Abstract Measure and Integration Theories

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Contents

In these notes we state and prove theorems that are of fundamental importance to abstract measure and integration theory. This compilation grew out of preparation for a midterm examination in a graduate course at McGill university. The theorems proven in this text are those that I think could appear on the exam.

We begin by studying abstract measure spaces and derive fundamental properties for these. We also present various methods for the construction of measures and, in doing-so, define Lebesgue-Stieltjes measures on $\mathbb{R}$ and study their regularity.

Later, we develop an abstract theory of integration. This involves convergence theorems, $L^1(X)$-spaces and modes of convergence. We conclude with an exploration of signed measures.

1 Measure Spaces

Throughout this section X will denote a non-empty set. We shall denote the powerset of X by $\mathscr{P}(X)$. If $\mathcal{A} \subseteq \mathscr{P}(X)$, we say that $\mathcal{A}$ is an algebra on X if each of the following hold true:

1. $X \in \mathcal{A}$,
2. $\mathcal{A}$ is closed under complements,
3. $\mathcal{A}$ is closed under finite unions.

This same family $\mathcal{A}$ is called a σ-algebra on X if the third property holds up to countable unions. σ-algebras will be the fundamental building blocks for our measure spaces. If X is a set and $\mathcal{M}$ a σ-algebra on X then we call the space $(X, \mathcal{M})$ a measurable space.

1.1 Elementary Sets and Algebras

There is a structure that, although considerably weaker than algebras, is of particular interest. A collection of subsets $\mathcal{E} \subseteq \mathscr{P}(X)$ is called an elementary family if

1. $\varnothing \in \mathcal{E}$,
2. $E, F \in \mathcal{E}$ implies $E \cap F \in \mathcal{E}$,
3. If $E \in \mathcal{E}$ then E^{C} can be expressed as the disjoint union of finitely many elements of $\mathcal{E}$.

As we will see below, these elementary families "induce" algebras.

PROPOSITION 1.1. *Let $\mathcal{E}$ be an elementary family and denote by $\mathcal{A}$ the collection of all finite disjoint unions of elements in $\mathcal{E}$. Then $\mathcal{A}$ is an algebra.*

Proof. Clearly, $\varnothing \in \mathcal{A}$. We first show that $\mathcal{A}$ is closed under finite unions. For this, it suffices to show that any finite union of elements in $\mathcal{A}$ belongs to $\mathcal{A}$. If $A, B \in \mathcal{E}$

then we can write $A \cup B = A \sqcup (B \cap A^{\mathsf{C}})$ where the notation '$\sqcup$' states that the union is disjoint. Since $A \in \mathcal{E}$ we can write

$$B \cap A = \bigsqcup_{j=1}^{k} (B \cap E_j), \quad E_j \in \mathcal{E}.$$

This shows that $A \cup B$ belongs to $\mathcal{A}$ by definition. Assume that $(n-1)$-unions of elements in $\mathcal{E}$ belong to $\mathcal{A}$ and let $\{E_j\}_{j=1}^{n}$ be a sub-collection of $\mathcal{E}$. By assumption, we may assume that $E_1, E_2, \ldots, E_{n-1}$ are disjoint. Let us write

$$\bigcup_{j=1}^{n} E_j = E_n \sqcup \bigsqcup_{j=1}^{n-1} E_j \setminus E_n.$$

Now, $E_j \setminus E_n = E_j \cap E_n^{\mathsf{C}} = \bigsqcup_{\ell=1}^{m} E_j \cap F_\ell$, where $F_\ell \in \mathcal{E}$. This is possible since $E_n \in \mathcal{E}$. Using this with the above, we obtain

$$\bigcup_{j=1}^{n} E_j = E_n \sqcup \bigsqcup_{j=1}^{n} \bigsqcup_{\ell=1}^{m} E_j \cap F_\ell$$

which clearly belongs to $\mathcal{A}$. It remains only to show that $\mathcal{A}$ is closed under complements. If $A \in \mathcal{A}$ then we can write $A = \bigsqcup_{j=1}^{n} E_j$ for $E_j \in \mathcal{E}$ so that

$$A^{\mathsf{C}} = \bigcap_{j=1}^{n} E_j^{\mathsf{C}} = \bigcap_{j=1}^{n} \bigsqcup_{\ell=1}^{m_j} F_{j,\ell}$$

where $F_{j,\ell} \in \mathcal{E}$. Since $\mathcal{E}$ is closed under intersections and $\mathcal{A}$ is closed under finite unions (by the above) it follows that $A^{\mathsf{C}} \in \mathcal{A}$. □

Hence, we know how to construct an algebra from an elementary set. We follow this up with a similar result for σ-algebras.

PROPOSITION 1.2. *Let $\mathcal{A}$ be an algebra on a non-empty set X. Then $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable disjoint unions.*

Proof. One direction is completely trivial. For the converse suppose that $\mathcal{A}$ is an algebra closed under countable disjoint unions. We need only check that $\mathcal{A}$ is closed under countably infinite unions. Let $\{A_j\}_{j=1}^{\infty}$ be a sub-collection of $\mathcal{A}$ Define

$$B_1 := A_1, \quad B_j := A_j \setminus \bigcup_{k=1}^{j-1} A_j.$$

Since $\mathcal{A}$ is an algebra, we clearly have $B_j \in \mathcal{A}$ for all j. Note also that by construction the B_j are pairwise disjoint. Hence, $\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} B_j \in \mathcal{A}$. □

PROPOSITION 1.3. *Let I be an index family and suppose $\mathcal{M}_i$ is a σ-algebra on X for all $i \in I$. Then $\bigcap_{i \in I} \mathcal{M}_i$ is again a σ-algebra on X.*

The proof is straightforward and left as an exercise to the reader. Nonetheless, this is crucial notion. It allows to define a "minimal" σ-algebra on any family $\mathcal{F} \subseteq \mathscr{P}(X)$. Certainly, if $\mathcal{F}$ is a collection of subsets of X we define

$$\mathsf{M}(\mathcal{F}) := \bigcap_{\mathcal{M} \supseteq \mathcal{F}} \mathcal{M}, \quad \text{where } \mathcal{M} \text{ is a } \sigma\text{-algebra on } \mathcal{F}.$$

This intersection is non-empty since $\mathscr{P}(X)$ is always a σ-algebra containing $\mathcal{F}$. We call $\mathsf{M}(\mathcal{F})$ the σ-algebra *generated* by $\mathcal{F}$.

DEFINITION 1. Let $(X, \mathcal{T})$ be a topological space. The Borel algebra, denoted $\mathcal{B}_X$, is the σ-algebra generated by $\mathcal{T}$.

PROPOSITION 1.4. *Let $\mathcal{B}_{\mathbb{R}}$ be the σ-algebra on $\mathbb{R}$ generated by the usual topology. This is also generated by either of the following families:*

$$\mathcal{F}_1 := \{(a,b) : a, b \in \mathbb{R},\ a < b\}\,; \quad \mathcal{F}_2 := \{(-\infty, a) : a \in \mathbb{R}\}\,;$$

$$\mathcal{F}_3 := \{(a, \infty) : a \in \mathbb{R}\}\,; \quad \mathcal{F}_4 := \{[a,b] : a < b,\ a, b \in \mathbb{R}\}\,.$$

Proof. Since $\mathcal{B}_{\mathbb{R}}$ clearly contains $\mathcal{F}_1$ it is immediate that $\mathcal{B}_{\mathbb{R}} \supseteq \mathsf{M}(\mathcal{F}_1)$. For the reverse inclusion, note that any open set O in $\mathbb{R}$ may be expressed as the countable disjoint union of open intervals. Thus, $O \in \mathsf{M}(\mathcal{F}_1)$ which shows that $\mathcal{B}_{\mathbb{R}} = \mathsf{M}(\mathcal{F}_1)$.

If $a, b \in \mathbb{R}$ with $a < b$ then

$$(a,b) = (-\infty, b) \setminus (-\infty, a)$$

which shows that $\mathcal{F}_1 \subseteq \mathsf{M}(\mathcal{F}_2)$ whence $\mathsf{M}(\mathcal{F}_1) \subseteq \mathsf{M}(\mathcal{F}_2)$. For the reverse inclusion, it suffices to observe that any interval $(-\infty, a)$ is the countable disjoint union of open intervals. One may proceed in similar ways for the remaining families. $\square$

1.2 Measures

Given a measurable space $(X, \mathcal{M}, \mu)$, a non-negative set function on the space is a function

$$\tau : \mathcal{M} \longrightarrow [0, \infty].$$

We of course use the convention $0 \cdot \infty = 0$.

DEFINITION 2. Let $(X, \mathcal{M})$ be a measurable space and μ a non-negative set function on the space. We say that μ is a measure on $(X, \mathcal{M})$ whenever

1. $\mu(\varnothing) = 0$,
2. If $\{E_j\}_j$ is a countable disjoint sub-collection of $\mathcal{M}$

$$\mu\left(\bigcup_j E_j\right) = \sum_j \mu(E_j).$$

The triple $(X, \mathcal{M}, \mu)$ is then called a measure space.

As we shall soon see, measures have some nice continuity and regularity, as well as some expectedly intuitive properties. The results that follow should convince the reader that measures are suitable generalizations of volume in $\mathbb{R}^n$. Before proceeding to these continuity results, we give a useful formula.

LEMMA 1.5. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $E, F \in \mathcal{M}$ with $\mu(E) < \infty$. If $F \subseteq E$ then*

$$\mu(E \setminus F) = \mu(E) - \mu(F).$$

Proof. Write $E = (E \cap F) \sqcup (E \setminus F) = F \sqcup (E \setminus F)$ so that $\mu(E) = \mu(F) + \mu(E \setminus F)$. □

THEOREM 1.6. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

1. *If $E, F \in \mathcal{M}$ are such that $E \subseteq F$ then $\mu(E) \leq \mu(F)$.*
2. *If $\{E_j\}_j$ is a countable family in $\mathcal{M}$ then $\mu(\bigcup_j E_j) \leq \sum_j E_j$.*
3. *If $\{E_j\}_j \subseteq \mathcal{M}$ is a countable family such that $E_1 \subseteq E_2 \subseteq \dots$ then*

$$\mu(\bigcup_{j=1}^{\infty} E) = \lim_{n \to \infty} \mu(E_n).$$

4. *If $\{F_j\}_j \subseteq \mathcal{M}$ is a countable family such that $\mu(F_1) < \infty$ and $F_1 \supseteq F_2 \supseteq \dots$ then*

$$\mu\left(\bigcap_{j=1}^{\infty} F_j\right) = \lim_{n \to \infty} \mu(F_n).$$

Proof. For (1) we write $F = E \sqcup (F \setminus E)$ such that $\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(F)$. Similarly, for (2) we define $A_1 := E_1$ and $A_n := E_n \setminus \bigcup_{j=1}^{n-1} E_j$. Clearly $A_n \in \mathcal{M}$ for all n and

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{n=1}^{\infty} F_j\right) = \sum_{n=1}^{\infty} \mu(F_n)$$

where we have used that the F_n's are disjoint and their union is $\bigcup_{j=1}^{\infty} E_j$. By the first part we have $\mu(F_n) \leq \mu(E_n)$ which yields (2).

To prove (3) let $E_1 := F_1$ and $F_n := E_n \setminus E_{n-1}$ for $n > 1$. Since the E_n's are increasing, we find that $\{F_n\}_n$ is a countable disjoint family in $\mathcal{M}$ such that $\bigcup_{j=1}^{\infty} E_j = \bigsqcup_{n=1}^{\infty} F_n$. Thus,

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{n=1}^{\infty} \mu(F_n) = \lim_{N \to \infty} \sum_{n=1}^{N} \mu(F_n).$$

By observing that $E_N = \bigsqcup_{n=1}^{N} F_n$ we obtain from the above that

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{N \to \infty} \mu(E_N).$$

For the final point, define $F_j := E_1 \backslash E_j$ for all j. Then $\{F_j\}_j$ is a countable increasing family in $\mathcal{M}$ so that, after applying (3),

$$\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \lim_{n\to\infty} \mu(F_j) = \mu(E_1) - \lim_{n\to\infty} \mu(E_n)$$

where $\mu(E_n)$ tends to a real number as $n \to \infty$ since it is monotone decreasing sequence of real numbers. It is not difficult to see that

$$\bigcup_{j=1}^{\infty} F_j = E_1 \setminus \bigcap_{j=1}^{\infty} E_j.$$

Since $\bigcap_{j=1}^{\infty} E_j \subseteq E_1$ we obtain that $\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \mu(E_1) - \mu\left(\bigcap_{j=1}^{\infty} E_j\right)$. Therefore,

$$\mu(E_1) - \mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \mu(E_1) - \lim_{n\to\infty} \mu(E_n).$$

Since $\mu(E_1) < \infty$ we can subtract □

Having established preliminaries we should introduce notation and convention. The σ-algebra $\mathcal{M}$ is called the collection of measurable sets and any set $E \in \mathcal{M}$ is called measurable. A measurable set E is said to be a *null set* if $\mu(E) = 0$. The measure space $(X, \mathcal{M}, \mu)$ is called **complete** if all subsets of null sets are measurable. The space is called finite if $\mu(X) < \infty$. Similarly, $(X, \mathcal{M}, \mu)$ is said to be σ-finite is there exists a countable collection $\{X_j\}_j$ in $\mathcal{M}$ with $\mu(X_j) < \infty$ and $X = \bigcup_{j=1}^{\infty} X_j$.

1.3 Outer-Measures and Premeasures

An outer-measure is a non-negative set function μ^* on the measurable space $(X, \mathscr{P}(X))$ that satisfies:

1. $\mu^*(\varnothing) = 0$,
2. $\mu^*(E) \leq \mu^*(F)$ whenever $E \subseteq F$,
3. $\mu^*(\bigcup_j E_j) \leq \sum_j \mu^*(E_j)$ for any countable family $\{E_j\}_j$ of subsets in X.

An outer-measure is defined for all subset of X but, unfortunately, is not typically a measure. Nonetheless, outer-measures are of tremendous use in the construction of measures. Moreover, any outer-measure can be restricted to a particular σ-algebra upon which it is a complete measure. This is what the following theorem establishes.

THEOREM 1.7 (Carathéodory). *Let μ^* be an outer-measure on $(X, \mathscr{P}(X))$. We say a set $A \subseteq X$ is μ^*-measurable if*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^{\mathsf{c}}), \quad \forall E \subseteq X.$$

Let $\mathcal{M}$ be the collection of all μ^*-measurable sets. Then $(X, \mathcal{M}, \mu^*)$ is a complete measure space.

Proof. We first note that $\mathcal{M}$ is non-empty since $\varnothing \in \mathcal{M}$ by definition of an outer-measure. Moreover, it is clear by the definition that $\mathcal{M}$ is closed under complements since A is μ^*-measurable if and only if A^{C} is. Let now $A, B \in \mathcal{M}$ be given. If $E \subseteq X$ we may write

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^{\mathsf{C}}) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^{\mathsf{C}}) + \mu^*(A^{\mathsf{C}} \cap B) + \mu^*(E \cap (A \cup B)^{\mathsf{C}}).\end{aligned}$$

Note that $A \cup B \subseteq (A \cap B) \cup (A^{\mathsf{C}} \cap B) \cup (A \cap B^{\mathsf{C}})$ whence, by monotonicity of μ^*:

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^{\mathsf{C}}).$$

Since the reverse inequality is always true (sub-additivity of μ^* together with the identity $E = (E \cap A) \sqcup (E \cap A^{\mathsf{C}})$); we find that $A \cup B$ belongs to $\mathcal{M}$. Therefore $\mathcal{M}$ is closed under finite unions and is therefore an algebra on X. By virtue of Proposition 1.2 it suffices to check that $\mathcal{M}$ is closed under countable disjoint unions to prove that $\mathcal{M}$ is a σ-algebra.

