On the Global Compactness of Positive Palais-Smale Sequences for p-Laplace Equations with Critical Nonlinearities in Smoothly Bounded Domains

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Abstract In these notes, we give a mostly self-contained treatment of a global compactness result for p-Laplace equations with critical nonlinearities in $\mathbb{R}^N$. Namely, we will prove a representation theorem for positive Palais-Smale sequences that was first established by Meruci-Willem in 2010.

Contents

1 Introduction

In these notes, we discuss some recent developments regarding the global compactness of Palais-Smale sequences for p-Laplace equations with critical nonlinearities. More precisely, we shall investigate the limiting behaviour of minimizing sequences to energy functionals for a large class of critical p-Laplace equations. In particular, we introduce an argument used by Meruci-Willem [5] which simplifies the approach taken by Struwe in [8] and [9] for the $p = 2$ case. Put informally, the key step in this new argument will be to focus on the dual space of $\mathcal{D}^{1,p}(\mathbb{R}^N)$, a homogeneous Sobolev space, rather than the "averaged behaviour" of the given Palais-Smale sequence.

Global compactness results are now indispensable, and highly sought after tools, for the analysis of partial differential equations. Indeed, they can be used to establish the existence of non-trivial solutions to various problems (e.g. blow-up solutions to Schrödinger equations and solutions to Yamabe equations – see Palatucci-Pisante [6] for more details). However, despite their numerous applications, these compactness results are interesting in their own right. Fundamentally, they describe an asymptotic expansion of the Palais-Smale sequence in the energy space $\mathcal{D}^{1,p}(\mathbb{R}^N)$, which we shall introduce shortly.

Henceforth, $N \geq 3$ is an integer, $\mu > 0$ is any real number, and Ω is a bounded domain in $\mathbb{R}^N$ having smooth boundary $\partial\Omega$. Unless stated otherwise, we will assume that $1 < p < N$. We denote by p' the conjugate Hölder exponent of p, and by p^* the Sobolev conjugate of p, i.e. p' and p^* satisfy:

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{and} \quad p^* := \frac{Np}{N-p}. \tag{1.1}$$

Note that $p' \in (1, \infty)$ and that $p^* > p$. Finally, Δ_p stands for the p-Laplace operator:

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = \nabla \cdot \left(|\nabla u|^{p-2}\,\nabla u\right), \tag{1.2}$$

which we interpret in the weak sense.[1]

Before proceeding further, we will require some machinery from functional analysis and several simple preliminary tools.

1. Let U be an open set in $\mathbb{R}^N$ and assume that u is a sufficiently regular function satisfying $\Delta_p u = 0$ in U. Testing against $\varphi \in C_c^\infty(U)$, an integration by parts shows that

$$0 = \int_U \varphi \Delta_p u = \int_U \varphi \operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = -\int_U |\nabla u|^{p-2}\,\nabla u \cdot \nabla\varphi. \tag{F.1}$$

In light of this, we say that $u \in W^{1,p}(U)$ satisfies $\Delta_p u = 0$ in U if the expression in (F.1) vanishes for all $\varphi \in C_c^\infty(U)$.

1.1 The Setup

In what follows, $\mathcal{X}$ and $\mathcal{Y}$ each denote normed vector spaces over the real numbers and $A : \mathcal{X} \to \mathcal{Y}$ denotes an operator, not necessarily linear. For the sake of convenience, $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ is the real vector space of all bounded linear operators $\mathcal{X} \to \mathcal{Y}$. If $\mathcal{Y}$ is complete, then $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ is a Banach space. Let us now reiterate the notion of a Fréchet derivative:

Definition 1. An operator $A : \mathcal{X} \to \mathcal{Y}$ is said to be *Fréchet differentiable* at a point $x \in \mathcal{X}$ if there exists a bounded linear operator $DA(x) : \mathcal{X} \to \mathcal{Y}$ such that

$$\lim_{\substack{h \to 0 \\ h \in \mathcal{X}}} \frac{\|A(x+h) - A(x) - DA(x)h\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} = 0.$$

The operator $DA(x)$ is called the *Fréchet derivative* of A at x. The *operator valued* operator

$$DA : \mathcal{X} \to \mathscr{L}(\mathcal{X}, \mathcal{Y})$$

given by $x \mapsto DA(x)$ is called the *Fréchet derivative* of A, whenever it exists. If this operator exists, we say that A is *Fréchet differentiable* on $\mathcal{X}$. In the case that DA is continuous, we write $A \in C^1(\mathcal{X}, \mathcal{Y})$.

It is not hard to see that a Fréchet differentiable operator $\mathcal{X} \to \mathcal{Y}$ is automatically continuous. Having given this definition, we are ready to formulate the setup of our main result. For N, p, μ and Ω as above, we are interested in solutions to the problem

$$\begin{cases} -\Delta_p u + a\,|u|^{p-2}\,u \equiv \mu\,|u|^{p^*-2}\,u & \text{in } \Omega, \\ u \geq 0 & \text{a.e. in } \Omega, \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{1.3}$$

where $a \in L^{N/p}(\mathbb{R}^N)$ is arbitrary, but fixed. Naturally, the above is simply the weak formulation of the problem

$$\begin{cases} -\Delta_p u + a\,|u|^{p-2}\,u \equiv \mu\,|u|^{p^*-2}\,u & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u \equiv 0 & \text{on } \partial\Omega \end{cases}$$

with the boundary condition interpreted in the trace sense. When studying the problem above, solutions to the following *limiting problem* will arise naturally:

$$\begin{cases} -\Delta_p u \equiv \mu\,|u|^{p^*-2}\,u & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{1.4}$$

Next, let us consider the functional on $W_0^{1,p}(\Omega)$ given by:

$$\phi : W_0^{1,p}(\Omega) \to \mathbb{R}, \quad u \mapsto \int_\Omega \left(\frac{|\nabla u|^p}{p} + a\frac{|u|^p}{p} - \mu \frac{|u|^{p^*}}{p^*} \right). \tag{1.5}$$

Through elementary techniques, it can be shown that ϕ is Fréchet differentiable on $W_0^{1,p}(\Omega)$ (and hence continuous on this space) with derivative equal to

$$\langle \phi'(u), h \rangle = \int_\Omega \left(|\nabla u|^{p-2} \nabla u \cdot \nabla h + a\,|u|^{p-2}\,uh - \mu\,|u|^{p^*-2}\,uh \right), \tag{1.6}$$

for all $u, h \in W_0^{1,p}(\Omega)$. The function ϕ is the energy functional associated with (1.3), and the *functional-valued* operator ϕ' will be how we test to see whether or not a function $u \in W_0^{1,p}(\Omega)$ satisfies (1.3) in Ω. Much like in (F.1), $u \in W_0^{1,p}(\Omega)$ will be called a *weak solution* to the critical equation

$$-\Delta_p u + a\,|u|^{p-2}\,u \equiv \mu\,|u|^{p^*-2}\,u \quad \text{in } \Omega \tag{1.7}$$

provided $\phi'(u)$ vanishes on all of $W_0^{1,p}(\Omega)$. On the other hand, to properly define the energy functional associated to (1.4), we must first introduce the notion of a "homogeneous Sobolev space".

Definition 2. Given an open set $U \subseteq \mathbb{R}^N$, we define $\mathcal{D}^{1,p}(U)$ to be those functions $u \in L^{p^*}(U)$ having weak derivatives in $L^p(U)$. More precisely,

$$\mathcal{D}^{1,p}(U) := \left\{ u \in L^{p^*}(U) : \nabla u \in L^p(U; \mathbb{R}^N) \right\}.$$

We then equip $\mathcal{D}^{1,p}(U)$ with the natural seminorm

$$\|u\| := \|\nabla u\|_{L^p(U)}, \quad \forall u \in \mathcal{D}^{1,p}(U).$$

For general open sets U in $\mathbb{R}^N$, our definition makes $\mathcal{D}^{1,p}(U)$ into a real locally convex topological vector space. When $U = \mathbb{R}^N$, it is easy to see that $\|\cdot\|$ is actually a *norm* on $\mathcal{D}^{1,p}(U)$, and thus $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is a real normed space. Finally, we denote by $\mathcal{D}_0^{1,p}(U)$ the closure of $C_c^\infty(U)$ in $\mathcal{D}^{1,p}(U)$. Note also that $W^{1,p}(\mathbb{R}^N)$ is a subset of $\mathcal{D}^{1,p}(\mathbb{R}^N)$ by the Sobolev embedding theorem.

Remark 1.1. A priori, it is not clear that $\mathcal{D}^{1,p}(\mathbb{R}^N)$ should be complete. After all, if (u_n) is a Cauchy sequence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, then (∇u_n) is Cauchy (and thus convergent) in $L^p(\mathbb{R}^N; \mathbb{R}^N)$. However, it is not obvious that this forces (u_n) to also be Cauchy in $L^{p^*}(\mathbb{R}^N)$, nor is it obvious that the would-be $L^{p^*}(\mathbb{R}^N)$ limit of (u_n) is weakly differentiable. Consequently, we encounter some difficulties when trying to show that (u_n) converges to *something* in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, as the $\mathcal{D}^{1,p}(\mathbb{R}^N)$-norm tests only the convergence of the gradients. But, by using a clever trick from Willem [10], we will later show that $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is in fact a reflexive Banach space.

