The Classification of Finite Dimensional C^*-algebras

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In this document, we will classify (up to isomorphism) all finite dimensional C^*-algebras with unity. Our proof follows that taught during a topics course at McGill university in the winter of 2018.

As a first step, we will prove a seemingly unrelated result which will play a crucial role in the classification presented in §2.

1 The First Step

Let $\mathcal{U}$ be a C^*-algebra with identity $\mathbb{1}$ and fix $A \in \mathcal{U}$. We define $\mathcal{U}_A$ to be the C^*-subalgebra of $\mathcal{U}$ generated by $\{\mathbb{1}, A\}$. In this section, we prove that $\mathcal{U}_A \cong C(\sigma(A))$. As usual, this will be achieved by way of multiple steps. We would first like to make it clear that $\sigma(A)$ is always non-empty and compact (by Problem 1 in the midterm) and as such $C(\sigma(A))$ is a Banach space over $\mathbb{C}$ (with the uniform norm) when equipped with the obvious pointwise operations.

To be more precise, we prove the following.

Theorem. *Let $\mathcal{U}$ be a C^*-algebra with identity $\mathbb{1}$ and let $A \in \mathcal{U}$ be self-adjoint. Letting $\mathcal{U}_A$ be the C^*-subalgebra generated by $\{A, \mathbb{1}\}$, one has that $\mathcal{U}_A \cong C(\sigma(A))$ as C^*-algebras.*

Proof. We will decompose this proof into multiple steps. First, we define

$$V := \left\{a_0\mathbb{1} + a_1A + \cdots + a_nA^n : n \in \mathbb{N}_0, a_j \in \mathbb{C}\right\}.$$

Step 1.1. *The set V forms a vector subspace of $\mathcal{U}_A$. Moreover, the topological closure $\overline{V}$ is equal to $\mathcal{U}_A$. In particular, V is a dense subspace of $\mathcal{U}_A$.*

Proof of Step 1.1. First, it is clear that V is a subset of $\mathcal{U}_A$ since $\mathbb{1}, A \in \mathcal{U}_A$ by construction. Therefore, $\overline{V} \subseteq \mathcal{U}_A$ since $\mathcal{U}_A$ is a closed subspace containing V. Also, it is obvious from the definition of V that V is closed under scalar multiplication and vector addition. Hence, V (and thus $\overline{V}$) is a vector subspace of $\mathcal{U}_A$. Let now $B, C \in V$ and write them as

$$B = \beta_0 \mathbb{1} + \cdots + \beta_n A^n, \quad C = \gamma_0 \mathbb{1} + \cdots + \gamma_m A^m$$

where $\beta_j, \gamma_j \in \mathbb{C}$. It is then clear that the product BC will be a finite sum of terms having the form $\alpha_k A^k$, with $k \in \mathbb{N}_0$ and $\alpha_k \in \mathbb{C}$. Thus, $BC \in V$ and V is closed under multiplication. Also, since $A^* = A$ we see that

$$B^* = \overline{\beta_0} \mathbb{1} + \cdots + \overline{\beta_n} A^n \in V.$$

Therefore, V is closed under involution. We now argue that $\overline{V}$ is a C^*-algebra. Notice that $\overline{V}$ is a closed subset of a complete metric space, and is therefore complete in its own right. Since $\overline{V}$ is a vector space, $\overline{V}$ is a Banach space. Now, for $B, C \in \overline{V}$ choose two sequences $(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ in V such that

$$B = \lim_{n\to\infty} B_n, \quad C = \lim_{n\to\infty} C_n.$$

Since these are convergent sequences, they are also bounded with respect to the norm. Thus, for every $n \in \mathbb{N}$ there holds

$$\|BC - B_n C_n\| = \|BC - BC_n + BC_n - B_n C_n\| \leq \|B\| \, \|C - C_n\| + \|B - B_n\| \, \|C_n\| \xrightarrow{n\to\infty} 0.$$

This implies that $B_n C_n \to BC$. But, $B_n C_n \in V$ for every n and therefore $BC \in \overline{V}$. Hence, $\overline{V}$ is closed under multiplication. Likewise, notice that

$$\|B^* - B_n^*\| = \|(B - B_n)^*\| = \|B - B_n\| \xrightarrow{n\to\infty} 0.$$

Now, since V is closed under involution, we have $B_n^* \in V$ for all n. Therefore, $B_n^* \in \overline{V}$ for all n and $B_n^* \to B^*$. This certainly means that $B^* \in \overline{V}$. We conclude from this that $\overline{V}$ is a C^*-algebra. Now, $V \subseteq \overline{V}$ and V contains $\{\mathbb{1}, A\}$ by construction. Therefore, $\overline{V}$ is a C^*-algebra containing $\{\mathbb{1}, A\}$ and so we must have $\overline{V} \supseteq \mathcal{U}_A$. On the other hand, at the start of the proof, we showed $\overline{V} \subseteq \mathcal{U}_A$. This completes the proof of the step. □

