{C}$. Observe that ℓ must also fix the origin: for $\ell(0) = \ell(0+0) = 2\ell(0)$.

Perhaps the simplest example of a linear functional from $\mathcal{H} \hookrightarrow \mathbb{C}$ is that induced by the map $f \mapsto \langle f, g\rangle$ where $g \in \mathcal{H}$ is fixed. The most surprising result, however, is that all continuous elements of $\mathcal{H}^*$ arise in this way.

Theorem 3.1 (Riesz Representation Theorem)**.** *Let $\ell : \mathcal{H} \to \mathbb{C}$ be a continuous linear functional. There exists a unique $g \in \mathcal{H}$ so that $\ell(f) = \langle f, g\rangle$ for all $f \in \mathcal{H}$.*

Proof of Theorem. We begin by defining a subspace of $\mathcal{H}$ called the **null-space**:

$$\mathcal{S} := \{f \in \mathcal{H} : \ell(f) = 0\} \tag{8}$$

We claim that $\mathcal{S} \leq \mathcal{H}$. Obviously, by linearity of ℓ one sees $\mathcal{S} < \mathcal{H}$. To see that $\mathcal{S}$ is closed, pick a sequence (f_n) in $\mathcal{S}$ and assume that $f_n \to f \in \mathcal{H}$ in norm. Then for all $n \in \mathbb{N}$ we have $\ell(f_n) = 0$. Now, it follows by continuity that $\lim \ell(f_n) = \ell(f) = 0$. Hence, $\mathcal{S} \leq \mathcal{H}$.

By our decomposition theorem we may write $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$. There are two cases to distinguish here:

(1) $\mathcal{S}^\perp = \varnothing$. Pick $g = 0$. Then $\ell(f) = \langle f, g\rangle = 0$ for all $f \in \mathcal{H}$ and we are done.

(2) Else $\mathcal{S}^\perp \neq \varnothing$ and thus we may select a non-trivial vector $h \in \mathcal{S}^\perp$ with $\|h\| = 1$. To see that this is possible, let $\widetilde{h} \neq 0$ be a vector in $\mathcal{H}$ and let $h := \widetilde{h}/\left\|\widetilde{h}\right\|$ which must lie in $\mathcal{S}^\perp$ for $\mathcal{S}^\perp$ is a vector subspace of $\mathcal{H}$. Now we shall define our candidate $g \in \mathcal{H}$ as follows:

$$g := \overline{\ell(h)}h$$

Now consider the vector $x := \ell(f)h - f\ell(h)$ for arbitrary $f \in \mathcal{H}$. Then, $x \in \mathcal{S}$ since $\ell(x) = \ell(f)\ell(h) - \ell(f)\ell(h) = 0$. This tells us that $\langle u, h\rangle = 0$. On the other hand,

$$0 = \langle u, h\rangle = \langle \ell(f)h - f\ell(h), h\rangle = \ell(f)\langle h, h\rangle - \ell(h)\langle f, h\rangle$$

Since $\|h\|^2 = 1$ we get $\ell(f) = \ell(h)\langle f, h\rangle = \langle f, \overline{\ell(h)}h\rangle = \langle f, g\rangle$.

It now only remains to prove uniqueness of this vector $g \in \mathcal{H}$. Assume $g, h \in \mathcal{H}$ are two vectors so that $\ell(f) = \langle f, g\rangle = \langle f, h\rangle$ for all $f \in \mathcal{H}$. Then we get

$$0 = \langle f, g - h\rangle \iff \langle g - h, f\rangle = 0$$

Taking an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ we find $\langle g - h, e_j\rangle = 0$ for all j and thus $g = h$ by our previous theorem.

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