Remark 1.2. Let $U \subset \mathbb{R}^N$ be a bounded domain with C^1-boundary. Then, the Sobolev inequality forces $W^{1,p}(U) \subseteq \mathcal{D}^{1,p}(U)$. On the other hand, $f \in L^{p^*}(U)$ implies $f \in L^p(U)$. Hence, $\mathcal{D}^{1,p}(U) \subseteq W^{1,p}(U)$. That is, $W^{1,p}(U)$ and $\mathcal{D}^{1,p}(U)$ consist of precisely the same functions. However, they will not possess the same topology. Indeed, this is because the Friedrich-Poincaré inequality may fail for those functions in $W^{1,p}(\Omega)$ having non-zero trace.

When dealing with the limiting problem (1.4), we will be focused on weak solutions belonging to the space $\mathcal{D}^{1,p}(\mathbb{R}^N)$. To formalize this, we introduce (as with problem (1.3)) an energy functional

$$\phi_\infty : \mathcal{D}^{1,p}(\mathbb{R}^N) \to \mathbb{R}, \quad u \mapsto \int_{\mathbb{R}^N} \left(\frac{|\nabla u|^p}{p} - \mu \frac{|u|^{p^*}}{p^*} \right).$$

This functional is Fréchet differentiable on $\mathcal{D}^{1,p}(\mathbb{R}^N)$ with derivative given by

$$\langle \phi_\infty'(u), h \rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla h - \mu |u|^{p^*-2} uh \right),$$

for $u, h \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. As before, we will say that $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ is a weak solution to the limiting problem

$$-\Delta_p u \equiv \mu |u|^{p^*-2} u \quad \text{in } \mathbb{R}^n \tag{1.8}$$

if $\phi_\infty'(u) = 0$ on all of $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Finally, we will also be treating solutions to the following problem:

$$\begin{cases} -\Delta_p u \equiv \mu |u|^{p^*-2} u \text{ in a halfspace } \mathbb{H}, \\ u \geq 0 \text{ a.e. in } \mathbb{H}, \\ u \in \mathcal{D}_0^{1,p}(\mathbb{H}), \end{cases} \tag{1.9}$$

which has a similar weak interpretation. To state our main result, we need only one last definition from the calculus of variations and critical point theory.

Definition 3. Let (u_n) be a sequence in $W_0^{1,p}(\Omega)$. We say that (u_n) is a *Palais-Smale* sequence (or simply a P.S.-sequence) for ϕ if (u_n) is bounded, $\phi(u_n)$ converges in $\mathbb{R}$, and

$$\phi'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega)$$

as $n \to \infty$. Here, $W^{-1,p'}(\Omega)$ denotes the topological dual of $W_0^{1,p}(\Omega)$, which is itself a Banach space.

As it turns out, the requirement that (u_n) be bounded is redundant. It can be shown (see Struwe [8] for a proof in the $p = 2$ case, or Willem [11] for the case $2 < p < 2^*$) that any sequence (u_n) in $W_0^{1,p}(\Omega)$ such that $\phi(u_n)$ converges and $\phi'(u_n) \to 0$ is necessarily bounded. However, since our goal is to illustrate the method of proof used by Mercuri-Willem, it will not harm us to assume a priori that a Palais-Smale sequence is bounded.

Having given this critical definition, a coherent formulation of our main result (Theorem 2 in Mercuri-Willem [5]) is within reach.

Theorem 1.1 (Mercuri-Willem)**.** *Let (u_n) be a Palais-Smale sequence for ϕ and assume additionally that*

$$(u_n)_- \to 0 \quad \text{in } L^{p^*}(\Omega) \quad \text{as } n \to \infty,$$

where $(u_n)_- := \max\{-u_n, 0\}$. After possibly passing to a subsequence, there exists a weak solution $v_0 \in W_0^{1,p}(\Omega)$ to the problem (1.3) and a finite (possibly empty) family $v_1, \dots, v_k$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ of weak solutions[2] to the limiting problem (1.4), together with associated sequences $(y_n^i)_{n\in\mathbb{N}} \subset \Omega$ and $(\lambda_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$\left\| u_n - v_0 - \sum\nolimits_{i=1}^k \left(\lambda_n^i\right)^{\frac{p-N}{p}} v_i\left(\frac{\cdot - y_n^i}{\lambda_n^i}\right)\right\| \to 0 \quad \text{as } n \to \infty, \tag{1.10}$$

$$\|u_n\|^p \to \sum_{i=0}^k \|v_i\|^p \quad \text{as } n \to \infty, \tag{1.11}$$

$$\phi(v_0) + \sum_{i=1}^k \phi_\infty(v_i) = \lim_{n\to\infty} \phi(u_n). \tag{1.12}$$

Moreover, there holds

$$\frac{\operatorname{dist}(y_n^i, \partial\Omega)}{\lambda_n^i} \to \infty, \quad \text{as } n \to \infty$$

for each $i = 1, \dots, k$.

Remark 1.3. For simplicity let us momentarily take $\mu = 1$; we then have a complete

2. These $v_1, \dots, v_k$ are often referred to as *bubbles* – see Figure 1.

classification of the bubbles. Namely, they all take the form

$$\left[\frac{\lambda^{\frac{1}{p-1}} N^{\frac{p-1}{p^2}} \left(\frac{N-p}{p-1}\right)^{1/p'}}{\lambda^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}}}\right]^{\frac{N-p}{p}} \quad \text{for some } x_0 \in \mathbb{R}^N \text{ and } \lambda > 0. \tag{1.13}$$

This classification was established by

- Caffarelli-Gidas-Spruck in 1989 for $p = 2$;
- Damascelli-Merchán-Montoro-Sciunzi in 2014 for $\frac{2N}{N+2} \leq p < 2$;
- Vétois 2016 + Damascelli-Ramaswamy in 2001 for the case $1 < p < \frac{2N}{N+2}$;
- Sciunzi 2015 for $2 \leq p < N$.

For more history on this classification result, we urge the reader to consult the introduction of the paper [7] of Sciunzi.

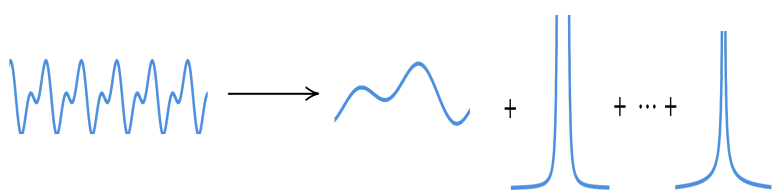

FIGURE 1: An illustration of the convergence occurring in Theorem 1.1. In the energy space $\mathcal{D}^{1,p}(\mathbb{R}^N)$, a subsequence of (u_n) converges to a solution of (1.3) and finitely many "bubbles" solving the limiting problem (1.4).

As previously stated, the argument we will use when proving Theorem 1.1 follows to a tee that outlined in Theorem 1.2 of Mercuri-Willem [5]. However, this argument has its roots in a paper of Brézis-Coron [3] and was later refined in the book "Minimax Methods" of Willem (where the case $2 < p < 2^*$ was treated – see [11]).

2 *Tools from Measure Theory and Functional Analysis*

In this section, we establish several results that will be used freely throughout the proof of Theorem 1.1 and its supporting lemmas. Although not directly related to the problem at hand, they are necessary tools that can be proven using only some elementary concepts from measure theory and functional analysis. We begin with a *very* useful convergence result reminiscent of the dominated convergence theorem. Throughout this section, we follow in large part both Willem [10] and Ziemer [12].

Theorem 2.1 (p-Bounded Convergence Theorem)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space, fix $1 < p < \infty$, and let (f_n) be a bounded sequence in $L^p(X, \mu)$. Let $f : X \to \mathbb{C}$ be measurable and assume that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for μ-a.e. $x \in X$. Then, $f \in L^p(X, \mu)$ and

$$\lim_{n\to\infty} \int_X |(f_n - f)\, g| \,\mathrm{d}\mu = 0$$

for every $g \in L^{p'}(X, \mu)$. In particular,

$$\lim_{n\to\infty} \int_X f_n g \,\mathrm{d}\mu = \int_X f g \,\mathrm{d}\mu$$

for all $g \in L^{p'}(X, \mu)$. Moreover, the measurability assumption on f can be dropped if $(X, \mathfrak{M}, \mu)$ is complete.

Proof. Fix $g \in L^{p'}(X, \mu)$. As a first observation, we necessarily have $f \in L^p(X, \mu)$. Indeed, we see from Fatou's lemma that

$$\int_X |f|^p \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X |f_n|^p \,\mathrm{d}\mu \le \sup_{n\geqslant 1} \|f_n\|^p_{L^p(X,\mu)} < \infty.$$

Hence, we may assume without loss of generality that $f = 0$ in $L^p(X, \mu)$. Thus, it suffices to check that

$$\limsup_{n\to\infty} \int_X |f_n g| \,\mathrm{d}\mu \le 0.$$

To this end, let $\eta > 0$ be given. For $n \in \mathbb{N}$, consider the measurable set

$$E_n := \left\{ x \in X : |f_n(x) g(x)| \le \eta \, |g(x)|^{p'} \right\}.$$

From the dominated convergence theorem, it is clear that

$$\lim_{n\to\infty}\int_{E_n}|f_n(x)g(x)|\,\mathrm{d}\mu=\lim_{n\to\infty}\int_X|f_n(x)g(x)|\,\mathbf{1}_{E_n}(x)\,\mathrm{d}\mu=0 \tag{2.1}$$

because $|g|^{p'}\in L^1(X,\mu)$, $f_n\to 0$ almost everywhere, and $|f_n(x)g(x)|\,\mathbf{1}_{E_n}(x)\le\eta\,|g(x)|^{p'}$ on X. Let $M>0$ be such that $\|f_n\|_{L^p(X,\mu)}\le M$ for every $n\ge 1$. Since

$$|f_n(x)g(x)|>\eta\,|g(x)|^{p'}$$

for each $x\in E_n^{\mathrm{c}}$, an application of Hölder's inequality gives

$$\begin{aligned}\int_{E_n^{\mathrm{c}}}|f_ng|\,\mathrm{d}\mu&\le\|f_n\|_{L^p(E_n^{\mathrm{c}},\mu)}\left(\int_{E_n^{\mathrm{c}}}|g|^{p'}\,\mathrm{d}\mu\right)^{1/p'}\\&\le\eta^{-1/p'}\,\|f_n\|_{L^p(X,\mu)}\left(\int_{E_n^{\mathrm{c}}}|f_ng|\,\mathrm{d}\mu\right)^{1/p'}\\&\le M\eta^{-1/p'}\left(\int_{E_n^{\mathrm{c}}}|f_ng|\,\mathrm{d}\mu\right)^{1/p'}.\end{aligned}$$