We now lend ourselves to a useful observation. Notably, that the elements of V commute with themselves. To see that this is so, let $B, C \in V$ be given and write them as

$$B = \beta_0 \mathbb{1} + \cdots + \beta_n A^n, \quad C = \gamma_0 \mathbb{1} + \cdots + \gamma_m A^m$$

It is therefore clear that

$$\begin{aligned}
BC &= (\beta_0 \mathbb{1} + \cdots + \beta_n A^n)\,(\gamma_0 \mathbb{1} + \cdots + \gamma_m A^m) \\
&= \sum_{k=0}^{n} \sum_{j=0}^{m} \beta_k \gamma_j A^{k+j} \\
&= (\gamma_0 \mathbb{1} + \gamma_1 A + \cdots + \gamma_m A^m)\,(\beta_0 \mathbb{1} + \beta_1 A + \cdots + \beta_n^n) \\
&= CB.
\end{aligned}$$

Thus, V can be viewed as commutative. This gives way to the second step of the proof.

Step 1.2. *Let $T : V \to C(\sigma(A))$ be given by the association $P(A) \mapsto P(\lambda)$, where $\lambda \in \sigma(A)$ is a free variable. Then T is well defined, linear, multiplicative and preserves both the norm and involution.*

Proof of Step 1.2. We first ensure that T is well defined. Obviously, elements of V can be viewed as polynomials (over $\mathbb{C}$) in A. Let now $P(A)$ and $Q(A)$ be any two such elements of V and suppose that $P(A) = Q(A)$. Then $P(A) - Q(A) \in V$ is normal (since V is commutative and closed under taking adjoints). This implies that

$$\rho\,(P(A) - Q(A)) = \|P(A) - Q(A)\| = 0.$$

But then it follows that

$$0 = \rho\,(P(A) - Q(A)) = \sup_{\lambda \in (P-Q)(\sigma(A))} |\lambda| = \sup_{\lambda \in \sigma(A)} |P(\lambda) - Q(\lambda)|$$

whence $P(\lambda) = Q(\lambda)$ for all $\lambda \in \sigma(A)$. This shows that T is well defined. It is also immediate from the definition that T is linear over $\mathbb{C}$.

Let now $P(A) \in V$ be a polynomial in A over $\mathbb{C}$. Since one has $A^* = A$ by hypothesis, direct calculation verifies that $P(A)^* = \overline{P}(A)$ whence

$$T\,(P(A)^*) = T\left(\overline{P}(A)\right) = \overline{P}(\lambda) \overset{\text{def}}{=} P(\lambda)^* = T(P(\lambda))^*.$$

Again, $P(A)$ is normal since V is closed under involutions and is commutative. This implies that

$$\|P(A)\| = \sup_{\lambda \in \sigma(P(A))} |\lambda| = \sup_{\lambda \in P(\sigma(A))} |\lambda| = \sup_{\lambda \in \sigma(A)} |P(\lambda)| = \|P(\lambda)\|$$

where this last term is clearly $\|T(P(A))\|$. Therefore, T is as required and this step is complete. $\square$

We now wind down with the final step in the proof of this theorem.

Step 1.3. *The map T described in Step 1.2 extends to an isomorphism $\mathcal{U}_A \to C(\sigma(A))$.*

Proof of Step 1.3. Now, we have shown in Step 1.1 that V is dense in $\mathcal{U}_A$. Also, since T is norm preserving on V, it is bounded. We may therefore extend[1] T to a bounded linear operator on the whole of $\mathcal{U}_A$.

Let us also label this extension T; we now show that T is the desired isomorphism. First, T will preserve multiplication. Certainly, let $B, C \in \mathcal{U}_A$ be given and choose sequences $(B_n)_{n\in\mathbb{N}}, (C_n)_{n\in\mathbb{N}}$ in V such that

$$B = \lim_{n\to\infty} B_n \quad \text{and} \quad C = \lim_{n\to\infty} C_n.$$

Then, $B_nC_n \to BC$ whence by continuity of T:

$$T(BC) = \lim_{n\to\infty} T(B_nC_n) = \lim_{n\to\infty} T(B_n)T(C_n).$$

But, since $B_n \to B$ and $C_n \to C$ we also have $T(B_n) \to T(B)$ and $T(C_n) \to T(C)$. Using this with the above implies that

$$T(BC) = \lim_{n\to\infty} T(B_n)T(C_n) = T(B)T(C).$$

Also, $\|B^* - B_n^*\| = \|B - B_n\|$ and so $B_n^* \to B^*$. Again, invoking the continuity of T gives

$$T(B^*) = \lim_{n\to\infty} T(B_n^*) = \lim_{n\to\infty} T(B_n)^*.$$

But, by the same argument, $T(B_n)^* \to T(B)^*$ and so the above yields

$$T(B^*) = \lim_{n\to\infty} T(B_n)^* = T(B)^*.$$

Now, the norm composed with T returns a continuous function and therefore

$$\|T(B)\| = \lim_{n\to\infty} \|T(B_n)\| = \lim_{n\to\infty} \|B_n\| = \|B\|$$

by continuity of the norm. This shows that T is norm preserving on the whole of $\mathcal{U}_A$. Notice that this implies that T is injective. Certainly, if $T(B) = T(C)$ then $T(B - C) = 0$ whence

$$\|B - C\| = \|T(B - C)\| = 0.$$

[1] This is a general fact about Banach spaces. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces and $Q \subseteq \mathcal{B}_1$ a dense subspace. Suppose that we are given a bounded linear operator $T : Q \to \mathcal{B}_1$. Then T extends *uniquely* to a bounded linear operator $\mathcal{B}_1 \to \mathcal{B}_2$.