Let $\{A_j\}_{j=1}^{\infty}$ be a countable disjoint family in $\mathcal{M}$; for $n \in \mathbb{N}$ we define $B_n := \bigcup_{j \leq n} A_j$ and $B := \bigcup_{j=1}^{\infty} A_j$. Then $B_n \nearrow B$ as $n \to \infty$. Since A_n is μ^*-measurable for each n one clearly has for each $E \subseteq X$

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^{\mathsf{C}}) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1})\end{aligned}$$

where we have used the fact that the A_j are disjoint so that $A_j \subseteq A_n^{\mathsf{C}}$ for $j < n$. Simple induction then gives

$$\mu^*(E \cap B_n) = \sum_{j=1}^{n} \mu^*(E \cap A_j).$$

By sub-additivity we have $\mu^*(E) \leq \mu^*(E \cap B) + \mu^*(E \cap B^{\mathsf{C}})$ and thus it only remains to show the reverse inequality. Since $\mathcal{M}$ is known to be an algebra, one has $B_n \in \mathcal{M}$ and

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^{\mathsf{C}}) \geq \sum_{j=1}^{n} \mu^*(E \cap A_n) + \mu^*(E \cap B^{\mathsf{C}}).$$

Letting $n \to \infty$ we obtain

$$\mu^*(E) \geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(E \cap B^{\mathsf{C}}) \geq \mu^*\left(E \cap B\right) + \mu^*(E \cap B^{\mathsf{C}}).$$

This shows that B is μ^*-measurable so that $\mathcal{M}$ is a σ-algebra. To check that μ^* is a measure when restricted to $\mathcal{M}$ we take $E = B$ in the above to obtain:

$$\mu^*(E) = \mu^*(E) + \mu^*(\varnothing).$$

As for completeness of $(X, \mathcal{M}, \mu^*)$ suppose that $\mu^*(N) = 0$. Then for all $E \subseteq X$:

$$\mu^*(E) \leq \mu^*(E \cap N) + \mu^*(E \cap N^{\mathsf{C}}) \leq \mu^*(E \cap N^{\mathsf{C}}) \leq \mu^*(E).$$

The proof is now complete. □

In light of the theorem, we discussing an outer-measure, we will write μ instead of μ^* when it is restricted to $\mathcal{M}$.

DEFINITION 3. A premeasure on an algebra $\mathcal{A}$ is a map $\mu_0 : \mathcal{A} \longrightarrow [0, \infty]$ adhering to the following

1. $\mu(\varnothing) = 0$,

2. If $\{A_j\}_j$ is a countable family of sets in $\mathcal{A}$ such that $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ then

$$\mu_0(A) = \sum_j \mu_0(A_j).$$

Before proceeding further we would like to relate premeasures to outer-measures. In fact, any premeasure induces an outer-measure on $(X, \mathscr{P}(X))$ (and thus a measure!). Define for $E \subseteq X$:

$$\mu^*(E) := \inf \left\{ \sum_{j \in J} \mu_0(A_j) : E \subseteq \bigcup_{j \in J} A_j,\ A_j \in \mathcal{A} \text{ and } J \text{ countable} \right\}. \tag{1}$$

PROPOSITION 1.8. *μ^* is indeed an outer-measure on $(X, \mathscr{P}(X))$.*

Proof. Clearly, $\mu^*(\varnothing) = 0$. If $E \subseteq F$ then any cover of F is again a cover of E so that $\mu^*(E) \le \mu^*(F)$ as required. Suppose that $\{E_j\}_j$ is a countable family of sets in X. Letting $\varepsilon > 0$ be arbitrary, for each j we select a countable covering $\{A_{j,k}\}_k$ of elements in $\mathcal{A}$ such that

$$E_j \subseteq \bigcup_k A_{j,k} \quad \text{and} \quad \sum_k \mu_0(A_{j,k}) \le \mu^*(E_j) + \frac{\varepsilon}{2^j}.$$

If we define $E = \bigcup_j E_j$ then $E \subseteq \bigcup_{j,k} A_{j,k}$ so that, by definition

$$\mu^*(E) \le \sum_{j,k} \mu_0(A_{j,k}) \le \sum_j \left(\mu_0(E_j) + \frac{\varepsilon}{2^j} \right) \le \varepsilon + \sum_j \mu_0(E_j).$$

Since $\varepsilon > 0$ was arbitrary, we may let $\varepsilon \to 0^+$ to obtain $\mu^*(E) \le \sum_j \mu^*(E_j)$. This shows that μ^* is indeed a well defined outer-measure. □

Hence, premeasures give rise to outer-measures on $(X, \mathscr{P}(X))$. It it therefore natural to study its refinement into a measure (which we know to exist by Carathéodory's theorem). However, this structure is not worth much if the σ-algebra of μ^*-measurable sets does not contain our original algebra $\mathcal{A}$. This is precisely the question we answer below.

PROPOSITION 1.9. *Let μ^* be the outer-measure generated by a premeasure μ_0 on an algebra $\mathcal{A}$. Then each of the following hold*

1. *$\mu_0(A) = \mu^*(A)$ for all $A \in \mathcal{A}$,*

2. *Every $A \in \mathcal{A}$ is μ^*-measurable.*

Proof. Fix $A \in \mathcal{A}$ and note that $\{A\}$ is a covering of A by elements of $\mathcal{A}$. It follows that $\mu^*(A) \leq \mu_0(A)$. For the reverse inequality we fix $\varepsilon > 0$ and choose a countable cover $\{A_j\}_j$ of elements in $\mathcal{A}$ such that

$$A \subseteq \bigcup_j A_j \quad \text{and} \quad \sum_j \mu_0(A_j) \leq \mu^*(A) + \varepsilon \tag{2}$$

Claim. Let $Q, R \in \mathcal{A}$ be such that $Q \subseteq R$. Then $\mu_0(Q) \leq \mu_0(R)$.

Proof of Claim. Write $R = Q \sqcup (R \backslash Q)$ where $R \backslash Q \in \mathcal{A}$ by closure under complement and finite intersections[1]. Applying additivity, $\mu_0(R) = \mu_0(Q) + \mu_0(R \setminus Q) \geq \mu_0(Q)$.

Claim. If $\{Q_j\}_j$ is a countable sub-collection of $\mathcal{A}$ and $Q = \bigcup_j Q_j \in \mathcal{A}$ then

$$\mu_0(Q) \leq \sum_j \mu_0(Q_j).$$

Proof of Claim. Define $R_1 := Q_1$ and $R_j = Q_j \setminus \bigcup_{l=1}^{j-1} Q_j$ for all suitable j and observe that the R_j are pairwise disjoint elements of $\mathcal{A}$ ($\mathcal{A}$ is closed under complement and finite union) such that $\bigcup_j Q_j = \bigcup_j R_j$ which shows that $\bigcup_j R_j \in \mathcal{A}$. Hence,

$$\mu_0\left(\bigcup_j Q_j\right) = \mu_0\left(\bigcup_j R_j\right) = \sum_j \mu_0(R_j) \leq \sum_j \mu_0(Q_j).$$

Applying these claims to (2) we obtain

$$\mu_0\left(A \cap \bigcap_j A_j\right) \leq \sum_j \mu_0(A \cap A_j) \leq \sum_j \mu_0(A_j) \leq \mu^*(A) + \varepsilon.$$

Thus, $\mu_0(A) \leq \mu^*(A) + \varepsilon$. Letting $\varepsilon \to 0$ shows that $\mu^*(A) = \mu_0(A)$. We now show that A is μ^*-measurable. By sub-additivity, if $E \subseteq X$

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^{\mathsf{C}}).$$

For the reverse inequality, we fix $\varepsilon > 0$ and choose a family $\{A_j\}_j$ in $\mathcal{A}$ such that $E \subseteq \bigcup_j A_j$ with $\sum_j \mu_0(A_j) \leq \mu^*(E) + \varepsilon$. We then infer from our previous claims

$$\mu^*(E) + \varepsilon \geq \sum_j \mu_0(A_j) = \sum_j \mu_0(A_j \cap A) + \sum_j \mu_0(A_j \cap A^{\mathsf{C}})$$

where we have used the fact that $A_j = (A \cap A_j) \sqcup (A_j \cap A^{\mathsf{C}})$. Since $A \cap E \subseteq \bigcup_j (A_j \cap A)$ and $A^{\mathsf{C}} \cap E \subseteq \bigcup_j (A_j \cap A^{\mathsf{C}})$ this yields

$$\varepsilon + \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^{\mathsf{C}}).$$

Letting $\varepsilon \to 0$ concludes the proof. $\square$

[1] Let $A, B \in \mathcal{A}$ and note that $(A \cap B)^{\mathsf{C}} = A^{\mathsf{C}} \cup B^{\mathsf{C}}$ which belongs to $\mathcal{A}$ by closure under complements. Hence, $A \cup B \in \mathcal{A}$.

Combining all of these results will yield an important theorem.

THEOREM 1.10. *Let $\mathcal{A}$ be an algebra on X and μ_0 a premeasure on $\mathcal{A}$. Denote by μ^* the outer-measure it induces and let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Then $(X, \mathcal{M}, \mu^*)$ is a measure space and if ν is any measure defined on $\mathcal{M}$ agreeing with μ_0 on $\mathcal{A}$*

$$\nu(E) \leq \mu^*(E), \quad \forall E \in \mathcal{M}$$

with equality when $\mu^*(E) < \infty$. If $(X, \mathcal{M}, \mu)$ is σ-finite then $\mu^* \equiv \nu$ on $\mathcal{M}$.

Proof. Let $\mathfrak{M}$ be the σ-algebra consisting of all μ^* measurable sets in X. We know that this is a σ-algebra containing $\mathcal{A}$ (by earlier results) and thus $\mathcal{M} \subseteq \mathfrak{M}$. This shows that $(X, \mathcal{M}, \mu^*)$ is a measure space when μ^* is restricted to $\mathcal{M}$. Hence, we write only μ to mean μ^*.

Now suppose ν is another measure on $(X, \mathcal{M})$ agreeing with μ_0 on $\mathcal{A}$. If $E \in \mathcal{M}$ then for every countable family $\{A_j\}_j$ of elements in $\mathcal{A}$ such that $E \subseteq \bigcup_j A_j$

$$\nu(E) \leq \sum_j \nu(A_j) = \sum_j \mu_0(A_j)$$

whence $\nu(E) \leq \mu(E)$. If in addition $\mu(E) < \infty$ we can choose the A_j such that

$$\sum_j \mu_0(A_j) \leq \mu(E) + \varepsilon.$$

Therefore, defining $A = \bigcup_j A_j$ we find

$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A \setminus E) \leq \nu(E) + \mu(A \setminus E)$$

where we have used that $\bigcup_{j \leq n} A_j \nearrow A$ as $n \to \infty$ to conclude that

$$\mu(A) = \lim_{n\to\infty} \mu\left(\bigcup_{j\leq n} A_j\right) = \lim_{n\to\infty} \nu\left(\bigcup_{j\leq n} A_j\right) = \nu(A).$$

Note that $\mu(A) \leq \sum_j \mu_0(A_j) \leq \mu(E) + \varepsilon$. Since $\mu(E) < \infty$ we obtain

$$\mu(A \setminus E) = \mu(E) - \mu(A) \leq \varepsilon.$$

This finally yields

$$\mu(E) \leq \nu(E) + \varepsilon;$$

where we now take $\varepsilon \to 0^+$. Suppose now that $(X, \mathcal{M}, \mu)$ is σ-finite. There exists a family $\{X_j\}_j$ of measurable sets such that $X = \bigcup_j X_j$ and $\mu(X_j) < \infty$. We may assume, by our usual procedure, that this family is disjoint. For every $E \in \mathcal{M}$

$$\mu(E) = \mu\left(E \cap \bigcup_j X_j\right) = \sum_j \mu(E \cap X_j) = \sum_j \nu(E \cap X_j) = \nu(E).$$

□

1.4 Lebesgue-Stieltjes Measures on $\mathbb{R}$

For what remains of this section we shall develop useful measures on $\mathbb{R}$. Suppose we are given a function $F : \mathbb{R} \longrightarrow \mathbb{R}$ that is monotone increasing and right continuous. An h-interval is an interval of the form $(a, b]$ where $-\infty \le a < b \in \mathbb{R}$. Clearly, the intersection of two h-intervals is again an h-interval. Furthermore, $\mathbb{R} \setminus (a, b]$ is always a finite disjoint union of h-intervals. Thus, the collection of h-intervals together with $\varnothing$ is an elementary family on $\mathbb{R}$.

By previous results, the collection $\mathcal{A}$ of finite disjoint unions of elements in $\mathcal{E}$ is an algebra on $\mathbb{R}$. It obvious that the σ-algebra generated by $\mathcal{A}$ is $\mathcal{B}_{\mathbb{R}}$. We define a premeasure on $\mathcal{A}$ by letting

$$\mu\left(\bigsqcup_{j=1}^{k}(a_j, b_j]\right) := \sum_{j=1}^{k}\left[F(b_j) - F(a_j)\right].$$

It is left to the reader to verify that this is indeed a well defined premeasure on $\mathcal{A}$. By previous results, this will extend to a measure μ° on $\mathcal{B}_{\mathbb{R}}$. We let $(X, \mathcal{M}, \mu)$ be the completion (see the exercises) of this measure space. In the special case, $F(x) = x$, this is the Lebesgue measure. In general, these are called *Lebesgue-Stieltjes* measures on $\mathbb{R}$ generated by F.

LEMMA 1.11. *Let $(X, \mathcal{M}, \mu)$ be a Lebesgue-Stieltjes measure space generated by F. Then, for all $E \in \mathcal{M}$*

$$\mu(E) = \inf\left\{\sum_j \mu((a_j, b_j)) : E \subseteq \bigcup_j (a_j, b_j)\right\}$$

where these collections are countable.

Proof. We denote the quantity on the right hand side by $\eta(E)$. First let $\{(a_j, b_j)\}_j$ be a countable cover of E by open intervals in $\mathbb{R}$. For each j we may write $(a_j, b_j) = \bigsqcup_k (c_j^k, c_j^{k+1}]$ as a countable disjoint union of h-intervals. Therefore, $E \subseteq \bigcup_{j,k}(c_j^k, c_j^{k+1}]$ whence we obtain

$$\mu(E) \le \sum_{j,k} \mu((c_j^k, c_j^{k+1}]) = \sum_j \mu((a_j, b_j))$$

which implies that $\mu(E) \le \eta(E)$. To obtain equality, let $\varepsilon > 0$ and choose a collection countable $\{(a_j, b_j]\}_j$ such that $E \subseteq \bigcup_j (a_j, b_j]$ and

$$\sum_j \mu((a_j, b_j]) \le \mu(E) + \varepsilon.$$

For each j there exists $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < \varepsilon/2^j$. Since $E \subseteq \bigcup_j (a_j, b + \delta_j)$ we obtain

$$\begin{aligned}\eta(E) \le \sum_j \mu((a_j, b_j + \delta]) &= \sum_j \left[F(b_j + \delta) - F(b_j) + F(b_j) - F(a_j)\right] \\ &\le \varepsilon + \sum_j \mu((a_j, b_j]) \\ &\le 2\varepsilon + \mu(E).\end{aligned}$$

Letting $\varepsilon \to 0^+$ concludes the proof. $\square$

Having this lemma, one can establish the following "regularity results" for Lebsegue-Stieltjes measures on $\mathbb{R}$.

THEOREM 1.12. *Let $(X, \mathcal{M}, \mu)$ be a Lebesgue-Stieltjes measure space. For all $E \in \mathcal{M}$*

$$\mu(E) = \inf\{\mu(O) : O \supseteq E,\ O \text{ open}\}$$
$$= \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}.$$

Proof. If $O \supseteq E$ is open then $\mu(E) \leq \mu(O)$ by monotonicity so that $\mu(E) \leq \inf_O \mu(O)$. For the reverse inequality, fix $\varepsilon > 0$ and choose a countable collection $\{(a_j, b_j)\}_j$ whose union contains E such that

$$\sum_j \mu((a_j, b_j)) \leq \mu(E) + \varepsilon.$$

Then $O = \bigcup_j (a_j, b_j)$ is an open set containing E so that

$$\inf_{O \supseteq E} \mu(O) \leq \mu(E) + \varepsilon.$$

Letting $\varepsilon \to 0^+$ gives that $\mu(E) = \inf_{O \supseteq E} \mu(O)$. We shall now show that the measurable sets can be approximated from the "inside" by compact sets. For this, we first suppose that E is bounded. Then, $\mu(E) < \infty$. If E is closed it is also compact; in this case equality is obvious.