So, for any $n\ge 1$, this implies

$$\left(\int_{E_n^{\mathrm{c}}}|f_ng|\,\mathrm{d}\mu\right)^{1/p}=\left(\int_{E_n^{\mathrm{c}}}|f_ng|\,\mathrm{d}\mu\right)^{1-1/p'}\le M\eta^{-1/p'}$$

Or, rather, that

$$\int_{E_n^{\mathrm{c}}}|f_ng|\,\mathrm{d}\mu\le M^p\eta^{-p/p'},\qquad\forall n\in\mathbb{N}.$$

Combining this with (2.1), it follows that

$$\limsup_{n\to\infty}\int_X|f_ng|\,\mathrm{d}\mu\le\limsup_{n\to\infty}\int_{E_n^{\mathrm{c}}}|f_ng|\,\mathrm{d}\mu\le M^p\eta^{-p/p'}.$$

Finally, sending $\eta\to\infty$ verifies the assertion. □

This implies the following useful convergence criterion, which can be thought of as a "local version" of Theorem 2.1.

Corollary 2.2. *Let $(X,\mathfrak{M},\mu)$ be a finite measure space and let $f:X\to\mathbb{C}$ be a measurable function. Assume that (f_n) is a sequence of measurable functions such that*

$$\lim_{n\to\infty}f_n(x)=f(x)$$

for μ-a.e. $x\in X$. If (f_n) is bounded in $L^p(X)$ for some $1<p<\infty$, then $f_n\to f$ in $L^1(X)$.

In particular, we see that one can overcome the need for a dominating function when passing the limit inside the integral, provided our sequence (f_n) is bounded in some "higher" L^p-space.

Lemma 2.3. *Let (X, d) be a compact metric space and denote by $C(X)$ the real vector space of all continuous maps $X \to \mathbb{R}$, endowed with the norm*

$$\|\cdot\|_\infty := \sup_{x \in X} |\cdot(x)| < \infty.$$

If $\mathcal{F} \subseteq C(X)$ is equicontinuous and equibounded, then

$$\sup_{f \in \mathcal{F}} f \in C(X).$$

Namely, $\sup_{f \in \mathcal{F}} f$ is continuous on X.

Proof. First, we define a function $s : X \to \mathbb{R}$ via the rule

$$s(x) := \sup_{f \in \mathcal{F}} f(x);$$

since the family $\mathcal{F}$ is equibounded, this is clearly a well defined function. Next, consider any two points $x, y \in X$. We have

$$\sup_{f \in \mathcal{F}} f(x) = \sup_{f \in \mathcal{F}} (f(x) - f(y) + f(y)) \leq \sup_{f \in \mathcal{F}} (f(x) - f(y)) + \sup_{f \in \mathcal{F}} f(y)$$

whence

$$s(x) - s(y) = \sup_{f \in \mathcal{F}} f(x) - \sup_{f \in \mathcal{F}} f(y) \leq \sup_{f \in \mathcal{F}} (f(x) - f(y)) \leq \sup_{f \in \mathcal{F}} |f(x) - f(y)|.$$

Similarly,

$$s(y) - s(x) = \sup_{f \in \mathcal{F}} f(y) - \sup_{f \in \mathcal{F}} f(x) \leq \sup_{f \in \mathcal{F}} (f(y) - f(x)) \leq \sup_{f \in \mathcal{F}} |f(x) - f(y)|.$$

Combining these two inequalities yields

$$|s(x) - s(y)| = \left| \sup_{f \in \mathcal{F}} f(x) - \sup_{f \in \mathcal{F}} f(y) \right| \leq \sup_{f \in \mathcal{F}} |f(x) - f(y)|.$$

Let now $\varepsilon > 0$ be given. By equicontinuity, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{F}$, whenever $d(x, y) < \delta$. For any such $x, y \in X$, we must therefore have

$$|s(x) - s(y)| = \left| \sup_{f \in \mathcal{F}} f(x) - \sup_{f \in \mathcal{F}} f(y) \right| \leq \sup_{f \in \mathcal{F}} |f(x) - f(y)| \leq \sup_{f \in \mathcal{F}} \varepsilon = \varepsilon.$$

This shows that $s(x)$ is continuous and the proof is complete. □

Our only application of this lemma will be the following.

Proposition 2.4. *Let $U \subseteq \mathbb{R}^N$ be non-empty and let $f \in L^1(\mathbb{R}^N)$. For $r \geq 0$, we define the Lévy concentration function of f as*

$$Q(r) := \sup_{y \in U} \int_{B(y,r)} |f| . \tag{2.2}$$

The function $Q(r)$ is continuous and bounded on $[0, \infty)$.

Proof. For fixed $y \in U$, we define

$$Q_y : [0, \infty) \to \mathbb{R}, \quad r \mapsto \int_{B(y,r)} |f| .$$

Since $f \in L^1(\mathbb{R}^N)$, each Q_y is continuous at 0, i.e.

$$\lim_{r \searrow 0} Q_y(r) = 0.$$

By virtue of Lemma 2.3, it is enough to show that the family $\mathcal{F} := \left\{ Q_y \right\}_{y \in U}$ is equicontinuous and equibounded on any compact interval $[a, b] \subset [0, \infty)$. To see that this family is equibounded, let $y \in U$ and $r \geq 0$ be arbitrary. Clearly, one has the following:

$$\left| Q_y(r) \right| = \int_{B(y,r)} |f| \leq \int_{\mathbb{R}^N} |f| < \infty.$$

It follows that $\mathcal{F}$ is equibounded on $[0, \infty)$, and hence on $[a, b]$. Next, we establish the equicontinuity of $\mathcal{F}$ on $[a, b]$. Let $\varepsilon > 0$ be given. Since $f \in L^1(\mathbb{R}^N)$, there exists $\delta > 0$ such that

$$\int_E |f| < \varepsilon$$

whenever $E \subset \mathbb{R}^N$ is Lebesgue measurable with $m(E) < \delta$.[3] Now, for any $r, q \in [a, b]$ and every $y \in U$, it is easily seen that

$$\left|Q_y(r) - Q_y(q)\right| = \left|\int_{B(y,r)} |f| - \int_{B(y,q)} |f|\right| = \int_{B(y,r)\setminus B(y,q)} |f|$$

where we are assuming, without loss of generality, that $r \geq q$. Letting ω_N be the volume of the unit ball in $\mathbb{R}^N$, we have

$$\begin{aligned} m(B(y,r) \setminus B(y,q)) &= \omega_n(r^N - q^N) \\ &= \omega_n(r^{N-1} + r^{N-2}q + \cdots + rq^{N-2} + q^{N-1})(r-q) \\ &\leq C(r-q), \end{aligned}$$

for a constant $C > 0$ independent of $r, q \in [a, b]$.[4] Therefore, we will have

$$m(B(y,r) \setminus B(y,q)) < \delta$$

whenever $0 \leq (r - q) < \frac{\delta}{C}$. This implies that

$$\left|Q_y(r) - Q_y(q)\right| \leq \int_{B(y,r)\setminus B(y,q)} |f| < \varepsilon$$

for any such $r, q \in [a, b]$. This shows that the family $\mathcal{F}$ is both equicontinuous and equibounded on every $[a, b] \subset [0, \infty)$ whence the proof is complete. $\square$

2.1 Motivating the Homogeneous Space $\mathcal{D}^{1,p}(\mathbb{R}^N)$

Before discussing in detail the homogeneous space $\mathcal{D}^{1,p}(\mathbb{R}^N)$, let us first motivate its definition. Namely, we ask ourselves why one should care about functions in $L^{p^*}(\mathbb{R}^N)$ having weak derivatives in $L^p(\mathbb{R}^N; \mathbb{R}^N)$. Put informally, $\mathcal{D}^{1,p}(\mathbb{R}^N)$ seeks to fully take advantage of the Gagliardo-Nirenberg-Sobolev inequality

$$\|\varphi\|_{L^{p^*}} \leq C \|\nabla\varphi\|_{L^p(\mathbb{R}^N)} \tag{2.3}$$

which is valid for all $\varphi \in C_c^1(\mathbb{R}^N)$, and a constant $C > 0$ independent of φ. Perhaps even more importantly, we want to take advantage of the nice rescaling properties present

3. Here, $m(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^N$.
4. Note that this constant C will depend on the interval $[a, b]$.

in the Gagliardo-Nirenberg-Sobolev inequality. More precisely, fix $\varphi \in C_c^1(\mathbb{R}^N)$ and let $\lambda > 0$ be given. Define a new function $\varphi_\lambda \in C_c^1(\mathbb{R}^N)$ via the rule $\varphi_\lambda(x) := \lambda^{(N-p)/p}\varphi(\lambda x)$ and note that

$$\int_{\mathbb{R}^N} |\nabla\varphi_\lambda|^p \,\mathrm{d}m = \int_{\mathbb{R}^N} \left|\lambda \cdot \lambda^{(N-p)/p}\nabla\varphi(\lambda x)\right|^p \mathrm{d}x = \int_{\mathbb{R}^N} \lambda^N \left|\nabla\varphi(\lambda x)\right|^p \mathrm{d}x$$
$$= \int_{\mathbb{R}^N} |\nabla\varphi|^p \,\mathrm{d}m.$$