Thus, to establish that T is the desired isomorphism of C^*-algebras, we need only check that T is surjective. This will make use of the Stone-Weierstrass theorem.

Recall that $\sigma(A)$ is non-empty and compact. Also, since $A^* = A$, we must have $\sigma(A) \subset \mathbb{R}$. Let now $f \in C(\sigma(A))$ be given; by the Stone-Weierstrass theorem there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of polynomials defined on $\sigma(A)$ converging *uniformly* to f on $\sigma(A)$. Now, for each $n \in \mathbb{N}$ we may express this φ_n as

$$\varphi_n(\lambda) = \sum_{j=0}^{K(n)} \alpha_{n,j}\lambda^j, \quad \alpha_{n,j} \in \mathbb{C}.$$

For each $n \in \mathbb{N}$ we subsequently define

$$B_n := \varphi_n(A) = \sum_{j=0}^{K(n)} \alpha_{n,j}A^j.$$

Clearly, B_n "corresponds" to $\varphi_n(\lambda)$ in the sense that $\varphi_n(\lambda) = T(B_n)$ for all n. Now, using the fact that T is norm preserving:

$$\|B_n - B_m\| = \|T(B_n - B_m)\| = \|T(B_n) - T(B_m)\| = \|\varphi_n - \varphi_m\|_\infty .$$

Since $(\varphi_n)_{n\in\mathbb{N}}$ is convergent in $C(\sigma(A))$, it is in particular Cauchy. The above then shows that $(B_n)_{n\in\mathbb{N}}$ is Cauchy as well. Since $\mathcal{U}_A$ is complete, B_n converges to some $B \in \mathcal{U}_A$. By the continuity of T, it follows that

$$T(B) = \lim_{n\to\infty} T(B_n) = \lim_{n\to\infty} \varphi_n = f.$$

Therefore, T is surjective and hence is the desired isomorphism of algebras. □

2 The Classification of Finite Dimensional C^*-Algebras

Let $\mathcal{U}$ be a finite dimensional C^*-algebra with identity $\mathbb{1}$, our goal is to prove that there exist finite dimensional Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ such that

$$\mathcal{U} \cong \bigoplus_{j=1}^{n} \mathcal{B}(\mathcal{H}_j).$$

Again, this proof will make use of multiple lemmas/propositions. Notice also that if $\mathcal{H}$ is a finite dimensional Hilbert space, then $\mathcal{B}(\mathcal{H}) \cong M_d(\mathbb{C})$ for some integer d. Therefore, if the above holds, then we would also have that

$$\mathcal{U} \cong \bigoplus_{j=1}^{n} M_{d_j}(\mathbb{C}), \quad d_j \geq 1.$$

For what remains of this document, $\mathcal{U}$ is a **finite dimensional** C^*-algebra with identity $\mathbb{1}$. We begin with the following expected result.

Lemma 1. *Let (X, d) be a metric space and let $C(X)$ denote the space of all continuous functions $X \to \mathbb{C}$. If X is infinite, then $\dim(C(X)) = \infty$.*

Proof. Notice that the statement makes sense for $C(X)$ is a vector space over $\mathbb{C}$ with the usual pointwise operations. Fix $N \in \mathbb{N}$ and choose a finite subset $\{x_1, \dots, x_N\} \subset X$. For every index n notice that the function $x \mapsto d(x, x_n)$ is continuous on X. Therefore, so is the product

$$x \mapsto f_1(x) := \prod_{n=2}^{N} d(x, x_n).$$

Now, $f_1(x) = 0$ if and only if $x \in \{x_2, \dots, x_n\}$ and is non-zero at x_1. After normalization, we might as well assume that $f_1(x_1) = 1$. Proceeding similarly, for every index $j = 1, \dots, N$ we may find a function $f_j \in C(X)$ so that for $x \in \{x_1, \dots, x_N\}$ one has

$$f_j(x) := \begin{cases} 1, & x = x_j, \\ 0, & x \neq x_j. \end{cases}$$

These functions are clearly linearly independent, and thus $\dim(C(X)) \geq N$. As N was arbitrary, it follows that $C(X)$ is infinite dimensional. □

Proposition 2.1. *Let $A \in \mathcal{U}$ be self-adjoint. Then $\sigma(A)$ is finite and*

$$A = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda$$

where the P_λ are non-zero projections in $\mathcal{U}$ such that $P_\lambda P_{\lambda'} = 0$ for $\lambda \neq \lambda'$ and $\sum_{\lambda \in \sigma(A)} P_\lambda = \mathbb{1}$.