Otherwise, let $\overline{E}$ denote the closure of E and let $\varepsilon > 0$. Choose an open set $U \supseteq \overline{E} \setminus E$ such that

$$\mu(U) < \mu(\overline{E} \setminus E) + \varepsilon.$$

Consider the compact set $K := \overline{E} \setminus U = \overline{E} \cap U^{\mathsf{C}}$. Note also that $K \subseteq \overline{E} \cap (\overline{E}^{\mathsf{C}} \cup E) = E$. Now, we have

$$\mu(E) = \mu(K) + \mu(E \setminus K)$$

where $E \setminus K = E \cap U$. Hence,

$$\begin{aligned}\mu(K) = \mu(E) - \mu(E \cap U) &= \mu(E) - \mu(U) + \mu(U \setminus E) \\ &\geq \mu(E) - \mu(U) + \mu(\overline{E} \setminus E) \\ &\geq \mu(E) - \varepsilon.\end{aligned}$$

Now, suppose E is unbounded, for $n \in \mathbb{N}$ define

$$E_n := E \cap [-n, n]$$

so that $E_n \nearrow E$. It follows that $\mu(E_n) \nearrow \mu(E)$ as $n \to \infty$. If $\mu(E) = \infty$ then for each n there exists a compact $K_n \subseteq E_n \subseteq E$ such that

$$\mu(K_n) > \mu(E_n) - 1.$$

As $n \to \infty$ we have that $\mu(K_n) \to \infty$. That is, there is a sequence of compact sets contained in E of unbounded measures. If instead $\mu(E) < \infty$ then given $\varepsilon > 0$ there exists N so large that

$$\mu(E_N) > \mu(E) - \varepsilon.$$

For this E_N, there is a compact set $K \subseteq E_N \subseteq E$ such that

$$\mu(E_N) - \varepsilon < \mu(K).$$

Thus,

$$\mu(K) > \mu(E) - 2\varepsilon.$$

□

LEMMA 1.13. *Let E be Lebesgue-Stieltjes measurable. For each $\varepsilon > 0$, there exists an open set $O \supseteq E$ such that $\mu(E \setminus O) \leq \varepsilon$.*

Proof. Let $\varepsilon > 0$ and define for $k \in \mathbb{Z}$ the set $E_k := E \cap (k, k+1]$. Clearly, we may write $E = \bigsqcup_{k\in\mathbb{Z}} E_k$. From earlier results, we may choose for each k an open set $U_k \supseteq E_k$ such that

$$\mu(U_k) < \mu(E_k) + \frac{\varepsilon}{2 \cdot 2^{|k|}}.$$

Since $U_k \supseteq E_k$ and $\mu(E_k) < \infty$, it follows from the above that

$$\mu(U_k \setminus E_K) < \frac{\varepsilon}{2 \cdot 2^{|k|}}.$$

Define now $O := \bigcup_{k\in\mathbb{Z}} U_k$ and note that $O \supseteq E$. It is easy to see that

$$O \setminus E = \bigcup_{k\in\mathbb{Z}} U_k \setminus \bigcup_{k\in\mathbb{Z}} E_k \subseteq \bigcup_{k\in\mathbb{Z}} (U_k \setminus E_k).$$

Hence,

$$\mu(O \setminus E) \leq \sum_{k\in\mathbb{Z}} \mu(U_k \setminus E_k) \leq \frac{\varepsilon}{2} \sum_{k\in\mathbb{Z}} 2^{-|k|} = \varepsilon.$$

□

We recall some intermittent convention. A set $A \subseteq \mathbb{R}$ is called a F_σ set provided it is the countable union of closed sets. In like, we say A is a G_δ set if it can be written as the countable union of open sets. One can interpret a Lebesgue-Stieltjes measurable subset of $\mathbb{R}$ as Borel sets "modulo" a null set. This is what the final result of this section suggests.

THEOREM 1.14. *Let $(X, \mathcal{M}, \mu)$ be a Lebesgue-Stieltjes measure space and fix $E \subseteq \mathbb{R}$. The following are equivalent:*

1. *$E \in \mathcal{M}$,*
2. *There exists a G_δ set G and a null set N such that $E = G \setminus N$.*
3. *There exists a F_σ set F and a null set N such that $E = F \cup N$.*

Proof. Since $(X, \mathcal{M}, \mu)$ is complete, it is obvious that the last two statements imply the first. Suppose that $E \in \mathcal{M}$ and let $\mu(E) < \infty$. By the previous proposition, to each $k \in \mathbb{N}$ we can associate a compact set $K_n \subseteq E$ and an open set $U_n \supseteq E$ such that

$$\mu(U_n) - \frac{1}{n} < \mu(E) < \mu(K_n) + \frac{1}{n}.$$

Define $G := \bigcap_{n=1}^{\infty} U_n$ and $F := \bigcup_{n=1}^{\infty} K_n$. Note that $G \supseteq E$ and $F \subseteq E$. Therefore, $\mu(E) \leq \mu(G)$ and $\mu(F) \leq \mu(E)$. For each $n \in \mathbb{N}$

$$\mu(G) \leq \mu(U_n) \leq \mu(E) + \frac{1}{n}, \quad \mu(E) - \frac{1}{n} \leq \mu(K_n) \leq \mu(F).$$

Letting $k \to \infty$ gives $\mu(G) = \mu(E) = \mu(F)$ which implies that

$$G = E \sqcup (G \setminus E), \quad E = F \sqcup (E \setminus F)$$

where $G \setminus E$ and $E \setminus F$ have measure zero. We now extend the argument to the case $\mu(E) = \infty$. For fixed $k \in \mathbb{Z}$ we define the restriction $E_k := E \cap (k, k+1]$ and note that $E = \bigcup_{k \in \mathbb{Z}} E_k$. For each k there exists a F_σ set $H_k \subseteq E_k$ with

$$E_k = H_k \cup N_k$$

where N_k is some null set. Taking the union over the k we obtain

$$E = \bigcup_{k \in \mathbb{Z}} H_k \cup \bigcup_{k \in \mathbb{Z}} N_k$$

where $\mu(\bigcup_{k \in \mathbb{Z}} N_k) \leq \sum_{k \in \mathbb{Z}} \sum(N_k) = 0$. Since countable unions of F_σ sets is again F_σ, we have our representation of E. Let $n \in \mathbb{N}$ be given, by the previous lemma there exists an open set $U_n \supseteq E$ such that $\mu(U_n \setminus E) \leq 1/n$. Then $G := \bigcap_{n=1}^{\infty} O_n \supseteq E$ is a G_δ-set and

$$\mu(G \setminus E) \leq \mu(O_n \setminus E) \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$ concludes the proof. □

2 Integration Theory

In elementary real analysis one often encounters a rather primitive theory of integration. This theory masquerades as a refined and powerful theory. Indeed, the definitions are more robust and precise than those from calculus (where one is, typically, first introduced to integration). Nonetheless, despite the abstract constructions from elementary analysis, the Riemann integral is lacking in many areas. There are no satisfying convergence theorems; and limits of Riemann integrable functions need not be Riemann integrable.

The goal of this section is to develop a stronger theory of integration that does not rely heavily on the structure of the real line. This new theory will make heavy use of abstract measure theory and will apply to any measure space.

2.1 Measurable Functions

Let us fix two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. What does it mean for a map $f : X \longrightarrow Y$ to preserve the structure of both spaces?

DEFINITION 4. A function $f : X \longrightarrow Y$ is called measurable if $f^{-1}(B) \in \mathcal{M}$ for all $B \in \mathcal{N}$. That is, a measurable function is a map such that the pre-image of every measurable set is measurable.

Such functions are of crucial importance as they will be the foundation of our theory of integration. Unfortunately it is often quite difficult to check directly whether a function is measurable. In the case where $\mathcal{N}$ has a generating set $\mathcal{E}$, this task becomes simpler.

PROPOSITION 2.1. *Suppose $\mathcal{N}$ is the σ-algebra generated by $\mathcal{E}$. Then a function $f : X \longrightarrow Y$ is measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.*

Proof. One direction is trivial. Suppose that $f^{-1}(E) \in \mathcal{M}$ for all $\mathcal{E} \ni E$. Consider

$$\mathfrak{N} := \left\{ N \in \mathcal{N} : f^{-1}(N) \in \mathcal{M} \right\};$$

then $\mathcal{E} \subseteq \mathfrak{N}$. It is easy to show that $\mathfrak{N}$ is a σ-algebra on Y. Since $\mathfrak{N}$ contains $\mathcal{E}$, it follows that $\mathfrak{N} \supseteq \mathsf{M}(\mathcal{E}) = \mathcal{N}$. □

We will mostly be dealing with function that are either real or complex valued. However, what does it mean for a function $f : X \longrightarrow \mathbb{C}$ to be measurable? There is a subtle question here: which σ-algebra is it natural to endow $\mathbb{R}$ and $\mathbb{C}$ with? Both $\mathbb{R}$ and $\mathbb{C}$ are equipped with a very special topology and it would be wise to take advantage of this fact. That is, we want all "nice sets" to be measurable in the target space.

As mentioned in the previous section, there exists a σ-algebra on any topological space that includes all open (and thus closed) sets. It is these algebras that we shall endow our target spaces with.

DEFINITION 5. Given a measurable space $(X, \mathcal{M})$ we say a function $f : X \longrightarrow \mathbb{R}$ is measurable provided $f^{-1}(B) \in \mathcal{M}$ for all $B \in \mathcal{B}_{\mathbb{R}}$. We say $g : X \longrightarrow \mathbb{C}$ is measurable if $\Re f$ and $\Im f$ are measurable as real valued functions.

In simpler more concise terms, when the target set is $\mathbb{R}$ or $\mathbb{C}$, we view these as the measurable spaces $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ respectively.[2]

It will be convenient to allow functions to take extended real valued, i.e. for functions to take on the valued $\pm\infty$. Recalling the convention $\overline{\mathbb{R}} := [-\infty, \infty]$, it is important to associate a σ-algebra to $\overline{\mathbb{R}}$. We define

$$\mathcal{B}_{\overline{\mathbb{R}}} := \mathsf{M}\left(\{(a, \infty] : a \in \mathbb{R}\}\right).$$

Note that $\mathcal{B}_{\overline{\mathbb{R}}} \supset \mathcal{B}_{\mathbb{R}}$. We also expect measurable functions to behave nicely under basic algebraic manipulations.

PROPOSITION 2.2. *Let $f, g : X \longrightarrow \mathbb{R}$ be measurable and let $\alpha \in \mathbb{R}$. Then $f + g$, fg and αf are measurable.*

Proof. Since intervals (a, ∞) generate $\mathcal{B}_{\mathbb{R}}$ it suffices to check that

$$\{f + g > a\} = \{x \in X : f(x) + g(x) > a\}$$

is measurable for $a \in \mathbb{R}$. Certainly, fix $a \in \mathbb{R}$ and observe that by density of $\mathbb{Q}$ in $\mathbb{R}$

$$\begin{aligned} \{f + g > a\} = \{f > a - g\} &= \bigcup_{r \in \mathbb{Q}} \{f > r\} \cap \{r > a - g\} \\ &= \bigcup_{r \in \mathbb{Q}} \{f > r\} \cap \{g > a - r\} \end{aligned}$$

[2]This will not be the case when the domain is $\mathbb{R}$ or $\mathbb{C}$, as we shall see later on.

where $\{f > r\}, \{g > a - r\} \in \mathcal{M}$ for each $r \in \mathbb{Q}$. Thus, $\{f > r\} \cap \{g > a - r\} \in \mathcal{M}$ which proves that $f + g$ is measurable (since $\mathbb{Q}$ is countable and $\mathcal{M}$ is closed under countable unions). Now, if $\alpha = 0$ then $\alpha f \equiv 0$ which is clearly measurable. If instead $\alpha < 0$ then

$$\{\alpha f > a\} = \left\{f < \frac{a}{\alpha}\right\} \in \mathcal{M}$$

and if $\alpha > a$ then

$$\{\alpha f > a\} = \left\{f > \frac{a}{\alpha}\right\} \in \mathcal{M}.$$

We show that f^2 is measurable whenever f is. It suffices to check that $\{f \geq a\}$ is measurable whenever $a \geq 0$. In this case,

$$\{f \geq a\} = \{f \geq a\} \cup \{f \leq -a\}$$

which are both measurable. Thus, to see that fg is measurable we need only write

$$fg = \frac{(f+g)^2 - (f-g)^2}{4}.$$

□

Note that the above remain true whenever f, g are either $\overline{\mathbb{R}}$ valued or complex valued, provided we take care in defining $f(x) - g(x)$ when $f(x) = g(x) = \infty$. It is left as an exercise (see solutions at the end) to verify that $f + g$ is $\overline{\mathbb{R}}$ measurable if we let

$$(f+g)(x) = \begin{cases} 0, & \text{if } f(x) = \infty,\ g(x) = -\infty, \\ f(x) + g(x), & \text{otherwise.} \end{cases}$$

The following properties are incredibly useful in practice.

PROPOSITION 2.3. *Let $\{f_n\}_n$ be an $\overline{\mathbb{R}}$ valued sequence of functions defined on a measurable space $(X, \mathcal{M})$. Define*

$$g_1(x) := \sup_{n \in \mathbb{N}} f_n(x), \quad g_2(x) := \inf_{n \in \mathbb{N}} f_n(x)$$

and

$$g_3(x) := \limsup_{n \to \infty} f_n(x), \quad g_4(x) := \liminf_{n \to \infty} f_n(x).$$

Then $g_{1,2,3,4}$ are $\overline{\mathbb{R}}$-measurable on X.

Proof. We show only that g_1 and g_3 are $\overline{\mathbb{R}}$-measurable; a similar argument applies to g_2 and g_4. It is not hard to see from the definition that $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by rays of the form $(a, \infty]$ or $[-\infty, a)$. Now, for each $a \in \mathbb{R}$

$$\{g_1 > a\} = \bigcup_{n=1}^{\infty} \{f_n > a\}$$

which shows that g_1 is measurable. Note also that

$$g_3(x) = \inf_{k \in \mathbb{N}} \sup_{n \geq k} f_n(x)$$

and so it suffices to check that $g_2(x) = \inf_{n\in\mathbb{N}} f_n(x)$ is measurable to obtain that g_3 is measurable. This is clear from the observation that

$$\{g_2 < a\} = \bigcup_{n=1}^{\infty}\{f_n < a\}.$$

□

COROLLARY 2.4. *Let $\{f_n\}_n$ be a sequence of functions mapping to either $\overline{\mathbb{R}}$ or $\mathbb{C}$. If $f_n(x) \to f(x)$ on X then f is measurable.*

Proof. If f_n and f are $\overline{\mathbb{R}}$-valued then this follows from the fact that

$$\lim_{n\to\infty} f_n(x) = f(x) = \limsup_{n\to\infty} f_n(x).$$

If f_n and f are $\mathbb{C}$-valued then apply the above argument to both $\Re f$ and $\Im f$ since $f_n \to f$ if and only if $\Re f_n \to \Re f$ and $\Im f_n \to \Im f$. □

2.2 Complete Measures and Approximation by Simple Functions

Before we introduce the very important *simple functions*, we would like to characterize measure spaces in terms of their measurable functions. Henceforth, we assume that all functions are either $\mathbb{R}$ or complex valued, unless stated otherwise.

PROPOSITION 2.5. *Throughout this proposition all functions are real valued. Let $(X, \mathcal{M}, \mu)$ be a measure space. The following are equivalent to saying that $(X, \mathcal{M}, \mu)$ is complete:*

1. *If f is measurable and $g = f$ almost everywhere then g is measurable.*
2. *If $\{f_n\}_n$ is a sequence of measurable functions converging almost everywhere to a function f then f is measurable.*

Proof. We prove (1). If $(X, \mathcal{M}, \mu)$ is complete then for all $a \in \mathbb{R}$

$$\{g > a\} = \{g > a\} \cap \{f = g\} \cup \{g > a\} \cap \{g \neq f\}$$

where $\{g > a\} \cap \{g \neq f\}$ is a null set and thus measurable. By closure under complements, $\{g = f\}$ is measurable so that g is also measurable. Conversely, assume (1) holds true. If $N \in \mathcal{M}$ is a null set let $E \subseteq N$; define $f = \mathbb{1}_N(x)$ and $g = \mathbb{1}_E(x)$ so that $f(x) = g(x) = 0$ for $x \notin N$. Thus, $f = g$ almost everywhere. Then g is measurable whence

$$E = (\{g > 1\} \cup \{g < 1\})^{\mathsf{C}} \in \mathcal{M}.$$

Now suppose $(X, \mathcal{M}, \mu)$ is complete and let $\{f_n\}_n$ and f be as in (2). Then

$$f(x) = \limsup_{n\to\infty} f_n(x)$$

and the measurability of f follows by (1). If (2) holds true fix a null set N and let $E \subseteq N$ be given. For $n \in \mathbb{N}$ define

$$f_n(x) = \mathbb{1}_N(x), \quad f(x) := \mathbb{1}_E(x).$$

Then for all $x \notin N$ we have $f_n(x) = f(x) = 0$. Thus, $f_n \to f$ outside N (a null set). By assumption f is measurable which implies (by our previous argument) that $E \in \mathcal{M}$. □

We now turn towards simple functions, which are the "atoms" of integration theory. These will play a fundamental role in the definition of the integral. Given a measure space $(X, \mathcal{M}, \mu)$ a function $f : X \longrightarrow \mathbb{C}$ is called a **simple function** provided it is of the form

$$f(x) = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{E_j}(x), \quad \alpha_j \in \mathbb{C},\ E_j \in \mathcal{M}.$$

In particular, simple functions cannot take on infinite values. It is trivial to show that simple functions are measurable. If the α_j's are distinct and the E_j's are disjoint we say f is in **canonical form**. Since simple functions have finite image it is clear that all simple functions have a unique canonical form. Indeed, if f is simple then $\mathrm{Im}(f) = \{\zeta_1, \zeta_2, \ldots, \zeta_k\}$ is finite; then

$$\sum_{j=1}^{k} \zeta_j f^{-1}(\{\zeta_j\})$$

is the canonical form of f (as is easy to show). What is fundamental to the theory of measurable functions it the approximation theorem that we give below.