Similarly, we have

$$\int_{\mathbb{R}} |\varphi_\lambda|^{p^*} \mathrm{d}m = \int_{\mathbb{R}} \lambda^{\frac{N-p}{p}p^*} |\varphi(\lambda x)|^{p^*} \mathrm{d}x = \int_{\mathbb{R}} \lambda^N |\varphi(\lambda x)|^{p^*} \mathrm{d}x$$
$$= \int_{\mathbb{R}} |\varphi|^{p^*} \mathrm{d}m.$$

Equivalently, $\|\varphi_\lambda\|_{L^{p^*}(\mathbb{R}^N)} = \|\varphi\|_{L^{p^*}(\mathbb{R}^N)}$ and $\|\nabla\varphi_\lambda\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} = \|\nabla\varphi\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}$. Such an invariance under a particular rescaling is known as *homogeneity*. In other words, the $\mathcal{D}^{1,p}(\mathbb{R}^N)$-norm is homogeneous with exponent $\frac{N-p}{p}$ and so is the L^{p^*}-norm. However, it is easily seen that the L^p-norm is *not* homogeneous with respect to this same exponent. For this reason, we chose to omit it when defining the $\mathcal{D}^{1,p}(\mathbb{R}^N)$-norm.

2.2 The Topology of $\mathcal{D}^{1,p}(\mathbb{R}^N)$

Let us now discuss the topology of the homogeneous space $\mathcal{D}^{1,p}(\mathbb{R}^N)$, which will be of particular interest to us. We have already given this space a norm:

$$\|u\| := \|\nabla u\|_{L^p(\mathbb{R}^N)},$$

but we may also give it the norm:

$$\|u\|_* := \|u\|_{L^{p^*}(\mathbb{R}^N)} + \|\nabla u\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^{p^*}(\mathbb{R}^N)} + \|u\|.$$

Recalling the proof that $W^{1,p}(\mathbb{R}^N) = W_0^{1,p}(\mathbb{R}^N)$, it becomes clear that $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_*$ (see, for instance, Willem [10]). Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and choose a sequence (φ_n) in $C_c^\infty(\mathbb{R}^N)$ such that

$$\lim_{n\to\infty} \|u - \varphi_n\|_* = 0.$$

Then, we have both

$$\lim_{n\to\infty} \|u - \varphi_n\|_{L^{p^*}(\mathbb{R}^N)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\nabla u - \nabla \varphi_n\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} = 0.$$

By continuity of norms, this combined with (2.3) gives

$$\begin{aligned} \|u\|_{L^{p^*}(\mathbb{R}^N)} = \lim_{n\to\infty} \|\varphi_n\|_{L^{p^*}} &\leq C \lim_{n\to\infty} \|\nabla \varphi_n\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} \\ &\leq C \|\nabla u\|_{L^p(\mathbb{R}^N)} . \end{aligned}$$

In other words, the Gagliardo-Nirenberg-Sobolev inequality holds for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. In fact, this shows that $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_*$ on $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Indeed, for any $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ there holds

$$\begin{aligned} \|u\| \leq \|u\|_* &= \|\nabla u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^{p^*}(\mathbb{R}^N)} \\ &\leq (C+1) \|\nabla u\|_{L^p(\mathbb{R}^N)} \\ &= C' \|u\| \end{aligned}$$

with $C' > 0$ a constant depending only on N and p. We summarize our findings in the following theorem:

Theorem 2.5 (Properties of $\mathcal{D}^{1,p}(\mathbb{R}^N)$)**.** *If $1 < p < \infty$, then*

(1) $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is a Banach space;

(2) smooth functions of compact support are dense in $\mathcal{D}^{1,p}(\mathbb{R}^N)$;

(3) the Gagliardo-Nirenberg-Sobolev inequality holds in $\mathcal{D}^{1,p}(\mathbb{R}^N)$: there exists a constant $C > 0$ such that

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p(\mathbb{R}^N)} , \quad \forall u \in \mathcal{D}^{1,p}(\mathbb{R}^N); \tag{GNS}$$

(4) $\mathcal{D}^{1,p}(\mathbb{R}^N) \subseteq W^{1,p}_{loc}(\mathbb{R}^N)$.

In particular, bounded sequences in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ are bounded in $L^{p^*}(\mathbb{R}^N)$.

Proof. By our previous discussion, we need only verify the first point. To this end, we fix a Cauchy sequence (u_n) in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. By definition, we see that $(\partial_i u_n)$ is Cauchy in $L^p(\mathbb{R}^N)$ for each $i = 1, \ldots, N$. Utilizing the Gagliardo-Nirenberg-Sobolev inequality

(GNS), the seqeunce (u_n) is also Cauchy in $L^{p^*}(\mathbb{R}^N)$. Thus, we may find $u^0 \in L^{p^*}(\mathbb{R}^N)$ and $u^1, \dots, u^N \in L^p(\mathbb{R}^N)$ such that $\lim_{n\to\infty} u_n = u^0$ in $L^{p^*}(\mathbb{R}^N)$ and

$$u^i = \lim_{n\to\infty} \partial_i u_n \quad \text{in } L^p(\mathbb{R}^N),$$

for every $i = 1, \dots, N$. By definition of $\|\cdot\|$, the claim will follow if $\partial_i u^0 = u^i$ for each index i. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be given; two applications of Hölder's inequality shows that

$$\begin{aligned}\int_{\mathbb{R}^N} u^0 \partial_i \varphi = \lim_{n\to\infty} \int_{\mathbb{R}^N} u_n \partial_i \varphi &= - \lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi \partial_i u_n \\ &= - \int_{\mathbb{R}^N} \varphi u^i.\end{aligned}$$

Thus, $u^0 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and $\nabla u^0 = (u^1, \dots, u^N)$. Especially, $\nabla u_n \to \nabla u^0$ in $L^p(\mathbb{R}^N; \mathbb{R}^N)$. By definition, it follows that $u_n \to u^0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and the proof is complete. □

2.3 Understanding Weak Convergence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$

As mentioned previously, the topological dual of $\mathcal{D}^{1,p}(\mathbb{R}^N)$ will play an important role in the proof of Theorem 1.1. Consequently, we seek to properly understand weak convergence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Of course, since strong convergence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ only tests the L^p-convergence of the gradients, we should expect that weak convergence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ be characterized by the weak convergence of the gradients in $L^p(\mathbb{R}^N; \mathbb{R}^N)$. As it turns out, this is indeed the case.

Proposition 2.6. *Let (v_n) be a sequence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and fix $v \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. Then, $v_n \rightharpoonup v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ if and only if $\nabla v_n \rightharpoonup \nabla v$ in $L^p(\mathbb{R}^N; \mathbb{R}^N)$.*

Proof. We begin by defining the following map

$$T : \mathcal{D}^{1,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N; \mathbb{R}^N), \quad u \mapsto \nabla u.$$

Clearly, T is a linear isometry and is hence an embedding. Through T, we can identify $\mathcal{D}^{1,p}(\mathbb{R}^N)$ with the subspace $T(\mathbb{R}^N) := T(\mathcal{D}^{1,p}(\mathbb{R}^N))$ of $L^p(\mathbb{R}^N; \mathbb{R}^N)$. As a first step, we will show that $v_n \rightharpoonup v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ if and only if $\nabla v_n \rightharpoonup \nabla v$ in $T(\mathbb{R}^N)$.

Assume that $v_n \rightharpoonup v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$; given a continuous linear functional ϕ on $T(\mathbb{R}^N)$, we define

$$\varphi : \mathcal{D}^{1,p}(\mathbb{R}^N) \to \mathbb{R}, \quad \varphi(u) := \phi(\nabla u).$$

In other words, we set $\varphi := \phi \circ T$. Clearly, φ is a continuous linear functional on $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Since v_n converges weakly to v in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, we must have $\varphi(v_n) \to \varphi(v)$ as $n \to \infty$. However, this is equivalent to the statement

$$\lim_{n\to\infty} \phi(\nabla v_n) = \phi(\nabla v).$$

We infer that ∇v_n converges weakly to ∇v in $T(\mathbb{R}^N)$ as $n \to \infty$. Conversely, we assume that $\nabla v_n \rightharpoonup \nabla v$ in $T(\mathbb{R}^N)$ and fix a continuous linear functional φ on $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Consider the map

$$\phi : T(\mathbb{R}^N) \to \mathbb{R}, \quad \phi := \varphi \circ T^{-1}.$$

Again, it is obvious that ϕ is a continuous linear functional on $T(\mathbb{R}^N)$. Since ∇v_n converges weakly to ∇v in $T(\mathbb{R}^N)$, we see that $\phi(\nabla v_n)$ converges to $\phi(\nabla v)$. By definition, this is equivalent to having

$$\lim_{n\to\infty} \varphi(v_n) = \varphi(v).$$

To summarize, we have shown that weak convergence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is equivalent to the weak convergence of the gradients in $T(\mathbb{R}^N)$. To complete the proof, we need only show that weak convergence of the gradients in $T(\mathbb{R}^N)$ is equivalent to their weak convergence in all of $L^p(\mathbb{R}^N; \mathbb{R}^N)$. Here, the only non-trivial claim is that weak convergence in $L^p(\mathbb{R}^N; \mathbb{R}^N)$ implies weak convergence in $T(\mathbb{R}^N)$. However, this is a simple consequence of the Hahn-Banach theorem. Indeed, any continuous linear functional ϕ on $T(\mathbb{R}^N)$ can be extended to a continuous linear functional ψ on all of $L^p(\mathbb{R}^N; \mathbb{R}^N)$. Consequently, if $\nabla v_n \rightharpoonup \nabla v$ in $L^p(\mathbb{R}^N; \mathbb{R}^N)$, then

$$\psi(\nabla v_n) \to \psi(\nabla v) \quad \text{as } n \to \infty.$$

Clearly, this simply means that $\lim_{n\to\infty} \phi(\nabla v_n) = \phi(\nabla v)$ whence we have established that ∇v_n converges weakly to ∇v in $T(\mathbb{R}^N)$. This completes the proof. □