Proof. From the first part of this final, we have $\mathcal{U}_A \cong C(\sigma(A))$. Since $\dim(\mathcal{U}) < \infty$ and $\mathcal{U}_A \subseteq \mathcal{U}$, we see that $\dim(\mathcal{U}_A) = \dim(C(\sigma(A))) < \infty$. Invoking the previous lemma, it becomes clear that $\sigma(A)$ is also a finite set. Thus, $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ for some $\lambda_j \in \mathbb{R}$. Indeed, the λ_j are real since A is self-adjoint. Define now for $k = 1, \dots, n$

$$f_k(\lambda) := \begin{cases} 1, & \lambda = \lambda_k, \\ 0, & \text{otherwise}. \end{cases}$$

Since $\sigma(A)$ is finite, this is clearly a continuous function $\sigma(A) \to \mathbb{C}$. Also, it is clear from the definition that for each $\lambda \in \sigma(A)$:

$$\sum_{k=1}^{n} f_k(\lambda) = 1 \quad \text{and} \quad \sum_{k=1}^{n} \lambda_k f_k(\lambda) = \lambda.$$

Now, in the first problem we proved that $\mathcal{U}_A \cong C(\sigma(A))$. For $k = 1, \ldots, n$, let P_k be the element of $\mathcal{U}_A$ associated to f_k (according to the isomorphism provided in the first problem). Then, since such an isomorphism will preserve the structure of $\mathcal{U}_A$ we see that $P_k^2 = P_k$ since $f_k^2 \equiv f_k$. Also, since f_k maps to $\mathbb{R}$ and the isomorphism respects involution, we will also have $P_k^* = P_k$. Just the same, this isomorphism takes $\mathbb{1} \to 1$ whence one has

$$\sum_{k=1}^{n} P_k = \mathbb{1} \quad \text{and} \quad \sum_{k=1}^{n} \lambda_k P_k = A.$$

This proves the proposition. □

We recall some notation regarding projections. Let P_1, P_2 be projections in $\mathcal{U}$. We will write $P_1 \geq P_2$ if and only if $P_1 P_2 = P_2$. If P is a projection, then $P^2 = P$ so that $P \leq P$. If $P_1 \geq P_2$ and $P_2 \geq P_1$ then

$$P_1 P_2 = P_2 \quad \text{and} \quad P_2 P_1 = P_1.$$

Taking adjoints in the second equation gives $P_1 P_2 = P_1$ and so $P_1 = P_1 P_2 = P_2$. If $P_3 \leq P_2$ and $P_2 \leq P_1$ then

$$P_2 P_3 = P_3 \quad P_1 P_2 = P_2.$$

But then

$$P_1 P_3 = P_1(P_2 P_3) = (P_1 P_2) P_3 = P_2 P_3 = P_3.$$

This shows that '$\geq$' is a partial order.

Proposition 2.2. *Let P_1, P_2 be projections in $\mathcal{U}$. Then $P_1 \geq P_2$ if and only if $P_2 \mathcal{U} P_2 \subseteq P_1 \mathcal{U} P_1$. Therefore,*

$$P_1 = P_2 \iff P_2 \mathcal{U} P_2 = P_1 \mathcal{U} P_1.$$

Proof. Suppose first that $P_1 \geq P_2$. Then, by definition,

$$P_2 \mathcal{U} P_2 = P_1 P_2 \mathcal{U} P_2 P_1 \subseteq P_1 \mathcal{U} P_1.$$

Indeed, this is because $P_1P_2 = P_2$ implies that

$$P_2 = P_2^* = (P_1P_2)^* = P_2P_1.$$

Conversely, suppose that $P_2\mathcal{U}P_2 \subseteq P_1\mathcal{U}P_1$. Then $P_2\mathbb{1}P_2 = P_1AP_1$ for some $A \in \mathcal{U}$ whence

$$P_1P_2 = P_1[P_1AP_1] = P_1AP_1 = P_2.$$

This completes the proof. $\square$

Proposition 2.3. *Let $P \in \mathcal{U}$ be a projection such that $\dim(P\mathcal{U}P) > 1$. There exists a non-zero projection $Q \in \mathcal{U}$ such that $P \geq Q$ and $P \neq Q$.*