THEOREM 2.6. *Let $(X, \mathcal{M})$ be a measurable space. If $f : X \longrightarrow [0, \infty]$ is measurable, there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ of simple functions so that*

$$0 \le \varphi_1 \le \varphi_2 \le \cdots \le \varphi_n \le \cdots \le f$$

such that $\lim_{n\to\infty} \varphi_n(x) = f(x)$.

We shall not prove this theorem since the proof is extremely technical and not particularly enlightening. Furthermore, almost every graduate real analysis text will give the proof and it is therefore easy for the interested reader to look up a proof.

We are now capable of defining the integral of a non-negative function, which is the topic of the following subsection.

2.3 Integration of Non-Negative Measurable Functions

Let $(X, \mathcal{M}, \mu)$ be a measure space, which we fix for the remainder of this subsection. We denote by $L^+(X)$ the collection all measurable functions $f : X \longrightarrow [0, \infty]$. This set contains non-negative, real valued, simple functions as well.

DEFINITION 6. Let $f \in L^+(X)$ be a simple function and write $f(x) = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{E_j}(x)$ in its (unique) canonical form. The integral of f over X is defined as

$$\int_X f(x)\, \mathrm{d}\mu := \sum_{j=1}^{n} \alpha_j \mu(E_j).$$

If $A \in \mathcal{M}$ then one can define

$$\int_A f(x)\, \mathrm{d}\mu := \int_X \mathbb{1}_A(x) f(x)\, \mathrm{d}\mu = \sum_j \alpha_j \mu(E_j \cap A).$$

We shall sometimes write $\int_X f$ instead of $\int_X f(x)\,\mathrm{d}\mu$ when the context is clear. This is well defined since $\alpha_j \geq 0$ for all j (f is not allowed to be negative).One should not lose track of the purpose of this definition. Despite the abstractness of our work, it is still our ultimate goal to *generalize* the classical Riemann integral and, as such, there are certain properties we should verify.

THEOREM 2.7. *Let $\phi, \psi \in L^1(X)$ be simple functions. The following hold true*

1. *For all $c \geq 0$ one has $\int_X c\phi = c\int_X \phi$.*
2. *$\int_X(\phi+\psi) = \int_X \phi + \int_X \psi$.*
3. *If $\phi \leq \psi$ then $\int_X \phi \leq \int_X \psi$.*
4. *The map*
$$A \mapsto \int_A \phi(x)\,\mathrm{d}\mu \tag{3}$$
is a measure on $(X, \mathcal{M})$.

Proof. (1) is clear from the definition. Write ϕ and ψ in their canonical form

$$\phi(x) = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{E_j}, \quad \psi(x) = \sum_{k=1}^{m} \beta_j \mathbb{1}_{F_j}(x).$$

Note that $X = \bigcup_{j=1}^n E_j = \bigcup_{k=1}^m F_k$. Furthermore,

$$\begin{aligned}
\int_X \phi(x)\,\mathrm{d}\mu + \int_X \psi(x)\,\mathrm{d}\mu &= \sum_{j=1}^{n} \alpha_j \mu(E_j) + \sum_{k=1}^{m} \beta_k \mu(F_k) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{m} \alpha_j \mu(E_j \cap F_k) + \sum_{k=1}^{m}\sum_{j=1}^{n} \beta_k \mu(E_j \cap F_k) \\
&= \sum_{j=1,k=1}^{n,m} (\alpha_j + \beta_k)\mu(E_j \cap F_k)
\end{aligned}$$

where all the $E_j \cap F_k$ are disjoint. By grouping these based on whether or not the $(\alpha_j + \beta_k)$ are repeated, one finds that this is precisely $\int_X(\phi+\psi)$.

Now suppose that $\phi \leq \psi$. We have

$$\int_X \phi(x)\,\mathrm{d}\mu = \sum_{j=1}^{n} \alpha_j \mu(E_j) = \sum_{j=1}^{n}\sum_{k=1}^{m} \alpha_j \mu(E_j \cap F_k).$$

If $E_j \cap F_k \neq \varnothing$ then $\alpha_j \leq \beta_k$ so that

$$\int_X \phi(x)\,\mathrm{d}\mu \leq \sum_{j=1}^{n}\sum_{k=1}^{m} \beta_k \mu(E_j \cap F_k) = \sum_{k=1}^{m}\sum_{j=1}^{n} \beta_k \mu(E_j \cap F_k)$$

where this last line is precisely $\sum_{k=1}^{m} \beta_j \mu(F_k) = \int_X \psi$. It only remains to check that the mapping defined by (3) is a measure. Clearly, $\varnothing \mapsto 0$ as required. If $\{A_\ell\}_\ell$ is a countable family of disjoint sets in $\mathcal{M}$ we let $A := \bigsqcup_\ell A_\ell$ and observe that

$$A \mapsto \sum_{j=1}^{n} \alpha_j \mu(E_j \cap A) = \sum_{j=1}^{n} \sum_{\ell} \alpha_j \mu(E_j \cap A_\ell) = \sum_{\ell} \sum_{j=1}^{n} \alpha_j \mu(E_j \cap A_\ell)$$
$$= \sum_{\ell} \int_{A_\ell} \phi(x) \, \mathrm{d}\mu.$$

The proof is now complete. □

Keeping these properties in mind, we extend our integral to all of $L^+(X)$. This is done by "approximating" $f \in L^+(X)$ with simple functions (we know this can be done by our approximation theorem).

DEFINITION 7. Let $f : X \longrightarrow [0, \infty]$ be measurable. Then we define

$$\int_X f(x) \, \mathrm{d}\mu := \sup \left\{ \int_X \phi(x) \, \mathrm{d}\mu : 0 \leq \phi \leq f, \, \phi \text{ simple} \right\}.$$

If $A \in \mathcal{M}$ then

$$\int_A f(x) \, \mathrm{d}\mu := \int_X \mathbb{1}_A(x) f(x) \, \mathrm{d}\mu.$$

We verify once again that some fundamental properties hold true.

PROPOSITION 2.8. *Let $f, g \in L^1(X)$ and $c \geq 0$. Then $\int_X cf = c \int_X f$ and $\int_X f \leq \int_X g$ whenever $f \leq g$.*

Proof. If $c = 0$ then the claim is obvious (since $c \cdot f$ is simple). Otherwise, note that

$$\int_X cf(x) \, \mathrm{d}\mu = \sup \left\{ \int_X \phi(x) \, \mathrm{d}\mu : 0 \leq \phi \leq cf, \, \phi \text{ simple} \right\}$$
$$= \sup \left\{ \int_X c\phi(x) \, \mathrm{d}\mu : 0 \leq \phi \leq f, \, \phi \text{ simple} \right\}$$
$$= c \sup \left\{ \int_X \phi(x) \, \mathrm{d}\mu : 0 \leq \phi \leq f, \, \phi \text{ simple} \right\}$$
$$= c \int_X f(x) \, \mathrm{d}x.$$

Assume $f \leq g$ and let ϕ be a simple function such that $0 \leq \phi \leq f$. Since $f \leq g$ we obtain $\phi \leq g$ so that $\int_X \phi \leq \int_X g$ by definition of $\int_X g$. Since ϕ was arbitrary, it follows that $\int_X f \leq \int_X g$. □

Having established these essential properties of the integral on $L^+(X)$ we move on to study how the integral behaves under limiting procedures.

2.4 Convergence Theorems

A useful tool is proving that the integral on $L^+(X)$ is additive is the so-called *monotone convergence theorem*, which is also powerful in applications.

THEOREM 2.9 (Monotone Convergence). *Let $\{f_n\}_n$ be a sequence in $L^+(X)$ converging pointwise to a function f such that*

$$f_1 \le f_2 \le \cdots \le f_n \le \cdots \le f.$$

Then f is measurable and

$$\lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu = \int_X f(x)\,\mathrm{d}\mu.$$

Proof. Since $\int_X f_n$ is an increasing sequence of real numbers we know that it has a limit as $n \to \infty$ (possibly ∞). Noting that $f_n \le f$ implies $\int_X f_n \le \int_X f$ we obtain

$$\lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu \le \int_X f(x)\,\mathrm{d}\mu.$$

To prove the reverse inequality we let ϕ be a simple function with $0 \le \phi \le f$ and fix $\alpha \in (0,1)$. For $n \in \mathbb{N}$ we define

$$E_n := \{x \in X : f_n > \alpha\phi\}$$

which is measurable. Note that $X = \bigcup_{n=1}^{\infty} E_n$ for otherwise there would exist $x \notin E_n$ for all n which would imply that for some x $f_n(x) \le \alpha\phi(x)$. Letting $n \to \infty$ would yield

$$f(x) \le \alpha\phi(x) < f(x)$$

which is absurd. Thus, $X = \bigcup_{n=1}^{\infty} E_n$. Therefore, for each $n \in \mathbb{N}$

$$\int_X f_n(x)\,\mathrm{d}\mu \ge \int_{E_n} f_n(x)\,\mathrm{d}\mu \ge \alpha \int_{E_n} \phi(x)\,\mathrm{d}\mu.$$

Since $E_n \subseteq E_{n+1}$ for all n it follows that $E_n \nearrow X$. Thus, by Theorem 2.7 we are free to take $n \to \infty$ from which we obtain

$$\lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu \ge \alpha \int_X \phi(x)\,\mathrm{d}\mu.$$

We may now let $\alpha \nearrow 1$ to obtain

$$\lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu \ge \int_X \phi(x)\,\mathrm{d}\mu.$$

Taking the supremum over ϕ yields

$$\lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu \ge \int_X f(x)\,\mathrm{d}\mu$$

as was required. ∎

An important consequence of monotone convergence is the corollary that follows.

COROLLARY 2.10. *Let $\{f_n\}_n$ be a countable sequence in $L^+(X)$ and define $f(x) = \sum_n f_n(x)$. Then $f \in L^+(X)$ and*

$$\int_X f(x)\,\mathrm{d}\mu = \sum_n \int_X f_n(x)\,\mathrm{d}\mu.$$

Proof. We know already that f is measurable, we must only show that the integral is countably additive over this family. We first prove it for finite sums, and for this is suffices to check that $\int_X (f_1 + f_2) = \int_X f_1 + \int_X f_2$.

Let $\{\phi_n\}_n$, $\{\psi_n\}_n$ be sequences of simple functions in $L^+(X)$ such that $\phi_n \nearrow f_1$ and $\psi_n \nearrow f_2$ pointwise on X. Then, $(\phi_n + \psi_n) \nearrow (f_1 + f_2)$ so that, by monotone convergence,

$$\begin{aligned}\int_X (f_1(x) + f_2(x))\,\mathrm{d}\mu &= \lim_{n\to\infty} \int_X (\phi_n(x) + \psi_n(x))\,\mathrm{d}\mu \\ &= \lim_{n\to\infty} \int_X \phi_n(x)\,\mathrm{d}\mu + \lim_{n\to\infty} \int_X \psi_n(x)\,\mathrm{d}\mu \\ &= \int_X f_1(x)\,\mathrm{d}\mu + \int_X f_2(x)\,\mathrm{d}\mu.\end{aligned}$$

Now, if $\{f_n\}_n$ is a finite family we are done. Thus we may assume that it is countably infinite. For $N \in \mathbb{N}$ we now define $g_N(X) := \sum_{n=1}^N f_n(x)$ and note that $g_N \nearrow f$ as $N \to \infty$. By monotone convergence,

$$\begin{aligned}\int_X \sum_n f_n(x)\,\mathrm{d}\mu = \lim_{N\to\infty} \int_X \sum_{n\le N} f_n(x)\,\mathrm{d}\mu &= \lim_{N\to\infty} \sum_{n\le N} \int_X f_n(x)\,\mathrm{d}\mu \\ &= \sum_n \int_X f_n(x)\,\mathrm{d}\mu.\end{aligned}$$

□

LEMMA 2.11. *Let f be measurable and non-negative almost everywhere. Then $\int_X f = 0$ if and only if $f = 0$ almost everywhere.*

Proof. Assume first $f = 0$ almost everywhere and let $\{\phi_n\}_n$ be a sequence of non-negative simple functions increasing to f pointwise. Then ϕ_n vanishes almost everywhere so that by monotone convergence

$$\int_X f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_X \phi_n(x)\,\mathrm{d}\mu = 0.$$

Conversely, suppose $\int_X f = 0$. For $n \in \mathbb{N}$ we define

$$E_n := \left\{x \in X : f(x) > \frac{1}{n}\right\}$$

and observe that $\bigcup_{n=1}^\infty E_n = \{f > 0\} = \{f \neq 0\}$. However,

$$0 = \int_X f(x)\,\mathrm{d}\mu \ge \int_{E_n} f(x)\,\mathrm{d}\mu > \frac{1}{n}\mu(E_n)$$

which shows $\mu(E_n) = 0$ for all n. Thus, $\mu(\{f \neq 0\}) \le \sum_{n=1}^\infty \mu(E_n) = 0$. □

We may now present the stronger form of monotone convergence.

COROLLARY 2.12. *Let $\{f_n\}_n$ be an increasing sequence in $L^+(X)$ converging pointwise almost everywhere to a measurable function f. Then*

$$\lim_{n\to\infty}\int_X f_n(x)\,\mathrm{d}\mu = \int_X f(x)\,\mathrm{d}\mu.$$

Proof. There exists a set $E \in \mathcal{M}$ such that $\mu(E^{\mathsf{C}}) = 0$ and $f_n(x) \nearrow f(x)$ for $x \in E$. Since $(f - \mathbb{1}_E f) \geq 0$ and equal to zero almost everywhere

$$\int_X f(x)\,\mathrm{d}\mu = \int_X \mathbb{1}_E(x) f(x)\,\mathrm{d}\mu = \lim_{n\to\infty}\int_X \mathbb{1}_E(x) f_n(x)\,\mathrm{d}\mu = \lim_{n\to\infty}\int_X f_n(x)\,\mathrm{d}\mu$$

by the previous lemma. □

2.5 The Integration of Complex Valued Functions

Here we finish the construction of the integral. We focus first on $\mathbb{R}$-valued measurable functions on a measure space $(X, \mathcal{M}, \mu)$. If $f : X \longrightarrow \mathbb{R}$ is measurable (in the sense of $\mathcal{B}_{\mathbb{R}}$) we define

$$f^+(x) := \max\{f(x), 0\} \quad \text{and} \quad f^-(x) := \max(0, -f(x)).$$

Then $f^{\pm}$ are measurable and $f = f^+ - f^-$. Furthermore, $f^+ + f^- = |f|$. We say that f is **integrable** provided

$$\int_X |f(x)|\,\mathrm{d}\mu < \infty.$$

This is equivalent to saying that f^+ and f^- are both integrable. In this case, we define

$$\int_X f(x)\,\mathrm{d}\mu := \int_X f^+(x)\,\mathrm{d}\mu - \int_X f^-(x)\,\mathrm{d}\mu$$

which is a well defined real number. If f is $\mathbb{C}$-valued, we say f is integrable if and only if $\Re f$ and $\Im f$ are integrable[3] and define

$$\int_X f(x)\,\mathrm{d}\mu = \int_X \Re f(x)\,\mathrm{d}\mu + i\int_X \Im f(x)\,\mathrm{d}\mu.$$

It is left to the reader to check that the integral is linear over integrable functions. This is a purely mechanical and messy procedure.

DEFINITION 8. If $\xi \in \mathbb{C}$ we define

$$\mathsf{Sgn}(\xi) := \begin{cases} 0, & \text{if } \xi = 0, \\ \xi/|\xi|, & \text{if } x \neq 0. \end{cases}$$

We note that $\xi\overline{\mathsf{Sgn}(\xi)} = |\xi|$ if $\xi \neq 0$.

[3] This is again equivalent to saying that $|f|$ integrable!