The previous technical result will mainly be used when establishing the following:

Corollary 2.7. *If (v_n) is a bounded sequence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, there exists $v \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and a subsequence of (v_n) that converges weakly to v in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.*

Remark 2.1. Note that we cannot directly apply any familiar results about weak compactness in Banach spaces as $\mathcal{D}^{1,p}(\mathbb{R}^N)$ may not be reflexive, a priori. However, as will become apparent in the proof, we will be "saved" by the familiar properties of L^p-spaces. Certainly, if $\mathcal{X}$ is a Banach space, then $\mathcal{X}$ is reflexive if and only if the closed unit ball in

$\mathcal{X}$ is weakly compact[5]. It therefore follows that bounded sequences in $L^p(\mathbb{R}^N;\mathbb{R}^N)$ have a weakly convergent subsequence. Since $W^{1,p}(\Omega)$ is reflexive for $1 < p < \infty$, this reasoning also shows that bounded sequences in $W^{1,p}(\Omega)$ have weakly convergent subsequences.

Proof of Corollary. By definition of the $\mathcal{D}^{1,p}(\mathbb{R}^N)$-norm, the sequence of gradients (∇v_n) is bounded in $L^p(\mathbb{R}^N;\mathbb{R}^N)$. By Theorem 2.5, the sequence (v_n) is *also* bounded in $L^{p^*}(\mathbb{R}^N)$. After passing to a subsequence, we may assume that ∇v_n converges weakly to some $w = (w_1, \ldots, w_N)$ in $L^p(\mathbb{R}^N;\mathbb{R}^N)$. Passing to yet another subsequence, we may also assume that v_n converges weakly to $v \in L^{p^*}(\mathbb{R}^N)$. Next, we claim that v belongs to $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and that $\nabla v = w$. To see this, fix an index $i = 1, \ldots, N$ and let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be given. Weak convergence in $L^{p^*}(\mathbb{R}^N)$ gives

$$\int_{\mathbb{R}^N} v \partial_i \varphi = \lim_{n\to\infty} \int_{\mathbb{R}^N} v_n \partial_i \varphi = -\lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi \partial_i v_n.$$

Since ∇v_n converges weakly to w in $L^p(\mathbb{R}^N;\mathbb{R}^N)$, it is not hard to check that $\partial_i v_n$ also converges weakly to w_i in $L^p(\mathbb{R}^N)$. Using this with the above, we discover that

$$\int_{\mathbb{R}^N} v \partial_i \varphi = -\lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi \partial_i v_n = -\int_{\mathbb{R}^N} \varphi w_i.$$

We conclude that $v \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and that $\nabla v = w$. Finally, since $\nabla v_n \to w = \nabla v$ weakly in $L^p(\mathbb{R}^N;\mathbb{R}^N)$, it follows from Proposition 2.6 that $v_n \rightharpoonup v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. □

Corollary 2.8. *For $1 < p < \infty$, the homogeneous Sobolev space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is a reflexive Banach space.*

Before starting the proof of Theorem 1.1, we would like to partially strengthen the conclusion of Corollary 2.7. Obviously, it would be ideal to extract a subsequence that converges strongly in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Instead, we will have to be content with the following pointwise result. In light of Theorem 2.1, this turns out to be *almost* as good.

Theorem 2.9. *Let (v_n) be a bounded sequence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Then, (v_n) has a subsequence (v_{n_k}) such that $v_{n_k} \rightharpoonup v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $v_{n_k} \to v$ pointwise almost everywhere on $\mathbb{R}^N$.*

Proof. In light of Corollary 2.7, we may assume without loss of generality that v_n converges weakly to v in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. For each $k \geq 1$, we denote by B_k the open ball $B(0, k)$ in $\mathbb{R}^N$. We then consider the restrictions of the sequence (v_n) to B_k. In doing so, we obtain

5. The proof of this fact is essentially an application of the Banach-Alaoglu theorem.

a bounded sequence in $W^{1,p}(B_k)$, for each $k \in \mathbb{N}$. By the Rellich-Kondrachov theorem, the embedding

$$W^{1,p}(B_k) \hookrightarrow L^p(B_k)$$

is compact. Thus, for this ball B_k, there exists a subsequence $(v_{n,k})_{n\geqslant 1}$ such that

(i) $v_{n,k}$ converges *weakly* to some $u_k \in W^{1,p}(B_k)$ as $n \to \infty$;

(ii) $v_{n,k}$ converges *strongly* to u_k in $L^p(B_k)$ as $n \to \infty$; and

(iii) $v_{n,k}$ converges pointwise almost everywhere to u_k on B_k as $n \to \infty$.

By duality and the fact that $v_n \rightharpoonup v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, we see through Proposition 2.6 that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla v_{n,k} \cdot g = \int_{\mathbb{R}^N} \nabla v \cdot g$$

for each $g \in L^{p'}(\mathbb{R}^N; \mathbb{R}^N)$. In particular,

$$\lim_{n\to\infty} \int_{B_k} \nabla v_{n,k} \cdot g = \int_{B_k} \nabla v \cdot g$$

for all $g \in L^{p'}(B_k; \mathbb{R}^N)$. On the other hand, (i) ensures that

$$\lim_{n\to\infty} \int_{B_k} \nabla v_{n,k} \cdot g = \int_{B_k} \nabla u_k \cdot g$$

for every $g \in L^{p'}(B_k; \mathbb{R}^n)$. Combining these last two equations shows that

$$\int_{B_k} (\nabla u_k - \nabla v) \cdot g = 0, \quad \forall g \in L^{p'}(\mathbb{R}^N; \mathbb{R}^N).$$

By a density argument, it follows that $u_k = v + \alpha_k$ pointwise a.e. on B_k, where $\alpha_k \in \mathbb{R}$ is a constant. In other words, $v_{n,k}$ converges pointwise almost everywhere to $v + \alpha_k$ on B_k, as $n \to \infty$.

To complete the proof, we will now make use of a diagonal argument. For this family of balls $\{B_k\}_{k\geq 1}$, we inductively construct subsequences $\{(v_{n,k})\}_{k\geqslant 1}$ using the procedure described above (in such a way that $(v_{n,k+1})_{n\geqslant 1}$ is a subsequence of $(v_{n,k})_{n\geqslant 1}$). Therefore, there is a corresponding sequence $(\alpha_k)_{k\geq 1}$ of real numbers such that

$$v_{n,k} \to v + \alpha_k \quad \text{pointwise a.e. on } B_k \text{ as } n \to \infty,$$

for each k. Finally, for any fixed $k \in \mathbb{N}$, an application of Fatou's lemma yields

$$\begin{aligned}\alpha_k m(B_k)^{p^*} \leq \liminf_{n\to\infty} \int_{B_k} |v_{n,k} - v|^{p^*} &\leq \limsup_{n\to\infty} \int_{B_k} (|v_{n,k}| + |v|)^{p^*} \\ &\leq 2^{p^*-1} \limsup_{n\to\infty} \left(\int_{B_k} |v_{n,k}|^{p^*} + \int_{B_k} |v|^{p^*} \right) \\ &\leq 2^{p^*-1} \sup_{n\in\mathbb{N}} \left(\int_{\mathbb{R}^N} |v_{n,k}|^{p^*} + \int_{\mathbb{R}^N} |v|^{p^*} \right)\end{aligned}$$

where this last term is uniformly bounded in n and k since (v_n) is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $v \in L^{p^*}(\mathbb{R}^N)$. Noting that $m(B_k) \to \infty$ as $k \to \infty$, we infer that $\alpha_k \to 0$ as $k \to \infty$. Then, the diagonal subsequence $(v_{n,n})_{n\in\mathbb{N}}$ will converge pointwise almost everywhere to v on $\mathbb{R}^N$ as $n \to \infty$, which proves the claim. □

3 Preliminary Results

As mentioned above, we begin by stating two results established by Mercuri-Willem that will play a critical role in the proof of Theorem 1.1. The first is a rather technical, but elementary, identity:

Lemma A (Lemma 3.2 in [5])**.** *Let $1 < q < \infty$ and consider the function*

$$A : \mathbb{R}^N \to \mathbb{R}^N, \quad y \mapsto y\,|y|^{q-2}.$$

Let μ be a measure on Ω and assume that (u_n) is a bounded sequence in $L^q(\Omega, \mu)$ having the property that

$$\lim_{n\to\infty} u_n(x) = u(x)$$

for μ-a.e. $x \in \Omega$. Then,

$$\lim_{n\to\infty} \int_\Omega |A(u_n) - A(u_n - u) - A(u)|^{\frac{q}{q-1}}\, \mathrm{d}\mu = 0.$$