Proof. Notice that $P\mathcal{U}P$ is clearly a vector subspace of $\mathcal{U}$ with identity P. Now, $P^2 = P$

$$(PAP)(PBP) = P(APB)P \in P\mathcal{U}P.$$

Furthermore, since $P^* = P$, we have

$$(PAP)^* = PA^*P \in P\mathcal{U}P.$$

Since $P\mathcal{U}P$ is a subspace of a finite dimensional space $\mathcal{U}$, it is complete and hence a C^*-algebra. Let now $A \in P\mathcal{U}P$ be self adjoint and suppose that $\sigma(A)$ is a singleton $\{\lambda\}$. Then consider the functions $f(z) = z$ and $g(z) = \lambda$ on $\sigma(A)$. Relative to $\sigma(A)$, these are the same functions. But then, the first part of this final yields that

$$f(A) = A, \quad g(A) = \lambda P$$

since P is the identity of $P\mathcal{U}P$. This would imply also that $A = \lambda P$. Therefore, if every self-adjoint element of $P\mathcal{U}P$ had a singleton spectrum, then, since every such A would look like λP, we would have

$$P\mathcal{U}P = \{\lambda P : \lambda \in \mathbb{C}\}.$$

Indeed, this is because any $B \in P\mathcal{U}P$ can be written as $A_1 + iA_2$ for self adjoint A_1, A_2. But this contradicts the fact that $P\mathcal{U}P$ has dimension larger than 1. Hence, there exists a self adjoint element $A \in P\mathcal{U}P$ having two distinct points $\lambda, \lambda' \in \sigma(A)$. Following the notation of the first proposition, consider non-zero corresponding projections P_λ, P'_λ such that $P_\lambda P_{\lambda'} = 0$. Now, because $P_\lambda \in P\mathcal{U}P$ we can express it as

$$P_\lambda = PBP, \quad B \in \mathcal{U}$$

whence

$$PP_\lambda = PPBP = PBP = P_\lambda.$$

Likewise, we see that $PP'_\lambda = P_{\lambda'}$. In any case, we have shown that $P \geq P_\lambda$. Now, suppose that $P = P_\lambda$. Then, by what we have shown above

$$P_{\lambda'} = PP_{\lambda'} = 0$$

which contradicts our choice of $P_{\lambda'}$. Hence, $P \neq P_\lambda$ and the proposition has been proven. $\square$

Proposition 2.4. *There exists a projection P in $\mathcal{U}$ such that no non-zero projection $\theta \neq P$ satisfies $\theta \leq P$. In a sense, there exists a "minimal" non-zero projection.*

Proof. Since $\mathcal{U}$ has an identity $\mathbb{1}$ there will always exist a projection in $\mathcal{U}$. Let now P be a projection in $\mathcal{U}$ and assume that $\dim(P\mathcal{U}P) > 1$. There exists non-zero $Q \neq P$ with $Q \leq P$, by the previous proposition. Also, $Q \in P\mathcal{U}P$. Indeed, $Q \leq P$ implies that

$$PQ = Q \quad \text{and} \quad Q\mathcal{U}Q \subseteq P\mathcal{U}P.$$

Hence, $Q\mathbb{1}Q = Q^2 = Q$ lives in $P\mathcal{U}P$. Also, we must have

$$Q\mathcal{U}Q \subsetneq P\mathcal{U}P.$$

If we had equality, then it would follow that $Q \geq P$ which contradicts $P \neq Q$. This also yields $\dim(Q\mathcal{U}Q) < \dim(P\mathcal{U}P)$. We may repeat this procedure finitely many times to obtain a projection R such that $\dim(R\mathcal{U}R) = 1$. We claim that R is minimal. Suppose we can find a projection $\theta \leq R$ with $\theta \neq R$. But then we see that $\theta\mathcal{U}\theta \subsetneq R\mathcal{U}R$ so that $\dim(\theta\mathcal{U}\theta) = 0$. Hence, $\theta = 0$. This shows that R is minimal. $\square$

The argument used in this proof gives the following.

Corollary 1. *P is minimal if and only if $P\mathcal{U}P$ has dimension* 1.

Proposition 2.5. *Let P be a non-zero minimal projection and consider the vector space $\mathcal{U}P$. If $A, B \in \mathcal{U}P$ then $A = AP$ and $B = BP$. Indeed, $A = CP$ for $C \in U$ so that*

$$AP = CPP = CP = A.$$

Moreover,

$$A^*B = PA^*BP \in P\mathcal{U}P.$$

By minimality, $\dim(P\mathcal{U}P) = 1$ *and so $A^*B = \lambda P$ for a unique $\lambda \in \mathbb{C}$. We claim that*

$$\langle A, B\rangle = \lambda$$

is an inner product on $\mathcal{U}P$. Hence, $\mathcal{U}P$ is a finite dimensional inner-product space. Therefore, $\mathcal{U}P$ is a Hilbert space over $\mathbb{C}$.