PROPOSITION 2.13. *Let f be integrable. Then*

$$\left|\int_X f(x)\,\mathrm{d}\mu\right| \le \int_X |f(x)|\,\mathrm{d}\mu.$$

Proof. If $f \ge 0$ the result is immediate. If f is real valued for all $x \in X$ then this follows from the fact that

$$\left|\int_X f(x)\,\mathrm{d}\mu\right| = \left|\int_X f^+(x)\,\mathrm{d}\mu - \int_X f^-(x)\,\mathrm{d}\mu\right| \le \int_X |f(x)|\,\mathrm{d}\mu$$

where we have applied the identity to the non-negative functions $f^{\pm}$. For the general case we assume f is complex valued. If $\int_X f(x)\,\mathrm{d}\mu = 0$ we are done. Otherwise, define

$$\alpha := \overline{\mathsf{Sgn}\left(\int_X f(x)\,\mathrm{d}x\right)}.$$

By our previous observations,

$$\left|\int_X f(x)\,\mathrm{d}\mu\right| = \alpha\int_X f(x)\,\mathrm{d}\mu = \int_X \alpha f(x)\,\mathrm{d}\mu.$$

In particular, $\alpha\int_X f(x)\,\mathrm{d}\mu = \int_X \alpha f(x)\,\mathrm{d}\mu$ is real and non-negative. By our previous steps,

$$\left|\int_X f(x)\,\mathrm{d}\mu\right| = \Re\int_X \alpha f(x)\,\mathrm{d}\mu = \int_X \Re\left(\alpha f(x)\right)\,\mathrm{d}\mu \le \int_X |\alpha f(x)|\,\mathrm{d}\mu.$$

where $|\alpha| = 1$ by definition. Hence, the proof is complete. □

PROPOSITION 2.14. *Let f and g be integrable. Then $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ if and only if $\int_X |f-g| = 0$.*

Proof. One direction is trivial. Assume that $\int_E (f-g) = 0$ for all measurable sets E. Without loss of generality assume $f-g$ is real. If $(f-g) \neq 0$ almost everywhere then there exists a set of positive measure, say, E where $(f-g) \neq 0$. Without harm assume $u = (f-g)^+$ is non-vanishing on E. Then,

$$\int_E (f(x) - g(x))\,\mathrm{d}\mu = \int_E u(x)\,\mathrm{d}\mu > 0.$$

This is a contradiction. □

We conclude this section with two major results in analysis, Fatou's lemma and the Dominated Convergence Theorem. There are some who go so far as to call the latter the "fundamental theorem of analysis". It is the most powerful result in the theory of abstract integration.

LEMMA 2.15 (Fatou's Lemma). *Let $\{f_n\}_n$ be a sequence of functions in $L^+(X)$. Then,*

$$\int_X \liminf_{n\to\infty} f_n(x)\,\mathrm{d}\mu \le \liminf_{n\to\infty}\int_X f_n(x)\,\mathrm{d}\mu.$$

Proof. Let $g(x) = \liminf_{n\to\infty} f_n(x)$ and note that

$$g_k(x) = \inf_{n\geq k} f_n(x) \nearrow g(x), \quad \text{as } k \to \infty.$$

By monotone convergence,

$$\int_X g(x)\,\mathrm{d}\mu = \lim_{k\to\infty} \int_X g_k(x)\,\mathrm{d}\mu.$$

Now, $g_k(x) \leq f_j(x)$ for all $j \geq k$. Thus, for each $j \geq k$:

$$\int_X g_k(x)\,\mathrm{d}\mu \leq \int_X f_j(x)\,\mathrm{d}\mu$$

whence

$$\int_X g_k(x)\,\mathrm{d}\mu \leq \inf_{j\geq k} \int_X f_j(x)\,\mathrm{d}\mu.$$

Letting $k \to \infty$ we obtain

$$\int_X g(x)\,\mathrm{d}\mu = \lim_{k\to\infty} \int_X g_k(x)\,\mathrm{d}\mu \leq \liminf_{j\to\infty} \int_X f_j(x)\,\mathrm{d}\mu.$$

□

Fatou's lemma may be strengthened as follows.

COROLLARY 2.16. *Let $\{f_n\}_n$ be a sequence in $L^+(X)$ converging almost everywhere to $f \in L^+(X)$. Then*

$$\int_X f(x)\,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu.$$

Dominated convergence now readily follows.

THEOREM 2.17 (Dominated Convergence). *Let $\{f_n\}_n$ be a sequence of measurable functions converging almost everywhere to a measurable function f. If there exists an integrable function g such that $|f_n| \leq g$ for all n then*

$$\lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu = \int_X f(x)\,\mathrm{d}\mu.$$

Proof. We assume first that f_n and f are real. By replacing f with $\liminf_{n\to\infty} f_n(x)$ we may assume that $f_n \to f$ everywhere on X. Now, note that $g+f \geq 0$ and $g-f \geq 0$ everywhere. By Fatou's lemma;

$$\begin{aligned}\int_X (g(x) + f(x))\,\mathrm{d}\mu &\leq \liminf_{n\to\infty} \int_X (g(x) + f_n(x))\,\mathrm{d}\mu \\ &= \int_X g(x)\,\mathrm{d}\mu + \liminf_{n\to\infty} f_n(x)\,\mathrm{d}\mu.\end{aligned}$$

Similarly,

$$\int_X (g(x) - f(x))\,\mathrm{d}\mu \leq \int_X g(x) - \limsup_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu.$$

We obtain

$$\limsup_{n\to\infty} f_n(x)\,\mathrm{d}\mu \leq \int_X f(x)\,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu$$

which concludes this step. If instead f_n and f are complex valued we apply the procedure above to both $\Re f_n$ and $\Im f_n$. □

The final result of this section is the following.

THEOREM 2.18. *Let $\{f_n\}_n$ be a sequence of integrable functions such that*

$$\sum_n \int_X |f_n(x)| \, d\mu < \infty.$$

Then there exists an integrable function f equal to $\sum_n f_n(x)$ almost everywhere and moreover,

$$\int_X f(x) \, d\mu = \sum_n \int_X f_n(x) \, d\mu.$$

Proof. We define

$$f(x) := \liminf_{N \to \infty} \sum_{n \leq N} f_n(x);$$

then f is measurable and equal to $\sum_n f_n(x)$ almost everywhere (this series is also convergent at these points). Note that

$$\left| \sum_{n \leq N} f_n(x) \right| \leq \sum_{n=1}^{\infty} |f_n(x)|$$

which we know to be integrable. Thus, the equality of integrals follows by dominated convergence. □

2.6 Modes of Convergence

The previous topics give rise to different notions of what is means for a sequence of functions to converge. Indeed, one can view the integral as a metric on sequences. If $\{f_n\}_n$ is a sequence of integrable functions and f is integrable, we say that f_n converges to f in L^1 (this notation will be justified later) if

$$\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0.$$

We say measurable a sequence functions $\{f_n\}_n$ converges to a measurable function f in measure if for all $\varepsilon > 0$

$$\lim_{n \to \infty} \mu\left(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}\right) = 0.$$

Finally, a sequence of measurable functions $\{f_n\}_n$ is called Cauchy in measure if for each $\varepsilon > 0$

$$\mu\left(\{x \in X : |f_n(x) - f_m(x)| \geq \varepsilon\}\right) \xrightarrow{n,m \to \infty} 0.$$

PROPOSITION 2.19. *Let $\{f_n\}_n$ be a sequence of integrable functions converging to an integrable f in L^1. Then, $\{f_n\} \to f$ in measure. Moreover, $\{f_n\}_n$ is Cauchy in measure.*

Proof. We fix $\varepsilon > 0$, note that

$$\begin{aligned}
\lim_{n \to \infty} \int_X |f_n(x) - f(x)| \, d\mu &\geq \lim_{n \to \infty} \int_{\{x : |f_n(x) - f(x)| \geq \varepsilon\}} |f_n(x) - f(x)| \, d\mu \\
&\geq \varepsilon \lim_{n \to \infty} \mu\left(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}\right) = 0.
\end{aligned}$$

For this same $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$

$$\int_X |f_n(x) - f(x)| \, \mathrm{d}\mu < \frac{\varepsilon}{2}.$$

Thus, if $n, m \geq N$

$$\int_X |f_n(x) - f_m(x)| \, \mathrm{d}\mu \leq \int_X |f_n(x) - f(x)| \, \mathrm{d}\mu + \int_X |f_m(x) - f(x)| \, \mathrm{d}\mu$$

whence $\int_X |f_n - f_m| < \varepsilon$ for all $n, m \geq N$. So, if $\varepsilon > 0$ is given, for all n, m one has

$$\begin{aligned}\int_X |f_n(x) - f_m(x)| \, \mathrm{d}\mu &\geq \int_{\{x \in X : |f_n(x) - f_m(x)| \geq \varepsilon\}} |f_n - f_m| \, \mathrm{d}\mu \\ &\geq \varepsilon \mu \left(\{x \in X : |f_n(x) - f_m(x)| \geq \varepsilon\} \right).\end{aligned}$$

Letting $n, m \to \infty$ we can make the right-hand side arbitrarily small which yields the desired conclusion. □

It is well known that pointwise convergence is weak, and certainly does not imply uniform convergence. However, the following result does tell us that, on finite measure spaces, pointwise convergence implies "mostly uniform" convergence. We give a precise meaning to this statement below.

THEOREM 2.20 (Egoroff). *Let $(X, \mathcal{M}, \mu)$ be a finite*[4] *measure space and $\{f_n\}_n$ a sequence of measurable functions converging pointwise almost-everywhere to a measurable function f. For each $\varepsilon > 0$ there exists a measurable set $E \subseteq X$ with $mu(E) < \varepsilon$ such that $f_n \to f$ uniformly on E^{C}.*

Proof. Let $\varepsilon > 0$ be given; we may assume without loss of generality that $f_n \to f$ pointwise on X (neglecting a set of measure zero otherwise). Fix a pair $(k, n) \in \mathbb{N} \times \mathbb{N}$ and define

$$E_k(n) := \bigcup_{j \geq n} \left\{ x \in X : |f_j(x) - f(x)| \geq \frac{1}{k} \right\}.$$

For fixed k, we note that $E_k(n) \supseteq E_k(n+1)$ and that $E_k(n) \nearrow \varnothing$ as $n \to \infty$ since $f_n \to f$ on X. Since $\mu(X) < \infty$, for each $k \in \mathbb{N}$ there is n_k such that

$$\mu(E_k(n_k)) < \frac{\varepsilon}{2^k};$$

this is immediate from the fact that $0 = \lim_{n \to \infty} \mu(E_k(n))$ for each k. Now, we define

$$E = \bigcup_{k=1}^{\infty} E_k(n_k)$$

whence by sub-additivity we obtain $\mu(E) < \sum_{k=1}^{\infty} \varepsilon / 2^k = \varepsilon$. Let $\gamma > 0$ and take $k \in \mathbb{N}$ such that $1/k < \gamma$. If $x \notin E$ then $x \notin E_k(n_k)$ so that

$$|f_j(x) - f(x)| < \gamma, \quad \forall j \geq n_k.$$

Thus, $f_n \to f$ uniformly on E^{C}. □

[4] Recall that this means $\mu(X) < \infty$.

We conclude this section with the following technical result.

THEOREM 2.21. *Let $\{f_n\}_n$ be a sequence of measurable functions that is Cauchy in measure. There exists a measurable function f and a subsequence $\{f_{n_k}\}_k$ converging to f almost everywhere. Furthermore, $\{f_n\}_n \to f$ in measure.*

Proof. Since $\{f_n\}_n$ is Cauchy in measure we may extract a subsequence $\{g_j\}_j$ such that

$$\mu(E_j) \leq 2^{-j}$$

where we define

$$E_j := \left\{x \in X : |g_j(x) - g_{j+1}(x)| \geq 2^{-j}\right\}.$$

For $k \in \mathbb{N}$ we define $F_k = \bigcup_{j \geq k} E_j$ and note that $\mu(F_k) \leq \sum_{j \geq k} 2^{-j} = 2^{1-k}$. Now, if $x \notin F_k$ for $k \in \mathbb{N}$ note that for $i \geq j \geq k$:

$$|g_i(x) - g_j(x)| \leq \sum_{\ell=j}^{i-1} |g_{\ell+1}(x) - g_\ell(x)| \leq 2^{1-j}. \tag{4}$$

It follows that $\{g_j\}_j$ is pointwise Cauchy in $\mathbb{C}$ for $x \notin F_k$. We define $F = \bigcap_{k \in \mathbb{N}} F_k$ so that $\mu(F) \leq 2^{1-k}$ for all k. That is, $\mu(F) = 0$. If $x \notin F$ then $\{g_j\}_j$ is Cauchy in $\mathbb{C}$ and thus has a limit. If we define

$$g(x) = \liminf_{j \to \infty} g_j(x)$$

then g is measurable and convergent for almost all x. We shall now show that $g_j \to g$ in measure. Note that by letting $i \to \infty$ in (4) we obtain

$$|g(x) - g_j(x)| \leq 2^{1-j}, \quad x \notin F_k.$$

Let $\varepsilon > 0$ be given; if $k \in \mathbb{N}$ is fixed and $j \gg k$ we have

$$\{x : |g(x) - g_j(x)| \geq \varepsilon\} \subseteq \left\{x : |g(x) - g_j(x)| > 2^{1-j}\right\}$$

where this latter set is contained in F_k for all $j \gg k$. Hence,

$$\limsup_{j \to \infty} \mu\left(\{x : |g(x) - g_j(x)| \geq \varepsilon\}\right) \leq \mu(F_k) \leq 2^{1-k}.$$

Since k was arbitrary, we conclude that $g_j \to g$ in measure. It remains only to prove that $f_n \to g$ in measure as well. For this, we note that

$$\{x : |f_n(x) - g(x)| \geq \varepsilon\} \subseteq \left\{|f_n(x) - g_j(x)| \geq \frac{\varepsilon}{2}\right\} \cup \left\{|g_j(x) - g(x)| \geq \frac{\varepsilon}{2}\right\}.$$

By letting n and j be large, both these sets can be made arbitrarily small in measure. This concludes the proof. □

3 The Fubini-Tonelli Theorems

In practice it is often convenient to "interchange the order of integration". For those well grounded in calculus, this is the procedure that allows one to say

$$\int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}A = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x \right)\,\mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y \right)\,\mathrm{d}x.$$

Under which conditions is the above true? This is precisely the question that the Fubini-Tonelli theorems answer. In this section, we establish sufficient conditions for the above to hold on arbitrary measure spaces.

3.1 $L^1(X)$ Space

The time has come to give an algebraic structure to the set of all integrable functions, or almost. We know that the collection of all integrable functions over a measure space $(X, \mathcal{M}, \mu)$ is a vector space over $\mathbb{C}$. We wish to equip this space with a suitable norm. A suitable choice is

$$\|f\|_{L^1(X)} := \int_X |f(x)| \, d\mu.$$

Indeed, it satisfies all the properties of a norm except for $\|f\|_{L^1(X)} = 0$ if and only if $f = 0$. Here this "norm" vanishes whenever $f = 0$ almost everywhere. We reconcile this by introducing an equivalence relation on the collection of integrable functions.

If f and g are two integrable functions, we write $f \sim g$ if $(f - g) = 0$ almost everywhere. It is easy to check that this is a well defined equivalence relation on the space of integrable functions. We then define $L^1(X)$ to be the space of equivalence classes under this relation. If $[f], [g] \in L^1(X)$ and $\alpha, \beta \in \mathbb{C}$ we define

$$\alpha[f] + \beta[g] := [\alpha f + \beta g].$$

We know from earlier results that this is well defined. Hence, $L^1(X)$ is a vector space over $\mathbb{C}$. Given $[f] \in L^1(X)$ we define the L^1-norm.

$$\|[f]\|_{L^1(X)} := \int_X |f(x)| \, d\mu.$$

It is left as an exercise to the reader to check that $L^1(X)$ together with $\|\cdot\|_{L^1(X)}$ is a normed vector space. Typically, it is not usually practical to view $L^1(X)$ as a space of equivalence classes; more often than not we shall think of it as a function space whilst keeping in mind that equality in the norm holds up to measure zero.

3.1.1 Applications of Convergence Theorems

Let m be the Lebesgue measure on $\mathbb{R}$ and denote by $\mathcal{L}$ the σ-algebra of Lebesgue measurable sets. In these notes it is proven that continuous functions of compact support are dense in $L^1(\mathbb{R})$ (with respect to the L^1-norm). Using the previous facts, it is possible to show that measurable functions are, in a sense, mostly continuous.