Theorem B (Theorem 3.3 in [5])**.** *Let (Ω_k) be a sequence of open sets such that $\Omega_k \nearrow \Omega$. Assume that $q > 1$ and (v_n) is a sequence in $W^{1,q}(\Omega)$ such that $v_n \rightharpoonup v$ in $W^{1,q}(\Omega)$. Let $T : \mathbb{R} \to \mathbb{R}$ be given by the rule*

$$T(s) := \begin{cases} s & \text{if } |s| \leq 1, \\ \frac{s}{|s|} & \text{if } |s| > 1. \end{cases}$$

If for each $k \geq 1$ there holds

$$\lim_{n\to\infty} \int_{\Omega_k} \left(|\nabla v_n|^{q-2} \nabla v_n - |\nabla v|^{q-2} \nabla v\right) \cdot \nabla T(v_n - v) = 0,$$

then, after passing to a subsequence,

(1) $\nabla v_n \to \nabla v$ *pointwise almost everywhere in* Ω*;*

(2) $\lim_{n\to\infty} \left(\|\nabla v_n\|^q_{L^q(\Omega)} - \|\nabla(v_n - v)\|^q_{L^q(\Omega)}\right) = \|\nabla v\|_{L^q(\Omega)^q}$*; and*

(3) $|\nabla v_n|^{q-2} \nabla v_n - |\nabla(v_n - v)|^{q-2} \nabla(v_n - v) \to |\nabla v|^{q-2} \nabla v$ *in* $L^{\frac{q}{q-1}}(\Omega)$*.*

Furthermore, the theorem holds when $\Omega = \mathbb{R}^N$ *and* $W^{1,q}(\Omega)$ *is replaced by* $\mathcal{D}^{1,q}(\mathbb{R}^N)$*.*

Finally, we will require the following non-existence result:

Theorem C (Theorem 1.1 in [5])**.** *Any non-negative weak solution* $u \in \mathcal{D}^{1,p}_0(\mathbb{R}^N_+)$ *to* (1.9) *vanishes almost everywhere.*

3.1 Technical Lemmas

This subsection is devoted to establishing two special lemmas that can be viewed as the "base cases" for the iterative argument that will be used in the proof of Theorem 1.1. We note that these Lemmas are slight variants of those stated in Mercuri-Willem [5]. More precisely, we do not impose any assumption on the limiting behaviour of $(u_n)_-$.

Lemma 3.1. *Let* (u_n) *be a sequence in* $W^{1,p}_0(\Omega)$ *such that the following hold true:*

(1) u_n *converges weakly to* u *in* $W^{1,p}_0(\Omega)$*;*

(2) $u_n \to u$ *pointwise almost everywhere on* Ω*;*

(3) $\phi(u_n) \to c \in \mathbb{R}$ *as* $n \to \infty$*; and*

(4) $\phi'(u_n) \to 0$ *in* $W^{-1,p'}(\Omega)$ *as* $n \to \infty$*.*

After passing to a subsequence, one has $\nabla u_n \to \nabla u$ *pointwise almost everywhere on* Ω *and* $\phi'(u) = 0$*. Furthermore, by defining* $v_n := u_n - u$*, we obtain a sequence in* $W^{1,p}_0(\Omega)$ *with the property that* *(i)* $\lim_{n\to\infty} (\|u_n\|^p - \|v_n\|^p) = \|u\|^p$*;*

(ii) $\lim_{n\to\infty} \phi_\infty(v_n) = \lim_{n\to\infty} \phi(u_n) - \phi(u)$*; and*

(iii) $\phi'_\infty(v_n) \to 0$ *in* $W^{-1,p'}(\Omega)$ *as* $n \to \infty$*.*

Proof. We follow, with only slight modifications, the proof from Mercuri-Willem [5]. Define $T : \mathbb{R}^N \to \mathbb{R}^N$ as in Theorem B. Since Ω is bounded and $|T| \le 1$ on $\mathbb{R}$, the dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_\Omega |T(u_n - u)|^q = 0 \tag{3.1}$$

for all $0 < q < \infty$. Here, we have used the continuity of T and the assumption that $u_n \to u$ pointwise almost everywhere on Ω. Next, we claim that $T(u_n - u) \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$. Of course, this amounts to showing that, up to a subsequence,

$$T(v_n) \rightharpoonup 0 \quad \text{in } W_0^{1,p}(\Omega).$$

Note that this is well defined (see Ziemer [12]) because T is Lipschitz continuous whence $T(v_n) \in W^{1,p}(\Omega)$. In fact, because $|T(v_n)| \le |v_n|$ a.e. on Ω, we have

$$\text{Trace}(T(v_n)) = 0$$

which implies that $T(v_n) \in W_0^{1,p}(\Omega)$.

Since v_n is weakly convergent, it is bounded in $W_0^{1,p}(\Omega)$. Now, it is obvious that $(T(v_n))$ is bounded in $L^p(\Omega)$. By the chain-rule (which is valid because T is Lipschitz continuous – refer to Ziemer [12]), one has

$$\partial_i[T(v_n)] = T'(v_n)\partial_i v_n$$

for all $i = 1, \dots, N$. Since the derivative of a Lipschitz continuous function is bounded, it follows that $(\nabla T(v_n))$ is bounded in $L^p(\Omega)$ whence $(T(v_n))$ is bounded in $W_0^{1,p}(\Omega)$ as well. After passing to a subsequence, it is of no harm to assume that

$$T(v_n) \rightharpoonup \eta \quad \text{in } W_0^{1,p}(\Omega)$$

as $n \to \infty$. In fact, by compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, we might as well assume that $T(v_n) \to \eta$ *strongly* in $L^p(\Omega)$ and pointwise almost everywhere on Ω. Finally, since $u_n \to u$ pointwise almost everywhere on Ω, we have

$$\lim_{n\to\infty} T(v_n(x)) = 0 \quad \text{a.e. on } \Omega.$$

This means that $\eta = 0$ for almost every $x \in \Omega$. In particular, $T(v_n) \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$. Next, direct computation shows that

$$\begin{aligned}\int_\Omega \left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u\right) \cdot \nabla T(u_n - u) &= \langle \phi'(u_n), T(u_n - u)\rangle \\ &\quad - \underbrace{\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla T(u_n - u)}_{I_1} \\ &\quad - \underbrace{\int_\Omega \left(a(x)\, |u_n|^{p-2} u_n - \mu\, |u_n|^{p^*-2} u_n\right) T(u_n - u)}_{I_2}.\end{aligned}$$

Since $\phi'(u_n) \to 0$ strongly in $W^{-1,p'}(\Omega)$ and $T(u_n - u)$ is bounded in $W_0^{1,p}(\Omega)$, it is easy to see that $\langle \phi'(u_n), T(u_n - u)\rangle \to 0$ as $n \to \infty$. To deal with I_1, note that by Hölder's inequality,

$$\begin{aligned}|I_1| \le \int_\Omega |\nabla u|^{p-1} |\nabla T(u_n - u)| &\le \left(\int_\Omega |\nabla u|^{(p-1)\cdot\frac{p}{p-1}}\right)^{1/p'} \|\nabla T(u_n - u)\|_{L^p(\Omega)} \\ &\le \|u\|_{W^{1,p}(\Omega)}^{p/p'} \|T(u_n - u)\|_{W^{1,p}(\Omega)}.\end{aligned}$$

These estimates show that the map

$$f \mapsto \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla f$$

is a continuous linear functional on $W^{1,p}(\Omega)$. Using that $T(u_n - u) \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$, it follows that $I_1 \to 0$ as $n \to \infty$. For convenience, we now break I_2 into two parts. An easy application of Corollary 2.2 together with the boundedness of (u_n) in $L^{p^*}(\Omega)$ shows that

$$\int_\Omega \mu\, |u_n|^{p^*-2} u_n T(u_n - u) \to 0, \quad \text{as } n \to \infty.$$

Applying Hölder's inequality once again, we find that

$$\int_\Omega |a|\, |u_n|^{p-1} |T(u_n - u)| \le \|a\|_{L^{N/p}(\Omega)} \left(\int_\Omega \left(|u_n|^{p-1} |T(u_n - n)|\right)^{\frac{N}{N-p}}\right)^{\frac{N-p}{N}}$$

where $a \in L^{N/p}(\Omega)$ by assumption and

$$\int_\Omega \left(|u_n|^{p-1}\,|T(u_n-n)|\right)^{\frac{N}{N-p}} = \int_\Omega |u_n|^{p^*-\frac{N}{N-p}}\,|T(u_n-n)|^{\frac{N}{N-p}} \tag{3.2}$$

$$\leq \left[\int_\Omega |u_n|^{\left(p^*-\frac{N}{N-p}\right)q}\right]^{1/q}\left[\int_\Omega |T(u_n-u)|^{\frac{N}{N-p}q'}\right]^{1/q'} \tag{3.3}$$

$$= \left(\int_\Omega |u_n|^{p^*}\right)^{\frac{p-1}{p}}\left(\int_\Omega |T(u_n-u)|^{p^*}\right)^{1/p}, \tag{3.4}$$

where

$$q := \frac{p^*}{p^*-\frac{N}{N-p}} = \frac{(N-p)p^*}{(N-p)p^*-N} = \frac{Np}{Np-N} = \frac{p}{p-1} = p'.$$

By the same reasoning as before, we see that $\int_\Omega |a|\,|u_n|^{p-1}\,|T(u_n-u)| \to 0$ as $n\to\infty$ and therefore that $I_2 \to 0$. Putting all of this together, we find that

$$\lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^{p-2}\,\nabla u_n - |\nabla u|^{p-2}\,\nabla u\right)\cdot\nabla T(u_n-u) = 0.$$