Proof. By definition, $\langle A, B\rangle P = \lambda P = A^*B$. Therefore,

$$\langle A, \alpha B_1 + \beta B_2\rangle P = A^*(\alpha B_1 + \beta B_2) = \alpha A^* B_1 + \beta A^* B_2 = \alpha\langle A, B_1\rangle P + \beta\langle A, B_2\rangle P.$$

Since $P \neq 0$, this gives

$$\langle A, \alpha B_1 + \beta B_2\rangle = \alpha\langle A, B_1\rangle + \beta\langle A, B_2\rangle.$$

But, also, $\langle A, A\rangle P = A^*A \geq 0$. In particular, $\langle A, A\rangle = 0$ if and only if $A^*A = 0$ which is equivalent to $\|A\| = 0$. Now, notice that P is positive since

$$P = P^2 = PP = P^*P.$$

Let then $\lambda := \langle A, A\rangle$ so that $\lambda P = A^*A \geq 0$. Since $\sigma(\lambda P) = \lambda\sigma(P) \subseteq [0, \infty)$, it follows from the positivity of P that $\lambda \geq 0$. That is, $\langle A, A\rangle \geq 0$ as was required. □

Proposition 2.6. *Let P be a minimal non-zero projection. Then*

$$\mathcal{U}P\mathcal{U} \stackrel{def}{=} \operatorname{Span}\left(\{APB : A \in \mathcal{U}, B \in \mathcal{U}\}\right)$$

is a unital C^*-subalgebra of $\mathcal{U}$.

Proof. Clearly, $\mathcal{U}P\mathcal{U}$ is closed under scalar multiplication and vector addition; this makes $\mathcal{U}P\mathcal{U}$ a subspace of a finite dimensional vector space ($\mathcal{U}$) and so it must also be finite dimensional. As a vector subspace of a finite dimensional space, it is topologically closed and thus complete. It is clear from the definitions that $\mathcal{U}P\mathcal{U}$ is closed under multiplication and involution. This makes $\mathcal{U}P\mathcal{U}$ into a C^*-algebra.

Now, we have shown that $\mathcal{U}P$ is a finite dimensional inner product space with respect to the inner product introduced in the previous proposition. Since it has finite dimension, it is a Hilbert space. Let $C_1, \ldots, C_n$ be an orthonormal basis for $\mathcal{U}P$. Define

$$Q := \sum_{j=1}^{n} C_j C_j^*.$$

Now, $C_j \in \mathcal{U}P$. This means that $C_j = C_j P$ so that $C_j^* = P^* C_j^* = PC_j^*$. Therefore,

$$C_j C_j^* = C_j PPC_j^* = C_j PC_j^* \in \mathcal{U}P\mathcal{U}.$$

Especially, $C_j PC_j^* \in \mathcal{U}P\mathcal{U}$. Hence, $Q \in \mathcal{U}P\mathcal{U}$. Let now $A \in \mathcal{U}P$ be given and write

$$QA = \sum_{j=1}^{n} C_j C_j^* A = \sum_{j=1}^{n} C_j \left\langle C_j, A\right\rangle P = \sum_{j=1}^{n} \left\langle C_j, A\right\rangle C_j P = \sum_{j=1}^{n} \left\langle C_j, A\right\rangle C_j = A.$$

Here we have used that $C_j = C_jP$, by $C_j \in \mathcal{U}P$. This means that $QA = A$ for all $A \in \mathcal{U}P$. If $A \in \mathcal{U}P$ and $C \in \mathcal{U}$ this gives also

$$QAC = AC$$

whence $QA = A$ for all $A \in \mathcal{U}P\mathcal{U}$. Thus, Q is a left identity. Taking adjoints, we get

$$A^*Q^* = A^*, \quad \forall A \in \mathcal{U}P\mathcal{U}.$$

But, it is obvious that $Q^* = Q$ and so Q is also a right identity. This concludes the proof. □

Proposition 2.7. *Let P be a minimal non-zero projection. then $\mathcal{U}P\mathcal{U}$ is isomorphic to $\mathcal{B}(\mathcal{U}P)$.*

Proof. We construct a map $\pi : \mathcal{U}P\mathcal{U} \to \mathcal{B}(\mathcal{U}P)$. For an element $A \in \mathcal{U}P\mathcal{U}$ we define an operator

$$\pi(A) : \mathcal{U}P \to \mathcal{U}P$$

by $\pi(A)a := Aa$. Clearly, this is well defined, linear, and bounded. Hence, π is well defined. Clearly, $\pi(A + B) = \pi(A) + \pi(B)$ and also $\pi(A) \circ \pi(B) = \pi(AB)$. Moreover, for all $a, b \in \mathcal{U}P$ one has

$$\langle a, \pi(A^*)b\rangle P = a^*\pi(A^*)b = a^*A^*b = \langle Aa, b\rangle P = \langle \pi(A)a, b\rangle P.$$

Since $P \neq 0$, this gives that

$$\langle a, \pi(A^*)b\rangle = \langle \pi(A)a, b\rangle$$

and so $\pi(A)^* = \pi(A^*)$. This means that π is a $*$-morphism. It hence only remains to ensure that π is a bijection. Now, an arbitrary element of $\mathcal{U}P\mathcal{U}$ is a finite sum of terms having the form APB. Since AP, B^*P live in $\mathcal{U}P$ we may write them as