LEMMA 3.1. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $f : X \longrightarrow \mathbb{C}$ measurable. For each $\varepsilon > 0$, there exists a subset $E \subseteq X$, with $\mu(E^{\mathsf{C}}) < \varepsilon$, upon which f is bounded.*

Proof. For fixed $n \in \mathbb{N}$ we define

$$F_n := \{x \in X : |f(x)| \leq n\}.$$

Note that $F_n \subseteq F_{n+1}$ for all n and that $F_n \nearrow X$ as $n \to \infty$. Thus, we know that $\lim_{n\to\infty} \mu(F_n) = \mu(X) < \infty$. Choose N so large that

$$0 \leq \mu(X) - \mu(F_N) < \varepsilon.$$

Then, if we set $E = F_N$ it follows that $\mu(E^{\mathsf{C}}) = \mu(X \setminus F_N) = \mu(X) - \mu(F_N) < \varepsilon$. □

We are now prepared to prove Lusin's theorem, which makes heavy use of Egoroff's theorem.

THEOREM 3.2 (Lusin). *Let μ be the restriction of the Lebesgue measure to $[a,b]$ and fix a measurable function $f : X \longrightarrow \mathbb{C}$. For each $\varepsilon > 0$, there exists a measurable $E \subseteq [a,b]$ with $\mu(E^{\mathsf{C}}) < \varepsilon$ such that $f\mid_E$ is continuous.*

Proof. Since f is complex valued, it does not take infinite values. Hence, by the previous lemma, there exists a set $A \subseteq X$, with $\mu(A^{\mathsf{C}}) < \varepsilon$, upon which f is bounded. Defining $g(x) = f(x)\mathbb{1}_A(x)$, it follows that $g \in L^1([a,b])$ so that there exists a sequence $\{\phi_n\}_n$ of compactly supported continuous functions that convergence to g in the L^1-norm. By passing to a suitable subsequence, we may assume these converge almost everywhere to g on $[a,b]$.

By Egorov's theorem, there exists $F \subseteq [a,b]$ with $\mu(F) < \varepsilon$ such that $\phi_n \to g$ uniformly only F^{C}. Choose now a compact set $E \subseteq F^{\mathsf{C}} \cap A$ with

$$\mu((F^{\mathsf{C}} \cap A) \setminus E) < \varepsilon$$

Since E is compact, it follows that g is continuous (as the uniform limit of continuous functions) when restricted to E. Thus, $f\mid_E$ is continuous since $E \subseteq A$. It now remains only to note that

$$\begin{aligned}
\mu(E^{\mathsf{C}}) = \mu(X) - \mu(E) &= \mu(X) - \mu(F^{\mathsf{C}} \cap A) + \mu((F^{\mathsf{C}} \cap A) \setminus E) \\
&< \varepsilon + \mu(X) - \mu(F^{\mathsf{C}} \cap A) \\
&= \varepsilon + \mu(F \cup A^{\mathsf{C}}) \\
&\leq \varepsilon + \mu(F) + \mu(A^{\mathsf{C}}) \\
&\leq 3\varepsilon.
\end{aligned}$$

□

3.2 Product Measure Spaces

We fix two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. We want to know what it means to integrate over $X \times Y$? How can this be defined in a way that preserves the structures of both $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$?

DEFINITION 9. A rectangle in $X \times Y$ is a set of the form $E \times F$ where $E \in \mathcal{M}$ and $F \in \mathcal{N}$. We denote the collection of all rectangles in $X \times Y$ be $\mathfrak{R}$.

We wish to show that $\mathfrak{R}$ is an elementary family on $X \times Y$. Indeed, $\varnothing \in \mathfrak{R}$. If $A_1 = E_1 \times F_1$ and $A_2 = E_2 \times F_2$ are rectangles in $X \times Y$, note that

$$A_1 \cap A_2 = (E_1 \cap E_2) \times (F_1 \cap F_2) \quad \text{and} \quad A_1^{\mathsf{C}} = (X \times F_1^{\mathsf{C}}) \sqcup (E_1^{\mathsf{C}} \times F_1)$$

Therefore, by Proposition 1.1, if $\mathcal{A}$ is the collection of finite disjoint unions of elements in $\mathfrak{R}$, then $\mathcal{A}$ is an algebra on $X \times Y$. Clearly, $\mathcal{A} \supseteq \mathfrak{R}$. We denote by $\mathcal{M} \otimes \mathcal{N}$ the σ-algebra generated by $\mathcal{A}$. The pair $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ is then called the product space of $(X, \mathcal{M})$ and $(Y, \mathcal{N})$.

We now wish to extend our measures μ and ν to this new space. This is done by constructing a premeasure on $\mathcal{A}$, which will then induce an outer-measure on $X \times Y$. Define

$$\mu \times \nu : \mathcal{A} \longrightarrow [0, \infty], \quad E \times F \mapsto \mu(E)\nu(F).$$

Suppose that $E \times F$ can be expressed as the disjoint union of rectangles $\bigcup_{j=1}^{n}(E_j \times F_j)$. Notice that

$$\mathbb{1}_E(x)\mathbb{1}_F(y) = \mathbb{1}_{E\times F}(x,y) = \sum_{j=1}^{n} \mathbb{1}_{E_j \times F_j}(x,y) = \sum_{j=1}^{n} \mathbb{1}_{E_j}(x)\mathbb{1}_{F_j}(y).$$

Integrating with x fixed gives

$$\mathbb{1}_E(x)\nu(F) = \sum_{j=1}^{n} \mathbb{1}_{E_j}(x)\nu(F_j)$$

as well as $\mu(E)\nu(F) = \sum_{j=1}^{n} \mu(E_j)\nu(F_j)$. This shows that $\mu \times \nu$ is well defined on rectangles. This argument extends easily to show that $\mu \times \nu$ is well defined on any element $A \in \mathcal{A}$. In particular, this shows that $\mu \times \nu$ is a premeasure on $\mathcal{A}$.

Therefore, by previous results, $(\mu \times \nu)$ induces an outer-measure that is a measure when restricted to the σ-algebra generated by $\mathcal{A}$, i.e. $\mathcal{M} \otimes \mathcal{N}$. We denote this restriction by $\mu \times \nu$. It follows that $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ is a measure space in its own right.

Before we proceed, we introduce some notation. If $E \in \mathcal{M} \otimes \mathcal{N}$ and $(x, y) \in X \times Y$ we set

$$E_x := \{y \in Y : (x, y) \in E\}, \quad E^y = \{x \in X : (x, y) \in E\}.$$

Likewise, if f is defined on $X \times Y$

$$f_x(y) = f(x, y), \quad f^y(x) = f(x, y).$$

PROPOSITION 3.3. *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Then*

1. *For all $E \in \mathcal{M} \otimes \mathcal{N}$ one has $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$ for each $x \in X$ and $y \in Y$.*
2. *If f is $\mathcal{M} \otimes \mathcal{N}$-measurable then f_x is $\mathcal{N}$-measurable and f^y is $\mathcal{M}$-measurable for all $(x, y) \in X \times Y$.*

Proof. Suppose first that E is a rectangle of the form $E = A \times B$ for $A \in \mathcal{M}$ and $B \times \mathcal{N}$. If x is given then E_x is either $\varnothing$ or B (and likewise for E^y). Thus, (1) holds for rectangles. Let R be the collection of sets in $\mathcal{M} \otimes \mathcal{N}$ such that (1) holds true. This contains all disjoint finite unions of rectangles since

$$\left[\bigcup_j E_j\right]_x = \bigcup_j (E_j)_x.$$

We shall now show that R is a σ-algebra. By the above identity, it follows that R is closed under countable unions. Similarly,

$$(E^{\mathsf{C}})_x = (E_x)^{\mathsf{C}}$$

and likewise for y-sections. This shows that $R \supseteq \mathcal{M} \otimes \mathcal{N}$ by symmetry. □

Before we proceed with the proofs of the Fubini-Tonelli theorems, we require a preliminary technical result linking *monotone classes* and σ-algebras. A collection of subsets $\mathfrak{M}$ is called a monotone class provided it is closed under countable increasing unions and under countable decreasing intersections. It is not hard to show that arbitrary intersections of monotone classes is again a monotone class. Furthermore, σ-algebras are monotone classes. It thus makes sense to discuss a minimal monotone class containing a family of sets $\mathcal{F}$, whenever $\mathcal{F}$ is given. This "smallest" monotone class containing $\mathcal{F}$ is called the *monotone class generated by $\mathcal{F}$*.

LEMMA 3.4 (Monotone Class Lemma). *Let $\mathcal{A}$ be an algebra on a set and denote by $\mathfrak{C}$ the monotone class it generates. Then $\mathfrak{C} = \mathsf{M}(\mathcal{A})$.*

Proof. Since $\mathsf{M}(\mathcal{A})$ is also a monotone class containing $\mathcal{A}$ we immediately have $\mathsf{M}(\mathcal{A}) \supseteq \mathfrak{C}$; we shall show that $\mathfrak{C}$ is a σ-algebra to establish their equality. For $E \in \mathfrak{C}$ we define

$$C(E) := \{F \in \mathfrak{C} : E \setminus F,\ F \setminus E, \text{ and } E \cap F \in \mathfrak{C}\}.$$

It is purely mechanical to verify that $C(E)$ is itself a monotone class. Since $\mathcal{A}$ is an algebra, it is closed under finite intersections and complements, so that $\mathcal{A} \subseteq C(E)$ whenever $E \in \mathcal{A}$. In this case, we obtain $\mathfrak{C} \subseteq C(E)$. If $F \in \mathfrak{C}$ then $F \in C(E)$ which is equivalent to saying $E \in C(F)$. As $E \in \mathcal{A}$ was arbitrary, $\mathcal{A} \subseteq C(F)$ whence $\mathfrak{C} \subseteq C(F)$ for all $F \in \mathfrak{C}$. Therefore, $E \in C(X)$ for $E \in \mathfrak{C}$ which implies that $E^{\mathsf{C}} \in \mathfrak{C}$. Similarly, $E \cap F \in \mathfrak{C}$ if $E, F \in \mathfrak{C}$. This shows that $\mathfrak{C}$ is closed under complement and finite intersections. If $C_1, C_2, \ldots, C_n$ are elements of $\mathfrak{C}$ then

$$\left(\bigcup_j C_j\right)^{\mathsf{C}} = \bigcap_{j=1}^{n} C_j^{\mathsf{C}} \in \mathfrak{C}.$$

Thus, $\mathfrak{C}$ is an algebra. By closure under countable increasing unions, we obtain that $\mathfrak{C}$ is a σ-algebra which completes the proof. □

3.2.1 Product Algebras

Here we wish to generalize the notion of a product σ-algebra, which is sometimes useful when considering variants of Fubini-Tonelli in higher dimensions. This, of course, requires us to generalize the notion of a cartesian product of sets, as the usual coordinate notation is only feasible when dealing with products of countably many sets.

DEFINITION 10. Let Λ be an index set[5] and $\{X_\lambda\}_{\lambda\in\Lambda}$ a family of sets indexed by Λ. We define their cartesian product as

$$\prod_{\lambda\in\Lambda} X_\lambda := \left\{ f : \Lambda \longrightarrow \bigcup_{\lambda\in\Lambda} X_\lambda \mid f(\lambda) \in X_\lambda \right\}.$$

In the case where Λ is finite, one can view the elements of the cartesian products as n-tuples. As frequently encountered in abstract algebra (think of vector

[5]This index set need not be countable!

subspaces), this definition naturally gives rise to the notion of a *projection map*

$$\pi_\alpha : \prod_{\lambda \in \Lambda} X_j \longrightarrow X_\alpha, \quad f \mapsto f(\alpha)$$

where $\alpha \in \Lambda$ is fixed. Now, suppose we are given an associated family of σ-algebras on the family $\{X_\lambda\}_{\lambda \in \Lambda}$. More precisely, assume we have an indexed family of measurable spaces $\{(X_\lambda, \mathcal{M}_\lambda)\}_{\lambda \in \Lambda}$. Which structure should we endow $\prod_{\lambda \in \Lambda} X_\lambda$ with? To this end, we give the following definition.

DEFINITION 11. Given an indexed family of measurable spaces $\{(X_\lambda, \mathcal{M}_\lambda)\}_{\lambda \in \Lambda}$, we define the product σ-algebra on $\prod_{\lambda \in \Lambda} X_\lambda$ to be

$$\bigotimes_{\lambda \in \Lambda} \mathcal{M}_\lambda := \mathsf{M}\left(\left\{\pi_\alpha^{-1}(E_\alpha) : \alpha \in \Lambda,\ E_\alpha \in \mathcal{M}_\alpha\right\}\right).$$

We conclude this short digression with some basic properties of these general product algebras.

PROPOSITION 3.5. *Suppose that each M_λ is generated by some family $\mathcal{E}_\lambda$. If Λ is countable, then $\bigotimes_{\lambda \in \Lambda} \mathcal{M}_\lambda$ is generated by*

$$\left\{\prod_{\lambda \in \Lambda} E_\lambda : E_\lambda \in \mathcal{M}_\lambda\right\}.$$

Proof. Let $\mathfrak{M}$ denote the σ-algebra generated by $\left\{\prod_{\lambda \in \Lambda} E_\lambda : E_\lambda \in \mathcal{M}_\lambda\right\}$ and fix an element of the form $\pi_\alpha^{-1}(E_\alpha)$. This may, equivalently, be written as

$$\pi_\alpha^{-1}(E_\alpha) = \left\{f : \Lambda \longrightarrow \bigcup_{\lambda \in \Lambda} X_\lambda \mid f(\lambda) \in X_\lambda,\ f(\alpha) \in E_\alpha\right\} = \prod_{\lambda \in \Lambda} E_\lambda$$

where $E_\lambda = X_\lambda$ for $\lambda \neq \alpha$. This shows that $\bigotimes_{\lambda \in \Lambda} \mathcal{M}_\lambda \subseteq \mathfrak{M}$. For the reverse inclusion, note that if $E_\lambda \in \mathcal{M}_\lambda$ then

$$\prod_{\lambda \in \Lambda} E_\lambda = \left\{f : \Lambda \longrightarrow \bigcup_{\lambda \in \Lambda} E_\lambda : f(\lambda) \in E_\lambda\right\} = \bigcap_{\alpha \in \Lambda} \pi_\alpha^{-1}(E_\alpha).$$

Therefore, $\prod_{\lambda \in \Lambda} E_\lambda \in \bigotimes_{\lambda \in \Lambda} \mathcal{M}_\lambda$ since Λ is assumed to be countable. Therefore, $\mathfrak{M} \subseteq \bigotimes_{\lambda \in \Lambda} \mathcal{M}_\lambda$. □

PROPOSITION 3.6. *Suppose that each $\mathcal{M}_\lambda$ is generated by $\mathcal{E}_\lambda$. Then $\bigotimes_{\lambda \in \Lambda} \mathcal{M}_\lambda$ is generated by $\mathcal{F}_1$ where*

$$\mathcal{F} := \left\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha,\ \alpha \in \Lambda\right\}.$$

If Λ is countable and $X_\lambda \in \mathcal{E}_\lambda$ for all $\lambda \in \Lambda$ then $\bigotimes_{\lambda \in \Lambda} \mathcal{M}_\lambda$ is generated by

$$\left\{\prod_{\lambda \in \Lambda} E_\lambda : E_\lambda \in \mathcal{E}_\lambda\right\}.$$

Proof. We shall first show that $\mathcal{F}$ generates $\bigotimes_{\lambda \in \Lambda} \mathcal{M}_\lambda$. Since each $\mathcal{E}_\alpha$ is contained in $\mathcal{M}_\alpha$ it is clear that $\mathsf{M}(\mathcal{F}) \subseteq \bigotimes_{\lambda \in \Lambda} \mathcal{M}_\lambda$. To see the reverse inclusion, we need only note that the family

$$\left\{E \subseteq X_\alpha : \pi_\alpha^{-1}(E) \in \mathsf{M}(\mathcal{F})\right\}$$

is itself a σ-algebra containing $\mathcal{E}_\alpha$ (and thus $\mathcal{M}_\alpha$). Hence, $\pi_\alpha^{-1}(E) \subseteq \mathsf{M}(\mathcal{F})$ for all $E \in \mathcal{M}_\alpha$. This shows that, $\bigotimes_{\lambda \in \Lambda} \mathcal{M}_\lambda \subseteq \mathsf{M}(\mathcal{F})$. □

3.3 The Fubini-Tonelli Theorems: Proof

We begin by giving the following simple case of Fubini-Tonelli.