Applying Theorem B with $\Omega_k = \Omega$, we may extract a subsequence (also denoted (u_n)) such that $\nabla u_n \to \nabla u$ almost everywhere on Ω. By this same theorem (or, alternatively, Theorem A.3), we also have (i). Since (u_n) is bounded in $W^{1,p}(\Omega)$, the Sobolev inequality tells us that $\{|u_n|^p\}_{n\in\mathbb{N}}$ is bounded in $L^{\frac{N}{N-p}}(\Omega) \cong (L^{N/p}(\Omega))^*$. Recalling that $u_n \to u$ pointwise almost everywhere on Ω, an application of Theorem 2.1 shows that

$$\lim_{n\to\infty}\int_\Omega a(x)\,|u_n|^p = \int_\Omega a(x)\,|u|^p\,.$$

But then, using (i) together with the Brézis-Lieb lemma (see Theorem A.3 in Appendix A) implies that

$$\begin{aligned}
\phi_\infty(v_n) &= \int_\Omega\left(\frac{|\nabla v_n|^p}{p} - \mu\frac{|v_n|^{p^*}}{p^*}\right)\\
&= \frac{1}{p}\,\|u_n-u\|^p - \frac{\mu}{p^*}\,\|u_n-u\|^{p^*}_{L^{p^*}(\Omega)}\\
&= \frac{1}{p}\left(\|u_n\|^p - \|u\|^p\right) - \frac{\mu}{p^*}\left(\|u_n\|^{p^*}_{L^{p^*}(\Omega)} - \|u\|^{p^*}_{L^{p^*}(\Omega)}\right) + o(1)\\
&= \frac{1}{p}\left(\|u_n\|^p - \|u\|^p\right) - \frac{\mu}{p^*}\left(\|u_n\|^{p^*}_{L^{p^*}(\Omega)} - \|u\|^{p^*}_{L^{p^*}(\Omega)}\right)
\end{aligned}$$

$$+\int_\Omega a\left(|u_n|^p - |u|^p\right) + o(1)$$
$$= \phi(u_n) - \phi(u) + o(1)$$
$$= c - \phi(u) + o(1).$$

This proves (ii). Now, we claim that $\phi'(u) = 0$. More precisely, we check that $\langle \phi'(u), h\rangle = 0$ for all $h \in W_0^{1,p}(\Omega)$. Since $\phi'(u_n) \to 0$ strongly in $W^{-1,p'}(\Omega)$, it would actually suffice to show that

$$\lim_{n\to\infty} \langle \phi'(u_n), h\rangle = \langle \phi'(u), h\rangle, \quad \forall h \in W_0^{1,p}(\Omega).$$

Let $h \in W_0^{1,p}(\Omega)$ be given and write

$$\begin{aligned}\langle \phi'(u_n), h\rangle - \langle \phi'(u), h\rangle \\ &= \int_\Omega \left(|\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h + a\,|u_n|^{p-2} u_n h - \mu\,|u_n|^{p^*-2} u_n h\right) \\ &- \int_\Omega \left(|\nabla u|^{p-2} \nabla u \cdot \nabla h + a\,|u|^{p-2} uh - \mu\,|u|^{p^*-2} uh\right).\end{aligned}$$

As above, an application of Theorem 2.1 shows that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla h,$$

and

$$\lim_{n\to\infty} \int_\Omega \mu\,|u_n|^{p^*-2} u_n h = \int_\Omega \mu\,|u|^{p^*-2} uh$$

Next, we claim that the family $\left\{|u_n|^{p-2} u_n h\right\}_{n\in\mathbb{N}}$ is bounded in $L^{\frac{N}{N-p}}(\Omega)$, where $\frac{N}{N-p}$ is the Hölder conjugate exponent of $\frac{N}{p}$. As in (3.2)-(3.4), we have

$$\begin{aligned}\int_\Omega \left(|u_n|^{p-1} |h|\right)^{\frac{N}{N-p}} &= \int_\Omega |u_n|^{p^*-\frac{N}{N-p}} |h|^{\frac{N}{N-p}} \\ &\le \left[\int_\Omega |u_n|^{\left(p^*-\frac{N}{N-p}\right)q}\right]^{1/q} \left[\int_\Omega |h|^{\frac{N}{N-p}q'}\right]^{1/q'} \\ &= \left(\int_\Omega |u_n|^{p^*}\right)^{\frac{p-1}{p}} \left(\int_\Omega |h|^{p^*}\right)^{1/p}.\end{aligned}$$

After a final application of Theorem 2.1, we deduce that $\langle \phi'(u_n), h\rangle \to \langle \phi'(u), h\rangle$ for any fixed $h \in W_0^{1,p}(\Omega)$. Consequently, $\phi'(u) = 0$ in $W^{-1,p'}(\Omega)$. By similar convergence arguments, it readily follows from the Breźis-Lieb lemma (Theorem A.3) that

$$\phi'_\infty(v_n) = \phi'(v_n) + o(1) = \phi'(u_n) - \phi'(u) + o(1) = o(1)$$

when considered in $W^{-1,p'}(\Omega)$. This completes the proof. □

Finally, we cite a final result from Meruci-Willem which can be thought of as an analogue of the previous lemma for *rescalings* of the sequence (u_n).

Lemma 3.2. *Let (y_n) and (λ_n) be sequences in Ω and $(0, \infty)$, respectively, such that*

$$\frac{\operatorname{dist}(y_n, \partial\Omega)}{\lambda_n^1} \to \infty, \quad \text{as } n \to \infty. \tag{3.5}$$

Suppose further that we have a sequence (u_n) in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ such that the rescaled sequence

$$v_n(x) := \lambda_n^{\frac{N-p}{N}} u_n(\lambda_n x + y_n)$$

in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ satisfies

(1) $v_n \rightharpoonup v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and pointwise almost everywhere on $\mathbb{R}^N$;

(2) $\phi_\infty(u_n) \to c$ and $\phi'(u_n) \to 0$ in $W^{-1,p'}(\Omega)$ as $n \to \infty$.

Then $\phi'_\infty(v) = 0$. Furthermore, after passing to a subsequence, we have $\nabla v_n \to \nabla v$ almost everywhere on $\mathbb{R}^N$. Moreover,

(i) $\lim_{n\to\infty} (\|u_n\|^p - \|w_n\|^p) = \|v\|^p$;

(ii) $\phi_\infty(w_n) \to c - \phi_\infty(v)$ as $n \to \infty$; and

(iii) $\phi'_\infty(w_n) \to 0$ in $W^{-1,p'}(\Omega)$ as $n \to \infty$

where we define

$$w_n(z) := u_n(z) - \lambda_n^{\frac{p-N}{N}} v\left(\frac{z - y_n}{\lambda_n}\right).$$

Proof. We once again follow, with slight modification, the proof of Lemma 3.6 in Mercuri-Willem [5]. Note that (v_n) is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ since it converges weakly. Let $\{B_k\}_{k\in\mathbb{N}}$ be an exhaustion of $\mathbb{R}^N$ by open balls, centered at the origin, of radius $k \geq 1$. For any such k, we claim that $\phi'_\infty(v_n) \to 0$ strongly in $W^{-1,p'}(B_k)$. Given a test function $h \in C_c^\infty(\mathbb{R}^N)$, we define

$$h_n(z) := \lambda_n^{\frac{p-N}{p}} h\left(\frac{z-y_n}{\lambda_n}\right).$$

Suppose now that $h \in C_c^\infty(B_k)$. By the assumption in (3.5), we have $h_n \in C_c^\infty(\Omega)$ for all n sufficiently large. Hence, a change of variables together with the Friedrich-Poincaré inequality gives

$$\begin{aligned}
|\langle \phi'_\infty(v_n), h\rangle| = |\langle \phi'_\infty(u_n), h_n\rangle| &\leq \|\phi'_\infty(u_n)\|_{W^{-1,p'}(\Omega)} \|h_n\|_{W^{1,p}(\Omega)} \\
&\leq C\,\|\phi'_\infty(u_n)\|_{W^{-1,p'}(\Omega)} \|h_n\| \\
&= C\,\|\phi'_\infty(u_n)\|_{W^{-1,p'}(\Omega)} \|h\| \\
&\leq C\,\|\phi'_\infty(u_n)\|_{W^{-1,p'}(\Omega)} \|h\|_{W^{1,p}(B_k)}\,.
\end{aligned}$$

It follows that $\phi'_\infty(v_n) \to 0$ in $W^{-1,p'}(B_k)$ as $n \to \infty$, for each $k \geq 1$. Let $T : \mathbb{R} \to \mathbb{R}$ be as in Theorem B and fix $k \geq 1$. Let $\rho \in C_c^\infty(\mathbb{R}^N)$ be such that

$$\begin{cases} 0 \leq \rho \leq 1 & \text{in } \mathbb{R}^N, \\ \rho \equiv 1 & \text{on } B_k, \\ \rho \equiv 0 & \text{outside } B_{k+1}. \end{cases}$$

For $n \in \mathbb{N}$, we define

$$f_n : \mathbb{R}^N \to \mathbb{R}^N, \quad f_n := |\nabla v_n|^{p-2}\,\nabla v_n - |\nabla v|^{p-2}\,\nabla v.$$

It is not hard to check (see Lemma A.2 in Appendix A) that $f_n \cdot \nabla T(v_n - v) \geq 0$ on all of $\mathbb{R}^N$. Thus,

$$\int_{B_k} f_n \cdot \nabla T(v_n - v) = \int_{B_k} f_n \cdot \rho \nabla T(v_n - v) \leq \int_{\mathbb{R}^N} f_n \cdot \rho \nabla T(v_n - v).$$

