A SHORT PROOF OF THE BAIRE CATEGORY THEOREM

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This short note is devoted to a simple proof of the Baire Category Theorem for topological spaces that are homeomorphic to complete metric spaces. To be more precise, we shall establish the following result:

Theorem 1 (Baire's Category Theorem). *Let $(X, \mathfrak{T})$ be a topological space that is homeomorphic to a complete metric space. If $\{U_n\}_n$ is a countable collection of open dense sets then $\bigcap_n U_n$ is again dense in $(X, \mathfrak{T})$.*

To prove this result we will first need to establish the analogous statement when $(X, \mathfrak{T})$ is a complete metric space (X, d). We will then use elementary methods to lift the theorem to the case where $(X, \mathfrak{T})$ is homeomorphic to such a compete space. First, we would like to point out the following consequence of the theorem above:

Corollary 2. *Let $(X, \mathfrak{T})$ be a topological space that is homeomorphic to a complete metric space. Then X is not the countable union of nowhere dense sets.*

Proof. Let $\{E_n\}_n$ be a countable family of nowhere dense sets in $(X, \mathfrak{T})$ and let F_n denote the closure of E_n in X, for each n. By assumption, every F_n has empty interior. This means that for each open set $W \subseteq X$ one has $W \nsubseteq F_n$, i.e.

$$W \cap F_n^{\complement} \neq \varnothing.$$

It follows that $\left\{F_n^{\complement}\right\}_n$ is a countable family of open dense sets in $(X, \mathfrak{T})$. By the Baire Category Theorem above, we find that

$$\left(\bigcup_n F_n\right)^{\complement} = \bigcap_n F_n^{\complement} \neq \varnothing.$$

Hence, $\bigcup_n E_n \subseteq \bigcup_n F_n \neq X$. □

1. The Case of a Complete Metric Space

In this section we prove Theorem 1 in the case that $(X, \mathfrak{T})$ is a complete metric space (X, d). This is often referred to as the Baire Category Theorem and is stated below.

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Proposition 3. *Let (X, d) be a complete metric space and suppose $\{U_n\}_n$ is a countable collection of open dense sets in X. Then $\bigcap_n U_n$ is dense in X.*

Proof. Let $W \subseteq X$ be a non-empty open set, the statement amounts to showing that $W \cap \bigcap_n U_n$ is non-empty. To this end, notice that $W \cap U_1$ is non-empty (since U_1 is dense in X) and contains a point x_1. There then exists $0 < r_1 < 2^{-1}$ such that

$$\overline{B(x_1, r_1)} \subseteq U_1 \cap W.$$

We now proceed inductively as follows:

> Given x_k and r_k, we consider the intersection $B(x_k, r_k) \cap U_{k+1}$, which is non-empty by assumption. Thus, there exists a point
>
> $$x_{k+1} \in B(x_k, r_k) \cap U_{k+1},$$
>
> where this intersection is an open set. Once again, we may choose $r_{k+1} > 0$ such that
>
> $$\overline{B(x_{k+1}, r_{k+1})} \subseteq B(x_k, r_k) \cap U_{k+1}.$$
>
> Without harm, we may choose $r_{k+1} < 2^{-(k+1)}$.

Now, for every $n > 1$ we have found

$$\begin{aligned} x_n \in B(x_{n-1}, r_{n-1}) \cap U_n &\subseteq B(x_{n-2}, r_{n-2}) \cap U_{n-1} \cap U_n \\ &\subseteq B(x_1, r_1) \cap \bigcap_{k=2}^{n} U_k \\ &\subseteq W \cap \bigcap_{k=1}^{n} U_k. \end{aligned}$$

which means that we are done if the sequence of sets $\{U_n\}$ is finite. Otherwise, we have constructed a sequence $(x_n)_{n=1}^{\infty}$ of points in X. We now claim that this sequence is Cauchy. Indeed, let $\varepsilon > 0$ be given and let $N \in \mathbb{N}$ be such that $2^{-N} < \varepsilon/2$. If $n, m \geq N$ then

$$x_n, x_m \in B(x_N, r_N)$$

whence it follows that $d(x_n, x_m) < \varepsilon$. Since (X, d) is complete, there exists a point $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, for every $N \in \mathbb{N}$ our construction gives

$$x_n \in \overline{B(x_N, r_N)}, \quad \forall n \geq N.$$

Passing to the limit, we find that

$$x \in \overline{B(x_N, r_N)} \subseteq B(x_{N-1}, r_{N-1}) \cap W \subseteq U_N \cap W.$$

This implies that $x \in W \cap \bigcap_n U_n$ as was required. □

This completes the proof of the "classical" Baire Category Theorem. It remains only to lift this result to special topological spaces.

2. The Proof of Theorem 1

We recall that two topological spaces $(X, \mathfrak{T})$ and $(Y, \mathfrak{W})$ are called homeomorphic if there exists a bijective map

$$f : X \longrightarrow Y$$

such that both f and f^{-1} are continuous relative to their respective topologies. We summarize some properties of homeomorphisms that we will use below.

- A homeomorphism f is an open map. Indeed, let $U \subseteq X$ be open. Since f^{-1} is a bijective continuous map $Y \longrightarrow X$ it follows that

  $$f(U) = \left(f^{-1}\right)^{-1}(U)$$

  is open in Y.
- Homeomorphisms preserve dense subspaces: if D is dense in X then $f(D)$ is dense in Y. To see this, let W be a non-empty open set Y and note that $V := f^{-1}(W)$ is open and non-empty in X. Therefore,

  $$f(D) \cap W = f(D) \cap f(V) = f(D \cap V)$$

  is non-empty since D is dense in $(X, \mathfrak{T})$.

We are now equipped to prove Theorem 1.

Proof of Theorem 1. By assumption, there exists a complete metric space $(\hat{X}, d)$ and a homeomorphism

$$f : X \longrightarrow \hat{X}.$$

Since f is bijective, we know that $f\left(\bigcap_n U_n\right) = \bigcap_n f(U_n)$. Also, each $f(U_n)$ is an open and dense subset of $(\hat{X}, d)$. If we fix a non-empty open set $W \in \mathfrak{T}$, the set $f(W)$ is open in $(\hat{X}, d)$. By the classical Baire Category Theorem, it follows that

$$f(W) \cap \bigcap_n f(U_n) \neq \varnothing.$$

Using the bijectivity of f, we find that $f\left(W \cap \bigcap_n U_n\right) \neq \varnothing$ which completes the proof. ∎