PROPOSITION 3.7. *Let $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ be a σ-finite product space. Then for all $E \in \mathcal{M} \otimes \mathcal{N}$*

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\,\mathrm{d}\mu = \int_Y \mu(E^y)\,\mathrm{d}\nu.$$

Proof. We first argue when both $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are finite spaces. Let R be the collection of all sets in $\mathcal{M} \otimes \mathcal{N}$ for which the statement holds true. If $E = A \times B$ is a rectangle then $\nu(E_x) = \nu(B)\mathbb{1}_A(x)$ so that,

$$\int_X \nu(E_x)\,\mathrm{d}\mu = \mu(A)\nu(B) = (\mu \times \nu)(E).$$

Hence R is non-empty and contains all rectangles in $\mathcal{M} \otimes \mathcal{N}$. By additivity, it follows that R contains all finite disjoint unions of rectangles, i.e. $\mathcal{A}$. By the monotone class lemma, it suffices to check that R is a monotone class.

Let $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \ldots$ be a countable increasing family in R and define $E = \bigcup_{n=1}^{\infty} E_n$. Clearly, $E_n \nearrow E$ as $n \to \infty$. If y is fixed, note that

$$E_1^y \subseteq E_2^y \subseteq \cdots \subseteq E_n^y \subseteq \ldots$$

and that $\bigcup_{n=1}^{\infty} E_n^y = E^y$, by earlier calculations. Therefore,

$$(\mu \times \nu)(E) = \lim_{n \to \infty} (\mu \times \nu)(E_n) = \lim_{n \to \infty} \int_Y \mu(E_n^y)\,\mathrm{d}\nu.$$

For each fixed y, we have that $\mu(E_n^y)$ increases to E^y, thus by monotone convergence we find that

$$(\mu \times \nu)(E) = \int_y \mu(E^y)\,\mathrm{d}\nu$$

and likewise for E_x. Suppose now that $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq$ is a decreasing family in R with $F_n \searrow F := \bigcap_{n=1}^{\infty} F_n$. We know

$$(\mu \times \nu)(F) = \lim_{n \to \infty} (\mu \times \nu)(F_n) = \lim_{n \to \infty} \int_Y \mu(F_n^y)\,\mathrm{d}\nu.$$

Now, $\mu(F_n^y) \searrow \mu(F^y)$ as $n \to \infty$ for each fixed y. By dominated convergence (note that $\mu(F_1^y) < \mu(X)$ which is $L^1(\nu)$ by our assumptions that the spaces are finite) we obtain the desired conclusion.

Now, assume that $(X, \mathcal{M}, \nu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite spaces. We can then write $X \times Y$ as the union of an increasing sequence of rectangles $\{X_j \times Y_j\}_j$. Then,

$$(\mu \times \nu)(E) = \lim_{j \to \infty} (\mu \times \nu)(E \cap (X_j \times Y_j)) = \lim_{j \to \infty} \int_Y \mu\left[(E^y \cap (X_j \times Y_j)^y\right] \mathrm{d}\nu$$

where $\mu([E^y \cap (X_j \times Y_j)^y]) = \mu(E^y \cap X_j)\mathbb{1}_{Y_j}(y)$. Thus a final application of monotone convergence completes the proof, by symmetry. □

THEOREM 3.8 (Fubini-Tonelli). *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces.*

1. *Let $f \in L^+(X \times Y)$. Then $g(x) = \int_Y f_x(y)\,\mathrm{d}\nu$ and $h(y) = \int_X f^y(x)\,\mathrm{d}\mu$ belong to $L^+(X)$ and $L^+(Y)$ respectively. Moreover,*

$$\int_{X\times Y} f(x,y)\,\mathrm{d}(\mu\times\nu) = \int_X \left(\int_Y f(x,y)\,\mathrm{d}\nu(y)\right)\,\mathrm{d}\mu(x) = \int_Y \left(\int_X f(x,y)\,\mathrm{d}\mu(y)\right)\,\mathrm{d}\nu(x).$$

2. *If $f \in L^1(X \times Y)$ then $f_x \in L^1(Y)$ for almost all x and $f^y \in L^1(X)$ for almost all $x \in X$ and almost all $y \in Y$. Furthermore, the almost everywhere defined functions*

$$g(x) = \int_Y f_x(y)\,\mathrm{d}\nu(y), \quad h(y) = \int_X f^y(x)\,\mathrm{d}\mu(x)$$

are in $L^1(\mu)$ and $L^1(\nu)$, respectively, and

$$\int_{X\times Y} f(x,y)\,\mathrm{d}(\mu\times\nu) = \int_X \left(\int_Y f(x,y)\,\mathrm{d}\nu(y)\right)\,\mathrm{d}\mu(x) = \int_Y \left(\int_X f(x,y)\,\mathrm{d}\mu(y)\right)\,\mathrm{d}\nu(x).$$

Proof. We begin by proving the first point. By Proposition 3.7, we know that this first claim holds for characteristics of rectangles. By additivity, this will also hold for the characteristics of finite disjoint unions of rectangles. Hence, this first point holds true for non-negative simple functions.

Now let $\{f_n\}_n$ be sequence of functions in $L^+(X \times Y)$ increasing to f as $n \to \infty$. By virtue of monotone convergence,

$$g(x) = \int_Y f_x(y)\,\mathrm{d}\nu = \lim_{n\to\infty} \int_Y f_n(x,y)\,\mathrm{d}\nu$$

where $x \mapsto \int_Y f_n(x,y)\,\mathrm{d}\nu$ is measurable for each n. This shows that g, and h (by symmetry) are measurable. Hence,

$$\int_{X\times Y} f(x,y)\,\mathrm{d}(\mu\times\nu) = \lim_{n\to\infty}\int_{X\times Y} f_n(x,y)\,\mathrm{d}(\mu\times\nu) = \lim_{n\to\infty}\int_X \left(\int_Y f_n(x,y)\,\mathrm{d}\nu\right)\,\mathrm{d}\mu.$$

By monotone convergence,

$$\int_Y f_n(x,y)\,\mathrm{d}\nu \nearrow \int_Y f(x,y)\,\mathrm{d}\nu \quad \text{as } n\to\infty$$

which concludes the proof of this first point by symmetry with h. We now proceed to prove the second point. Let $f \in L^+(X \times Y)$. We may obviously suppose that f is $\overline{\mathbb{R}}$-valued by breaking f into $\Re f$ and $\Im f$. Since f is integrable if and only if f^+ and f^- are, we may assume by linearity of the integral that $f \geq 0$. By Tonelli,

$$\int_{X\times Y} f(x,y)\,\mathrm{d}(\mu\times\nu) = \int_X \left(\int_Y f(x,y)\,\mathrm{d}\nu(y)\right)\,\mathrm{d}\mu(x)$$

which shows that $\int_Y f_x\,\mathrm{d}\nu < \infty$ for almost all x. Likewise, $\int_X f^y\,\mathrm{d}\mu < \infty$ for almost all y. By linearity of the integrals, we are done. □

4 Signed Measures

In this section we explore generalization of measures to signed and complex measures. Let us fix a measurable space $(X, \mathcal{M})$; a function $\nu : \mathcal{M} \longrightarrow [-\infty, \infty]$ is said to be a *signed measure* (sometimes called a *charge*) on $(X, \mathcal{M})$ provided it satisfies each of the following:

1. $\nu(\varnothing) = 0$,
2. ν takes on at-most one of $\{\infty, -\infty\}$ on $\mathcal{M}$,
3. If $\{E_j\}_j$ is a countable disjoint family in $\mathcal{M}$ and $E = \bigsqcup_j E_j$:

$$\nu(E) = \sum_j \nu(E_j)$$

This last requirement is powerful. Note that if we relabel the E_j via any permutation then the sum $\sum_j \mu(E_j)$ is preserved. This implies that the series $\sum_j \nu(E_j)$ must convergence absolutely. We say a measurable set E is positive if $\nu(F) \geq 0$ for all measurable $F \subseteq E$. Similarly, a set $E \in \mathcal{M}$ is called null if $\nu(F) = 0$ for all measurable $F \subseteq E$. Finally, E is negative if $\nu(F) \leq 0$ for all measurable $F \subseteq E$.

PROPOSITION 4.1. *Let ν be a signed measure on $(X, \mathcal{M})$. If $\{E_j\}_j$ is a family in $\mathcal{M}$ increasing to E then $\nu(E) = \lim_{j\to\infty} \nu(E_j)$. Likewise, if $\{E_j\}_j$ decreases to E and $|\nu(E_1)| < \infty$ then $\nu(E) = \lim_{j\to\infty} \nu(E_j)$.*

Proof. Suppose $E_j \nearrow E$, then $E \in \mathcal{M}$ and we can define $F_1 = E_1$, $F_k := E_k \setminus \bigcup_{j<k} E_j$. These F_k are disjoint and their union is E. Hence, by absolute convergence of the series

$$\nu(E) = \lim_{n\to\infty} \sum_{j\leq n} \nu(F_j) = \lim_{n\to\infty} \nu\left(\bigcup_{j\leq n} F_j\right) = \lim_{n\to\infty} \nu(E_n).$$

Arguing as in Theorem 1.6 one can obtain the other result. □

PROPOSITION 4.2. *Let ν be a signed measure on $(X, \mathcal{M})$ and $\{P_j\}_j$ a countable sequence of positive sets in $\mathcal{M}$. Then $P = \bigcup_j P_j$ is positive. Furthermore, any measurable subset of a positive set is positive.*

Proof. It is clear that a measurable subset of a positive set is again positive. Define $Q_1 := P_1$ and $Q_k := P_k \setminus \bigcup_{j=1}^{k-1} P_k$ for $k \geq 2$. Then the $\{Q_j\}_j$ are pairwise disjoint elements of $\mathcal{M}$ whose union is P. If $E \subseteq P$ then $E \subseteq \bigsqcup_j Q_j$ so that

$$\nu(E) = \nu\left(\bigsqcup_j [E \cap Q_j]\right) = \sum_j \nu(E \cap Q_j).$$

By construction, $Q_j \subseteq P_j$ for all j which implies, by our first step, that Q_j is positive. Hence, $\nu(E \cap Q_j) \geq 0$ which implies that $\nu(E) \geq 0$. □

We shall now prove the Hahn Decomposition Theorem.

THEOREM 4.3 (Hahn Decomposition). *Let ν be a signed measure on $(X, \mathcal{M})$. There exists a positive set P and a negative set N such that $X = P \sqcup N$. Furthermore, if $X = P' \sqcup N'$ with P' positive and N' odd, $P \triangle P'$ and $N \triangle N'$ are null.*

Proof. We may suppose without loss of generality that $\nu < \infty$ on $\mathcal{M}$, replacing ν with $-\nu$ otherwise. Define

$$p := \sup \{\nu(P) : P \in \mathcal{M},\ P \text{ positive}\}.$$

By definition of the supremum, there exists a sequence of positive measurable sets $\{P_k\}_k$ such that $\nu(P_k) \to p$ as $k \to \infty$. Define $P = \bigcup_k P_k$ and note that P is positive with respect to ν. Thus, $\nu(P) \leq p$. To see that we also have equality, fix k and note that

$$\nu(P) = \nu(P_k) + \nu(P \setminus P_k) \geq \nu(P_k)$$

' since $P \supseteq P \setminus P_k$ is positive. Letting $k \to \infty$ we obtain $\nu(P) \geq p$. We now define $N := X \setminus P$. Clearly, $X = P \sqcup N$.

STEP 1. N does not contain any non-null positive (measurable) subsets.

Proof of Step 1. Assume for a contradiction that N contains a non-null positive subsets A. Without loss of generality assume $\nu(A) > 0$. Then $P \cap A = \varnothing$ and $P \cup A$ is positive. However,

$$\nu(P \cup A) = \nu(P) + \nu(A) > p$$

which is a contradiction.

STEP 2. If $A \subseteq N$ is measurable and $\nu(A) > 0$, there exists a measurable $B \subseteq A$ such that $\nu(B) > \nu(A)$.

Proof of Step 2. Let $\nu(A) > 0$. By step 1, this A is non-positive. Hence, there exists some set $C \subset A$ with $\nu(C) < 0$. Since $A = C \sqcup (A \setminus C)$ we find that

$$0 < \nu(A) = \nu(C) + \nu(A \setminus C) < \nu(A \setminus C) < \infty.$$

It suffices to take $B := C \setminus A$.

Suppose for a contradiction that N is non-negative. Then, there exists a subset $A \subseteq N$ with $\nu(A) > 0$. Let $n_1 \in \mathbb{N}$ be the minimal natural number for which there exists a subset $B \subseteq N$ with

$$\nu(B) > \frac{1}{n_1}$$

and let A_1 be any such subset. Then, $\nu(A_1) > 0$. Let $n_2 \in \mathbb{N}$ be the minimal integer such that there exists $B \subseteq A_1$ with

$$\nu(B) > \nu(A_1) + \frac{1}{n_2}$$

and let A_2 be any such B. In this way we proceed, constructing a sequence

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_k \supseteq \ldots$$

together with an associated sequence of natural numbers $\{n_k\}_k$. Let $A = \bigcap_{k=1}^{\infty} A_k$ and observe that $A \subseteq N$. Now, $\nu(A_1) \neq \pm\infty$ and $A_k \searrow A$ as $k \to \infty$. Thus,

$$\nu(A) = \lim_{k\to\infty} \nu(A_k) \geq \lim_{k\to\infty} \sum_{j\leq k} \frac{1}{n_j} = \sum_{j=1}^{\infty} \frac{1}{n_j}.$$

In particular, $\sum_{j=1}^{\infty} \frac{1}{n_j} < \infty$. It follows that $n_k \to \infty$ as $k \to \infty$. This also shows that $\nu(A) > 0$. Now, there exists $B \subseteq A$ such that

$$\nu(B) > \nu(A) + \frac{1}{n}$$

for some minimal $n \in \mathbb{N}$. By taking k sufficiently large, we can find an index such that $n < n_k$. Now, if $j \geq k$ then

$$\nu(A_j) > \nu(A_{j-1}) > \cdots > \nu(A_k).$$

Letting $j \to \infty$ gives $\nu(A) \geq \nu(A_k)$. Therefore,

$$\nu(B) > \nu(A_k) + \frac{1}{n};$$

which is a contradiction. Thus, N is negative with respect to ν. In other-words, there exists a Hahn-Decomposition of $(X, \mathcal{M}, \nu)$. Let now P', N' form another Hahn-Decomposition of $(X, \mathcal{M}, \nu)$ with P' being ν-positive and N' being ν-negative. We have that $P' \cap N' = \varnothing$ and

$$X = P \sqcup N = P' \sqcup N'.$$

Observe that $P \setminus P' \subseteq P$ and hence is positive. Furthermore, $P \setminus P' \subseteq N'$ and thus must be negative. In particular, $P \setminus P'$ is ν-null. By symmetry, $P \triangle P'$ is ν-null. The same argument shows that $N \triangle N'$ is ν-null as well. The proof is now complete. $\square$

Equipped with this result, we can prove the Jordan Decomposition theorem for signed measures. First, we require some notation. We say that two signed measures μ and ν are independent if there exists $E, F \in \mathcal{M}$ with $X = E \cup F$ and $E \cap F = \varnothing$ such that F is μ-null and E is ν-null.

THEOREM 4.4 (Jordan Decomposition). *Let $(X, \mathcal{M}, \nu)$ be a signed measure space. There exist measures ν^+ and ν^- on $(X, \mathcal{M})$ such that $\nu \equiv \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. Furthermore, these measures are uniquely determined.*

Proof. We shall first prove existence. We fix $\{P, N\}$ be a Hahn decomposition of $(X, \mathcal{M}, \nu)$ and define for $E \in \mathcal{M}$

$$\nu^+(E) := \nu(E \cap P), \quad \nu^-(E) := -\nu(E \cap N).$$

Then, $\nu^\pm$ are clearly measures such that $\nu \equiv \nu^+ - \nu^-$. Furthermore, it is clear from their construction that $\nu^+ \perp \nu^-$. We now prove uniqueness of this decomposition. Suppose that μ^+ and μ^- are two measures on $(X, \mathcal{M})$ such that

$$\nu \equiv \mu^+ - \mu^-, \quad \mu+ \perp \mu^-.$$

Let $E, F \in \mathcal{M}$ be two disjoint sets such that $E \cup F = X$, F is μ^+-null and E is μ^--null. Observe then that $E \cup F$ forms a Hahn decomposition of $(X, \mathcal{M}, \nu)$, thus $P \triangle E$ is ν-null so is $N \triangle F$. If $A \in \mathcal{M}$ then

$$\mu^+(A) = \mu^+(A \cap E) + \mu^+(A \cap E^{\mathsf{C}}) = \mu^+(A \cap E) = \mu^+(A \cap E) - \mu^-(A \cap E).$$

Thus, $\mu^+(A) = \nu(E \cap A)$. Note now that

$$E \cap A = (A \cap E \cap P) \sqcup (A \cap E \cap P^{\mathsf{C}})$$

so that $\mu^+(A) = \nu(A \cap E \cap P)$. In a similar vein,

$$A \cap P = (A \cap P \cap E) \sqcup (A \cap P \cap E^{\mathsf{C}})$$

so that $\nu(A \cap P) = \nu(A \cap P \cap E)$. Hence,

$$\mu^+(A) = \nu(A \cap P) = \nu^+(A).$$

This completes the proof by symmetry. □

5 Exercises with Solutions

In this section we solve some exercises related to the material presented in these notes. Many of the problems were selected from the assignments of Math 564 in the year 2017.

