A CONVERGENCE CRITERION FOR BOUNDED FAMILIES IN L^p-SPACES

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In this note we establish an often overlooked, yet elementary, result from integration theory which can be thought of as a counterpart to the famous dominated convergence theorem. Given a measure space $(X, \mathfrak{M}, \mu)$ and a sequence (f_n) of measurable functions converging pointwise a.e. to a measurable function f, the dominated convergence theorem says that

$$\lim_{n\to\infty} \int_X |f_n - f| \,\mathrm{d}\mu = 0$$

provided there exists $g \in L^1(X, \mu)$ such that $|f_n| \leq g$ a.e. on X. for *all* $n \in \mathbb{N}$. Although incredibly useful, there are many practical instances in which we cannot find a dominating function g for this sequence (f_n).

Even in such cases, one can often make use of the Brézis-Lieb lemma (see [1]), which we reiterate below for the sake of completeness.

Theorem A. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $0 < p < \infty$. Assume that (f_n) is a bounded sequence in $L^p(X, \mu)$ and that $f_n \to f$ a.e. on X, where f is measurable. Then, $f \in L^p(X, \mu)$ and*

$$\lim_{n\to\infty} \left(\int_X |f_n|^p \,\mathrm{d}\mu - \int_X |f_n - f|^p \,\mathrm{d}\mu \right) = \int_X |f|^p \,\mathrm{d}\mu.$$

Our goal here is to prove an alternative convergence criterion, which becomes particularly powerful when coupled with Theorem A and the dominated convergence theorem. More precisely, we shall establish the following result:

Theorem 1. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and fix $1 < p < \infty$. Let (f_n) be a bounded sequence in $L^p(X, \mu)$ and assume that f_n converges pointwise a.e. to a measurable function f. Let $p' \in (1, \infty)$ satisfy*

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{1}$$

Then,

$$\lim_{n\to\infty} \int_X |(f_n - f)g| \,\mathrm{d}\mu = 0, \quad \forall g \in L^{p'}(X, \mu).$$

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In particular,

$$\lim_{n\to\infty}\int_X f_n g\,\mathrm{d}\mu = \int_X fg\,\mathrm{d}\mu$$

for every $g \in L^{p'}(X,\mu)$.

We take note of the following immediate corollary:

Corollary. *Let* $(X,\mathfrak{M},\mu)$ *be a measure space and fix* $p \in (1,\infty)$. *Assume that* $f_n \rightharpoonup f$ *in* $L^p(X,\mu)$ *and that* $f_n \to f$ *pointwise a.e. on* X. *Then,*

$$\lim_{n\to\infty}\int_X |(f_n - f)g|\,\mathrm{d}\mu = 0$$

for all $g \in L^{p'}(X,\mu)$.

Remark. Theorem 1 has remarkable consequences when the measure space $(X,\mathfrak{M},\mu)$ is finite. In this case, the indicator function $\mathbf{1}_X$ of the entire space X belongs to $L^{p'}(X,\mu)$ and hence, by taking $g := \mathbf{1}_X$ in Theorem 1, we infer that

$$\int_X |(f_n - f)\mathbf{1}_X|\,\mathrm{d}\mu = \int_X |f_n - f|\,\mathrm{d}\mu \to 0$$

as $n \to \infty$. That is, $f_n \to f$ strongly in $L^1(X,\mu)$. Put informally, when the measure space is finite, one can do without the need for a dominating function provided the sequence (f_n) is bounded in a "higher" L^p-space. We formally highlight this phenomenon below.

Corollary. *Let* $(X,\mathfrak{M},\mu)$ *be a finite measure space and let* (f_n) *be a sequence of measurable functions converging pointwise almost everywhere to a measurable function*

$$f : X \to \mathbb{C}$$

as $n \to \infty$. *If* (f_n) *is bounded in* $L^p(X,\mu)$ *for some* $p \in (1,\infty)$, *then* $f_n \to f$ *strongly in* $L^p(X,\mu)$ *as* $n \to \infty$.

Let us now prove the aforementioned theorem.

Proof of Theorem 1. By invoking Fatou's lemma, we automatically have $f \in L^p(X,\mu)$:

$$\int_X |f|^p\,\mathrm{d}\mu \le \liminf_{n\to\infty}\int_X |f_n|^p\,\mathrm{d}\mu \le \sup_{n\geqslant 1}\|f_n\|^p_{L^p(X,\mu)} < \infty.$$

Hence, we may assume without loss of generality that $f = 0$ in $L^p(X,\mu)$. Thus, it suffices to check that

$$\limsup_{n\to\infty}\int_X |f_n g|\,\mathrm{d}\mu \le 0.$$

To this end, let us fix a constant $\alpha > 0$. For $n \in \mathbb{N}$, consider the measurable set

$$E_n := \left\{ x \in X : |f_n(x)g(x)| \le \alpha \, |g(x)|^{p'} \right\}.$$

From the dominated convergence theorem, it is clear that

$$\lim_{n\to\infty} \int_{E_n} |f_n(x)g(x)| \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X |f_n(x)g(x)| \, \mathbf{1}_{E_n}(x) \, \mathrm{d}\mu = 0 \tag{2}$$

because $|f_n(x)g(x)| \, \mathbf{1}_{E_n}(x) \le \alpha \, |g(x)|^{p'}$ on X. Let $M > 0$ be such that $\|f_n\|_{L^p(X,\mu)} \le M$ for every $n \ge 1$. Since

$$|f_n(x)g(x)| > \alpha \, |g(x)|^{p'}$$

for each $x \in E_n^{\complement}$, an application of Hölder's inequality gives

$$\begin{aligned} \int_{E_n^{\complement}} |f_n g| \, \mathrm{d}\mu &\le \|f_n\|_{L^p(E_n^{\complement},\mu)} \left(\int_{E_n^{\complement}} |g|^{p'} \, \mathrm{d}\mu \right)^{1/p'} \\ &\le \alpha^{-1/p'} \, \|f_n\|_{L^p(X,\mu)} \left(\int_{E_n^{\complement}} |f_n g| \, \mathrm{d}\mu \right)^{1/p'} \\ &\le M\alpha^{-1/p'} \left(\int_{E_n^{\complement}} |f_n g| \, \mathrm{d}\mu \right)^{1/p'}. \end{aligned}$$

So, for any $n \ge 1$, this implies

$$\left(\int_{E_n^{\complement}} |f_n g| \, \mathrm{d}\mu \right)^{1/p} = \left(\int_{E_n^{\complement}} |f_n g| \, \mathrm{d}\mu \right)^{1-1/p'} \le M\alpha^{-1/p'}$$

Or, rather, that

$$\int_{E_n^{\complement}} |f_n g| \, \mathrm{d}\mu \le M^p \alpha^{-p/p'}, \quad \forall n \in \mathbb{N}.$$

Combining this with (2), it follows that

$$\limsup_{n\to\infty} \int_X |f_n g| \, \mathrm{d}\mu \le \limsup_{n\to\infty} \int_{E_n^{\complement}} |f_n g| \, \mathrm{d}\mu \le M^p \alpha^{-p/p'}.$$

Finally, sending $\alpha \to \infty$ verifies the assertion. □

References

[1] Brezis, H., & Lieb, E. (1983). A Relation Between Pointwise Convergence of Functions and Convergence of Functionals. Proceedings of the American Mathematical Society, 88(3), 486-490. doi:10.2307/2044999