$$AP = \sum_{j=1}^{n} \alpha_j C_j, \quad B^*P = \sum_{j=1}^{n} \beta_j C_j.$$

Or, rather,

$$PB = (B^*P)^* = \sum_{j=1}^{n} \overline{\beta_j} C_j^*.$$

Now, since $P^2 = P$ it follows from the above that

$$APB = (AP)(PB) = \sum_{i,j=1}^{n} \gamma_{i,j} C_i C_j^*.$$

Thus, $\mathcal{U}P\mathcal{U}$ is spanned by the elements $C_iC_j^*$ for $i, j = 1, \dots, n$. We now argue that these elements are linearly independent. To this end, suppose that one has the relation $\sum_{i,j=1}^{n} \gamma_{i,j} C_i C_j^* = 0$ for $\gamma_{i,j} \in \mathbb{C}$. Then, for $a \in \mathcal{U}p$ one has that

$$0 = \sum_{i,j=1}^{n} \gamma_{i,j} C_i C_j^* a = \sum_{i,j=1}^{n} \gamma_{i,j} C_i \left\langle C_j, a \right\rangle P.$$

In the above, take $a := C_k$ for any $1 \leq k \leq n$ thereby yielding

$$0 = \sum_{i=1}^{n} C_i \sum_{j=1}^{n} \gamma_{i,j} \langle C_j, C_k \rangle P = \sum_{i=1}^{n} \gamma_{i,k} C_i P = \sum_{i=1}^{n} \gamma_{i,k} C_i.$$

By the linear independence of the C_i, it follows that $\gamma_{i,k} = 0$ for all i. Letting k vary also gives that $C_iC_j^*$ are linearly independent over $\mathbb{C}$. Hence, $C_iC_j^*$ forms a *basis* for $\mathcal{U}P\mathcal{U}$.

Now, notice that for $a \in \mathcal{U}P$

$$\pi(C_iC_j^*) = C_iC_j^* a = \left\langle C_j, a \right\rangle C_i.$$

This means that $\pi(C_iC_j^*) \equiv \langle C_j, \cdot \rangle C_i$. Let now $L : \mathcal{U}P \to \mathcal{U}P$ be a bounded linear operator. Then, this may be realized as

$$L(a) = \ell_1(a)C_1 + \dots + \ell_1(a)C_n$$

for bounded[2] linear functionals $\ell : \mathcal{U}P \to \mathbb{C}$. Now, by the Riesz-Representation theorem, every ℓ_k may be represented as

$$\ell_k(\cdot) \equiv \langle b, \cdot \rangle, \quad b \in \mathcal{U}P.$$

Now, this b will be itself a linear combination of the form $b = \sum_{j=1}^{n} \alpha_j C_j$. All of these facts combined show that the space $\mathcal{B}(\mathcal{U}P)$ is spanned by the collection

$$\left\{ \left\langle C_j, \cdot \right\rangle C_i \right\}_{i,j=1}^{n}.$$

However, this family contains n^2-elements. Since $\mathcal{B}(\mathcal{U}P)$ is isomorphic to $M_n(\mathbb{C})$ (in the sense of vector spaces), its dimension over $\mathbb{C}$ is n^2. Since we have found a spanning set having no larger than n^2-elements, it must be a linearly independent family (for otherwise we could remove an element, which would contradict $\dim(\mathcal{U}P) \leq n^2$). It follows that π is bijective and an isomorphism.

$\square$

[2]Linear functionals defined on finite dimensional complex vector spaces are always continuous, and hence bounded.

We now briefly move on towards a useful definition. A collection $\{P_1, \ldots, P_n\}$ of minimal non-zero projections is called **independent** if $P_i \mathcal{U} P_j = \{0\}$ whenever $i \neq j$. We will now show that any such collection is linearly independent and must satisfy $n \leq \dim(\mathcal{U})$.

Indeed, assume that $P_1, \ldots, P_n$ are linearly *dependent*. There must then exist a non-trivial linear combination equal to zero. Without loss of generality, we will assume P_1 is a non-trivial linear combination of $P_2, \ldots, P_n$. Then for suitable $\alpha_j \in \mathbb{C}$ one may write

$$P_1 = \alpha_2 P_2 + \cdots + \alpha_n P_n.$$

This implies that

$$P_1 \mathcal{U} P_1 = P_1 \mathcal{U}(\alpha_2 P_2 + \cdots + \alpha_n P_n) = \{0\}.$$

However, this implies that $P_1 \mathbb{1} P_1 = P_1 = 0$ which is absurd. Hence, these are linearly independent. As a result, we must clearly have $n \leq \dim(\mathcal{U})$. Let now $\{P_1, \ldots, P_n\}$ be a maximal collection of non-zero *independent* minimal projections.