PROBLEM 1. *Let $\mathcal{A}$ be an infinite σ-algebra on a non-empty set X. Prove that $\mathcal{A}$ is uncountable.*

Solution. We proceed by contradiction. Assume that $\mathcal{A}$ is countably infinite; then X is also infinite. Construct a map

$$F : X \longrightarrow \mathcal{A}, \quad x \mapsto \bigcap_{\substack{S \ni x \\ \mathcal{A} \ni S}} A.$$

Since $\mathcal{A}$ is countable, this is well defined. Suppose that $F(x) \cap F(y) \neq \varnothing$ for $x, y \in X$. We must have $x \in F(y)$ for otherwise $x \in F(x) \setminus F(y) \subsetneqq F(x)$ which contradicts the construction of F (since $F(x) \setminus F(y) \in \mathcal{A}$). This implies that $F(x) \subseteq F(y)$ and $F(y) \subseteq F(x)$ by symmetry. Consider the (countable) set

$$\mathcal{P} := \{F(x) : x \in X\} \subseteq \mathcal{A}.$$

If $E \in \mathcal{A}$ then $E = \bigcup_{x \in E} F(x)$ which implies that $\mathcal{P}$ is countable infinite, for otherwise $\mathcal{A}$ would be finite. indeed, every element of $\mathcal{A}$ is the union of elements in $\mathcal{P}$ and $\mathcal{A}$ is not finite. We shall now construct an injection

$$\Phi : \mathscr{P}(\mathcal{P}) \hookrightarrow \mathcal{A}, \quad \{F(x_k)\}_{k \in K} \mapsto \bigsqcup_{k \in K} F(x_k)$$

where K must be a countable (possibly finite) index set. Since the elements of $\mathcal{P}$ are disjoint, this is clearly an injection which contradicts the countability of $\mathcal{A}$. □

We recall that a probability space is a measure space $(X, \mathcal{M}, \mu)$ of finite measure such that $\mu(X) = 1$.

PROBLEM 2. *Let* $(X, \mathcal{M}, \mu)$ *be a probability space. Let* $E_1, E_2, \ldots, E_n \in \mathcal{M}$ *be such that* $\sum_{k=1}^{n} \mu(E_k) > n - 1$. *Prove that* $\mu(\bigcap_{k=1}^{n} E_k) > 0$.

Solution. Let $E \in \mathcal{M}$ be given and observe that $\mu(X) = \mu(E \sqcup E^{\mathsf{C}}) = \mu(E) + \mu(E^{\mathsf{C}})$. Thus, $\mu(E^{\mathsf{C}}) = 1 - \mu(E)$. Taking $E = \bigcup_{k=1}^{n} E_k^{\mathsf{C}}$, we obtain

$$\mu\left(\bigcap_{k=1}^{n} E_k\right) = 1 - \mu\left(\bigcup_{k=1}^{n} E_k^{\mathsf{C}}\right) \geq 1 - \sum_{k=1}^{n} \mu(E_k^{\mathsf{C}}) = 1 - \sum_{k=1}^{n} \left[1 - \mu(E_k)\right].$$

Since $1 - \sum_{k=1}^{n} \left[1 - \mu(E_k)\right] = 1 - n + \sum_{k=1}^{n} \mu(E_k) > 0$ we are done. □

PROBLEM 3. *Let* μ^* *be an outer-measure on a non-empty set* X *and* $\{A_j\}_{j \in \mathbb{N}}$ *a sequence of disjoint* μ^* *measurable sets. Prove that for all* $E \subseteq X$

$$\mu^*(E \cap A) = \sum_{j \in \mathbb{N}} \mu^*(E \cap A_j)$$

where $A = \bigcup_{j=1}^{\infty} A_j$.

Solution. We claim that for all $n \in \mathbb{N}$ we have

$$\mu^*\left(E \cap \bigcup_{j=1}^{n} A_j\right) \geq \sum_{j=1}^{n} \mu^*(E \cap A_j).$$

The case $n = 1$ is trivial. Assume that the above holds up-to $n - 1$; we prove it for n. Since $\bigcup_{j=1}^{n-1} A_j$ is μ^*-measurable

$$\begin{aligned}
\mu^*\left(E \cap \bigcup_{j=1}^{n} A_j\right) &= \mu^*\left(E \cap \bigcup_{j=1}^{n} A_j \cap \bigcup_{j=1}^{n-1} A_j\right) + \mu^*\left(E \cap \bigcup_{j=1}^{n} A_j \cap \bigcap_{j=1}^{n-1} A_j^{\mathsf{C}}\right) \\
&= \mu^*\left(E \cap \bigcup_{j=1}^{n-1} A_j\right) + \mu^*(E \cap A_n) \\
&\geq \sum_{j=1}^{n} \mu^*(A_j \cap E).
\end{aligned}$$

For all n, $\mu^*(E \cap A) \geq \mu^*\left(E \cap \bigcup_{j=1}^{n} A_j\right) \geq \sum_{j=1}^{n} \mu^*(E \cap A_j)$. Letting $n \to \infty$ we obtain

$$\mu^*(E \cap A) \geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j).$$

We are done since the inequality '$\leq$' is trivially true. □

PROBLEM 4. *Let* m *be the Lebesgue measure on* $\mathbb{R}$ *and let* $E \subseteq \mathbb{R}$ *be a set of positive Lebesgue measure. Prove that for all* $\alpha \in (0, 1)$ *there exists an open interval* I *such that*

$$m(E \cap I) > \alpha m(I). \tag{5}$$

Proof. We first assume that $\mu(E) < \infty$. Let $\alpha \in (0,1)$ and take $\beta = 1/\alpha > 1$. There exists an open set $O \supseteq E$ such that

$$m(O) < (\beta - 1)m(E) + m(E) = \frac{1}{\alpha}m(E).$$

That is, $\alpha m(O) < m(E)$. We may write $O = \bigsqcup_j I_j$ as the countable disjoint union of open intervals. We claim some j satisfies (5). Otherwise,

$$m(E \cap I_j) \leq \alpha m(I_j)$$

for all I_j. This implies that

$$m(E) = m(E \cap O) = \sum_j m(E \cap I_j) \leq \alpha \sum_j m(I_j) = \alpha m(O)$$

which contradicts the choice of O. If $\mu(E) = \infty$ there must exist an interval $(n, n+1]$, $n \in \mathbb{Z}$, such that $E \cap (n, n+1]$ has non-zero measure. Simply apply this procedure to that set. □

PROBLEM 5. *Let m be the Lebesgue measure on $\mathbb{R}$ and $E \subseteq \mathbb{R}$ such that $m(E) > 0$. Prove that*

$$E - E := \{x - y : x, y \in E\}$$

contains an interval centered about 0.

Solution. Without loss of generality we let $\mu(E) < \infty$. Consider the function

$$f(x) = \int_{\mathbb{R}} \mathbb{1}_E(y)\mathbb{1}_E(y - x)\,\mathrm{d}y < \infty.$$

Note that f is continuous on $\mathbb{R}$ since for any sequence $\{x_n\}_n$ such that $x_n \to x$ as $n \to \infty$:

$$|f(x_n) - f(x)| \leq \int_{\mathbb{R}} \mathbb{1}_E(y)\,|\mathbb{1}_E(y - x_n) - \mathbb{1}_E(y - x)|\,\mathrm{d}y$$

which tends to zero as $n \to \infty$ by dominated convergence. Note also that $f(0) = m(E) > 0$. Hence, $0 \in f^{-1}((0,\infty))$ which is open by continuity. Hence, there exists an interval $(-\delta, \delta) \subseteq f^{-1}((0,\infty))$. We claim now that $(-\delta, \delta) \subseteq E - E$. If this inclusion does not hold, there exists z such that $f(z) > 0$ but $z \neq y - x$ for all $y, x \in E$. In this case, $y - z \neq x$ for all $x \in E$ whence $f(z) = 0$: contradiction. □

PROBLEM 6. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^+(X)$. For $E \in \mathcal{M}$ let*

$$\eta(E) = \int_X f(x)\,\mathrm{d}\mu.$$

Show that η is a measure on $(X, \mathcal{M})$ and that for all $g \in L^+(X)$

$$\int_X g(x)\,\mathrm{d}\eta = \int_X f(x)g(x)\,\mathrm{d}\mu.$$

Solution. We first check that η is a measure on $(X, \mathcal{M})$. By definition, $\nu(\varnothing) = 0$. If $\{A_j\}_j$ is a countable collection of disjoint measurable sets

$$\eta\left(\bigsqcup_j A_j\right) = \int_{\bigsqcup_j A_j} f(x)\,\mathrm{d}\mu = \int_X \mathbb{1}_{\bigsqcup A_j(x)} f(x)\,\mathrm{d}\mu = \int_X \sum_j \mathbb{1}_{A_j}(x) f(x)\,\mathrm{d}\mu$$
$$= \sum_j \int_X \mathbb{1}_{A_j}(x) f(x)\,\mathrm{d}\mu$$
$$= \sum_j \eta(A_j).$$

Thus, $(X, \mathcal{M}, \eta)$ is a measure space and one can integrate "with respect to" η. Suppose g is a non-negative simple function of the form

$$g(x) = \sum_{j=1}^N \alpha_j \mathbb{1}_{A_j}(x), \quad \alpha_j \in \mathbb{C}.$$

Then,

$$\int_X g(x)\,\mathrm{d}\eta = \sum_{j=1}^n \alpha_j \int_X \mathbb{1}_{A_j}(x)\,\mathrm{d}\eta = \sum_{j=1}^N \alpha_j \eta(A_j) = \sum_{j=1}^N \alpha_j \int_X f(x)\mathbb{1}_{A_j}(x)\,\mathrm{d}\mu$$
$$= \int_X f(x) \sum_{j=1}^N \alpha_j \mathbb{1}_{A_j}(x)\,\mathrm{d}\mu$$
$$= \int_x f(x) g(x)\,\mathrm{d}\mu.$$

Now let $g \in L^+(X)$ be arbitrary and choose a sequence of simple functions $\{\varphi_n\}_n$ increasing to g pointwise. Then by monotone convergence

$$\int_X g(x)\,\mathrm{d}\eta = \lim_{n\to\infty} \int_X \varphi_n(x)\,\mathrm{d}\eta = \lim_{n\to\infty} \int_X f(x)\varphi_n(x)\,\mathrm{d}\mu = \int_X f(x)g(x)\,\mathrm{d}\mu$$

since $f(x)\varphi_n(x) \nearrow f(x)g(x)$ as $n \to \infty$. □

PROBLEM 7. *Deduce monotone convergence from Fatou's lemma.*

Solution. Let $\{f_n\}_n$ be a sequence of measurable functions increasing pointwise to $f \in L^+(X)$. By Fatou's lemma

$$\limsup_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu \leq \int_X f(x)\,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu.$$

□

PROBLEM 8 (Generalized Dominated Convergence). *Let $\{f_n\}_n, \{g_n\}_n, f, g$ be in $L^1(X)$ with $f_n \to f$ and $g_n \to g$. Assume each $|f_n| \leq g_n$ and $\int_X g_n \to \int_X g$ as $n \to \infty$. Show that*

$$\lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu = \int_X f(x)\,\mathrm{d}\mu.$$

Proof. We have both $g_n - f_n \geq 0$ and $g_n + f_n \geq 0$. Thus, by Fatou's lemma

$$\int_X (g_n + f_n)\,\mathrm{d}\mu \leq \liminf_{n\to\infty} \left(\int_X g_n\,\mathrm{d}\mu + \int_X f_n\,\mathrm{d}\mu \right)$$

and

$$\int_X (g_n - f_n)\,\mathrm{d}\mu \leq \liminf_{n\to\infty} \left(\int_X g_n\,\mathrm{d}\mu + \int_X -f_n\,\mathrm{d}\mu \right).$$

These two combined statements give

$$\limsup_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu \leq \int_X f(x)\,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu.$$

□

PROBLEM 9. *Suppose $f_n, f \in L^1(X)$ and that $f_n \to f$. Prove that*

$$\lim_{n\to\infty} \int_X f_n\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu \iff \lim_{n\to\infty} \int_X |f_n - f|\,\mathrm{d}\mu = 0.$$

Proof. The $\Longleftarrow$ implication is trivial. Suppose $\int_X f_n\,\mathrm{d}\mu \to \int_X f\,\mathrm{d}\mu$ as $n \to \infty$. Define

$$g_n(x) := |f_n(x)| + |f(x)|$$

and note that $|f_n - f| \leq g_n(x)$. Furthermore, $g_n(x) \to g(x) = 2\,|f(x)|$. Clearly,

$$\int_X g_n(x)\,\mathrm{d}\mu \xrightarrow{n\to\infty} \int_X g(x)\,\mathrm{d}\mu$$

by assumption. Thus, an application of the previous problem gives the desired result. □

PROBLEM 10. *Let $\{r_n\}_n$ be an enumeration of the rationals and define*

$$f(x) := \frac{1}{\sqrt{x}} \mathbb{1}_{(0,1)}(x).$$

Set

$$g(x) := \sum_n 2^{-n} f(x - r_n).$$

Prove that g is Lebesgue integrable on $\mathbb{R}$, and that any function equal to g almost everywhere is discontinuous at all points and unbounded on any interval. Conclude that g^2 is not locally integrable.

Solution. We first show that $g \in L^1(\mathbb{R})$ with respect to the Lebesgue measure m. Since all summands are non-negative

$$\int_{\mathbb{R}} g(x)\,\mathrm{d}m = \sum_{n=1}^{\infty} \int_{\mathbb{R}} 2^{-n} f(x - r_n)\,\mathrm{d}m = \sum_{n=1}^{\infty} 2^{-n} \int_{\mathbb{R}} f(x)\,\mathrm{d}m$$

where we we have used translation invariance of the Lebesgue integral in this last step. It thus suffices to check that $\int_{\mathbb{R}} f(x)\,\mathrm{d}x$ is finite. Certainly, by monotone convergence

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}m = \int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}m = \lim_{N\to\infty}\int_{1/N}^1 \frac{1}{\sqrt{x}}\,\mathrm{d}m = \lim_{N\to\infty}\left(2-\frac{2}{\sqrt{N}}\right) < \infty.$$

Now, let $Z \subset \mathbb{R}$ be a null-set and suppose $h : \mathbb{R} \longrightarrow [0,\infty)$ is equal to g for $x \notin Z$. It suffices to check that h is unbounded on any interval, since then it is necessarily discontinuous at all points.[6] Let $I = (a,b)$ be a non-empty open interval and choose $r \in \mathbb{Q}\cap I$. This corresponds to some r_N in our enumeration of $\mathbb{Q}$. Let $\delta > 0$ and note that there exists a point $x \in (r_N, r_N+\delta)$ such that $h(x) = g(x)$. Indeed, if this were not the case then $(r_N, r_N+\delta) \subseteq Z$ which would contradict the fact that $m(Z) = 0$. Hence, there exists a sequence $\{x_j\}_j$ with $x_j > r_N$ and $x_j \to r_N$ as $j \to \infty$ such that $g(x_j) = h(x_j)$. Thus,

$$\lim_{j\to\infty} h(x_j) = \lim_{j\to\infty} g(x_j) \geq 2^{-N} \lim_{j\to\infty} f(x_j - r_N) = \infty.$$

Since $g \in L^1(\mathbb{R})$ it follows that $g < \infty$ for almost all x. Hence, $g^2 < \infty$ for almost every x. However, $g^2 \notin L^1_{\mathrm{loc}}(\mathbb{R})$ since for every non-empty open interval $I = (a,b)$ one can choose $r_N \in \mathbb{Q}\cap I$. This implies that

$$\int_{\mathbb{R}} g(x)^2\,\mathrm{d}m \geq \int_{(r_N,r_N+1)} g(x)\,\mathrm{d}m \geq \int_{(r_N,r_N+1)} \frac{1}{x-r_N}\,\mathrm{d}m = \int_0^1 \frac{1}{y}\,\mathrm{d}y \tag{6}$$

which is unbounded. □

[6] If h is continuous at $x \in \mathbb{R}$ then for $\varepsilon = 1$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ whenever $|x-y| < \delta$. Thus, for $y \in (x-\delta, x+\delta)$ one has that $f(y) \leq f(x) + 1$.