Proposition 2.8. *$\mathcal{U}$ is isomorphic (as a C^*-algebra) to the direct sum $\bigoplus_{j=1}^n \mathcal{U} P_i \mathcal{U}$. Moreover, invoking the previous proposition, one therefore has that*

$$\mathcal{U} \cong \bigoplus_{j=1}^{n} \mathcal{U} P_j \mathcal{U} \cong \bigoplus_{j=1}^{n} \mathcal{B}(\mathcal{U} P_j).$$

Proof. We recall that $\mathcal{U} P_k \mathcal{U}$ is a C^*-algebra having an identity Q_k. If $A \in \mathcal{U} P_i \mathcal{U}$ and $B \in \mathcal{U} P_j \mathcal{U}$ for $i \neq j$ then

$$A = \sum_{k=1}^{N_1} A_k^{(1)} P_i B_k^{(1)}, \quad B = \sum_{l=1}^{N_2} A_l^{(2)} P_j B_l^{(2)}.$$

Therefore, since $i \neq j$.

$$AB = \sum_{k,l} A_k^{(1)} \underbrace{\left(P_i B_k^{(1)} A_l^{(2)} P_j\right)}_{=0} B_l^{(2)} = 0.$$

Let now $Q := \mathbb{1} - \sum_1^n Q_j$. Now, notice that

$$Q^2 = \left(\mathbb{1} - \sum\nolimits_{j=1}^n Q_j\right)\left(\mathbb{1} - \sum\nolimits_{j=1}^n Q_j\right) = \mathbb{1} - 2\sum_{j=1}^n Q_j + \sum_{i,j=1} Q_i Q_J = \mathbb{1} - 2\sum_{j=1}^n Q_j + \sum_{j=1}^n Q_j^2$$
$$= \mathbb{1} - 2\sum_{j=1}^n Q_j + \sum_{j=1}^n Q_j$$
$$= \mathbb{1} - \sum_{j=1}^n Q_j$$
$$= Q.$$

Therefore, Q is a projection. If Q is non-zero as well then we can choose (by the proof of Proposition 2.4) a minimal non-zero projection P such that $Q \geq P$. Now, since $P \leq Q$ we may write for any index k

$$P\mathcal{U}P_k = PQ\mathcal{U}P_k = PQQ_k\mathcal{U}P_k;$$

here we have used that Q_k acts as an identity on $\mathcal{U}P_k\mathcal{U} \supseteq P_k\mathcal{U}$. However,

$$QQ_k = Q_k - \sum_{j=1}^n Q_jQ_k = Q_k - Q_k^2 = Q_k - Q_k = 0$$

whence $P\mathcal{U}P_k = \{0\}$, for all indices k. Thus, $\{P, P_1, \ldots, P_n\}$ is independent which contradicts the maximality of $\{P_1, \ldots, P_n\}$. This contradiction gives $Q = 0$ which implies

$$\sum_{j=1}^n Q_j = \mathbb{1}.$$

Given $A \in \mathcal{U}$, we have $A = \mathbb{1}A = \sum_{j=1}^n Q_jA =: \sum_{j=1}^n A_j$. Furthermore, we have that

$$A_j = Q_jA = \sum_{k=1}^m C_{j,k}C_{j,k}^*A = \sum_{k=1}^m C_{j,k}P_jC_{j,k}^*A \in \mathcal{U}P_j\mathcal{U}$$

where $\{C_{j,k}\}_k$ is a basis for $\mathcal{U}P_j$. Here we may add this factor of P_j into the summand since $C_{j,k} \in \mathcal{U}P_j$ whence

$$C_{j,k} = c_{j,k}P_j = c_{j,k}P_j^2 = C_{j,k}P_j$$

for some $c_{j,k} \in \mathcal{U}$. This shows that every element $A \in \mathcal{U}$ may be written as a sum $A_1 + \cdots + A_n$ where $A_j \in \mathcal{U}P_j\mathcal{U}$. This decomposition is also unique. Indeed, if A may also be expressed as

$$A = B_1 + \cdots + B_n, \quad B_j \in \mathcal{U}P_j\mathcal{U}$$

then $0 = (A_1 - B_1) + \cdots + (A_n - B_n)$ with $A_j - B_j \in \mathcal{U}P_j\mathcal{U}$. Therefore, uniqueness follows if we can show that

$$0 = A_1 + \cdots + A_n, \quad A_j \in \mathcal{U}P_j\mathcal{U}$$

implies $A_j = 0$ for all j. However, multiplying through by Q_j gives $0 = A_jQ_j = A_j$ which is precisely this result. Therefore, the map

$$\pi : \mathcal{U} \to \bigoplus_{j=1}^{n} \mathcal{U}P_j\mathcal{U}, A \mapsto (A_1, \cdots, A_n)$$

is linear (clearly) and also a bijection. We now argue that π is a $*$-morphism. Since the families $\mathcal{U}P_i\mathcal{U}$ are "orthogonal", if we take $A, B \in \mathcal{U}$ and write them as

$$A = A_1 + \cdots A_n, \quad B = B_1 + \cdots + B_n, \quad A_j, B_j \in \mathcal{U}P_j\mathcal{U},$$

it follows that

$$AB = A_1B_1 + \cdots + A_nB_n.$$

Thus, $\pi(AB) = \pi(A)\pi(B)$. In just the same way,

$$\pi(A^*) = \pi(A_1^* + \cdots + A_n^*) = (A_1^*, \ldots, A_n^*)$$

so that $\pi(A^*) = \pi(A)^*$. Since π is injective and a $*$-morphism, it preserves norm and the proof is complete. $\square$