Now, assuming that $\int_{\mathbb{R}^N} f_n \cdot \rho\nabla T(v_n - v) \to 0$ as $n \to \infty$, Theorem B would ensure the existence of a subsequence (v_n) with $\nabla v_n \to \nabla v$ almost everywhere on $\mathbb{R}^N$. In fact, by homogeneity, this same theorem would immediately give (i). A simple calculation verifies that

$$\int_{\mathbb{R}^N} f_n \cdot \rho \nabla T(v_n - v) = \underbrace{\int_{\mathbb{R}^N} f_n \cdot \nabla \left[T(v_n - v)\rho\right]}_{=:I_1} - \underbrace{\int_{\mathbb{R}^N} T(v_n - v) f_n \cdot \nabla \rho}_{=:I_2}.$$

As in the previous lemma, we are now reduced to verifying that $I_1, I_2 \to 0$ as $n \to \infty$. For I_2, an application of Hölder's inequality gives the estimate

$$|I_2| \leq \int_{\mathbb{R}^N} |T(v_n - v) f_n \cdot \nabla \rho| \leq \|f_n\|_{L^{\frac{p}{p-1}}(\mathbb{R}^N;\mathbb{R}^N)} \|T(v_n)\nabla\rho\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}$$

where

$$\int_{\mathbb{R}^N} |T(v_n - v)|^p \, |\nabla \rho|^p \to 0 \quad \text{as } n \to \infty$$

by the dominated convergence theorem. Hence, $I_2 \to 0$. To see that $I_1 \to 0$ as well, we first note that

$$I_1 = \langle \phi'_\infty(v_n), \rho T(v_n - v)\rangle + \mu \underbrace{\int_{\mathbb{R}^N} |v_n|^{p^*-2} \, v_n \rho T(v_n - v)}_{=:J_1} \tag{3.6}$$

$$- \int_{\mathbb{R}^N} \underbrace{|\nabla v|^{p-2} \, \nabla v \cdot \nabla \left[\rho T(v_n - v)\right]}_{=:J_2}. \tag{3.7}$$

Now, ρ has compact support and T is a bounded Lipschitz function with bounded derivative. Since $\phi'_\infty(v_n) \to 0$ strongly in $W^{-1,p'}(B_k)$ for every $k \geq 1$, it is not hard to see that

$$\langle \phi'_\infty(v_n), \rho T(v_n - v)\rangle, \quad \text{as } n \to \infty.$$

To deal with J_1, we first apply Hölder's inequality:

$$\begin{aligned} |J_1| &\leq \left\| |v_n|^{p^*-1} \right\|_{L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N;\mathbb{R}^N)} \|\rho T(v_n - v)\|_{L^{p^*}(\mathbb{R}^N)} \\ &= \|v_n\|_{L^{p^*}(\mathbb{R}^N)}^{p^*-1} \left(\int_{\mathbb{R}^N} |\rho| \, |T(v_n - v)|^{p^*} \right)^{1/p^*} \end{aligned}$$

where

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\rho| \, |T(v_n - v)|^{p^*} = 0$$

by the dominated convergence theorem. It follows that $J_1 \to 0$ as $n \to \infty$. Treating J_2, we write

$$J_2 = \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \cdot \nabla [\rho T(v_n - v)] = \int_{\mathbb{R}^N} |\nabla v|^{p-2} \rho \nabla v \cdot \nabla T(v_n - v) + \int_{\mathbb{R}^N} |\nabla v|^{p-2} T(v_n - v) \nabla v \cdot \nabla \rho.$$

As before, it is clear from Theorem 2.1 that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v|^{p-2} T(v_n - v) \nabla v \cdot \nabla \rho = 0.$$

On the other hand,

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2} \rho \nabla v \cdot \nabla T(v_n - v) = \int_{\mathbb{R}^N} |\nabla v|^{p-2} \rho \nabla v \cdot T'(v_n - v) \nabla (v_n - v)$$

for each $n \in \mathbb{N}$. Using the definition of T, we see that

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2} \rho \nabla v \cdot T'(v_n - v) \nabla (v_n - v) = \int_{\mathbb{R}^N} |\nabla v|^{p-2} \rho \nabla v \cdot \nabla (v_n - v) - \int_{|v_n - v| \geq 1} |\nabla v|^{p-2} \rho \nabla v \cdot \nabla (v_n - v)$$

where this first term converges to zero because $v_n \rightharpoonup v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. As for the second term, invoking Hölder's inequality shows that

$$\left| \int_{|v_n - v| \geq 1} |\nabla v|^{p-2} \rho \nabla v \cdot \nabla (v_n - v) \right| \leq M \|v_n - v\| \left(\int_{E_n} |\nabla v|^p \right)^{(p-1)/p}$$

with $E_n := \{x \in \operatorname{supp}(\rho) : |v_n(x) - v(x)| \geq 1\}$. Using that ρ has compact support and $v_n \to v$ a.e. on $\mathbb{R}^N$, an application of Proposition A.1 ensures that $m(E_n) \to 0$ as $n \to \infty$. Consequently, this second term also tends to zero. It follows that $J_2 \to 0$ and so $I_1 \to 0$.

By our earlier remarks, we may now assume that $\nabla v_n \to \nabla v$ almost everywhere on $\mathbb{R}^N$ and that (i) holds. Much like in Lemma 3.1 (and, of course, Mercuri-Willem [5]), the Brézis-Lieb lemma together with (i) implies that

$$\begin{aligned} \phi_\infty(w_n) = \phi_\infty(v_n - v) &= \phi_\infty(v_n) - \phi_\infty(v) + o(1) \\ &= \phi_\infty(u_n) - \phi_\infty(v) + o(1) \\ &= c - \phi_\infty(v) + o(1) \end{aligned}$$

whence we have (ii). Next, let us fix $h \in C_c^\infty(\mathbb{R}^N)$. Clearly, we have $\langle \phi'_\infty(v_n), h\rangle \to 0$ as $n \to \infty$. On the other hand,

$$\begin{aligned}\langle \phi'_\infty(v_n), h\rangle - \langle \phi'_\infty(v), h\rangle = &\int_{\mathbb{R}^N} \left(|\nabla v_n|^{p-2}\nabla v_n \cdot \nabla h - \mu |v_n|^{p^*-2} v_n h\right)\\ &- \int_{\mathbb{R}^N} \left(|\nabla v|^{p-2}\nabla v \cdot \nabla h - \mu |v|^{p^*-2} v h\right).\end{aligned}$$

Next, note that the $\left\{|\nabla v_n|^{p-1}\right\}_{n\geq 1}$ and $\{|v_n|^{p^*-1}\}_{n\geq 1}$ are bounded in the dual spaces of $L^p(\mathbb{R}^N;\mathbb{R}^N)$ and $L^{p^*}(\mathbb{R}^N)$, respectively. Since h has compact support, applying Theorem 2.1 in a now routine manner gives

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla h = \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \cdot \nabla h$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |v_n|^{p^*-2} v_n h = \int_{\mathbb{R}^N} |v|^{p^*-2} v h.$$

This tells us that $\phi'_\infty(v) = 0$. Finally, convergence arguments that are by now familiar show that

$$\langle \phi'_\infty(w_n), g\rangle \to 0 \quad \text{as } n \to \infty$$

uniformly in $g \in C_c^\infty(\Omega)$ satisfying $\|g\| = 1$. This establishes (iii) and the proof is complete. $\square$

4 *The Proof of Theorem 1.1*

We are now ready to illustrate the method used by Mercuri-Willem [5] in the proof of Theorem 1.1. In the spirit of their proof, we shall divide the proof into several steps for the sake of readability. Recall that $\|\cdot\|$ by default denotes the $\mathcal{D}^{1,p}(\mathbb{R}^N)$-norm and *not* the $W^{1,p}(\Omega)$-norm.

Step 1. *There exists a subsequence, also denoted (u_n), and a solution v_0 to the problem (1.3) such that $u_n \rightharpoonup v_0$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, and u_n converges pointwise almost everywhere to v_0 on Ω. Furthermore, the sequence in $W_0^{1,p}(\Omega)$ given by*

$$u_n^1 := u_n - v_0$$

is bounded in $W_0^{1,p}(\Omega)$ and satisfies

(1) $\nabla u_n^1 \to 0$ *almost everywhere on* Ω*;*

(2) $\left\|u_n^1\right\|^p = \|u_n\|^p - \|v_0\|^p + o(1)$*;*

(3) $\lim_{n\to\infty} \phi_\infty(u_n^1) = \lim_{n\to\infty} \phi(u_n) - \phi(v_0)$*; and*

(4) $\phi'_\infty(u_n^1) \to 0$ *in* $W^{-1,p'}(\Omega)$ *as* $n \to \infty$*.*

Proof of Step 1. Since the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$, we may obviously extract a subsequence converging weakly to some v_0 in $W_0^{1,p}(\Omega)$. Furthermore, since Ω is a smoothly bounded domain, the embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

is compact by the Rellich-Kondrachov theorem. Passing to yet another subsequence, we might as well assume that u_n converges *strongly* to some function w in $L^p(\Omega)$ and that $\lim_{n\to\infty} u_n(x) = w(x)$ for almost every $x \in \Omega$. In particular, u_n converges weakly to w in $L^p(\Omega)$. However, weak convergence in $W_0^{1,p}(\Omega)$ implies weak convergence in $L^p(\Omega)$. This means that $u_n \rightharpoonup v_0$ in $L^p(\Omega)$; by the uniqueness of weak limits we must then have $v_0 = w$ almost everywhere in Ω. Consequently, because $u_n \to v_0$ a.e. on Ω and $(u_n)_- \to 0$ in $L^{p^*}(\Omega)$, we find that $v_0 \geq 0$ a.e. on Ω.

By virtue of Lemma 3.1, it is immediate that $\phi'(v_0) = 0$, i.e. v_0 is a weak solution to problem (1.3). Consequently, it remains only to verify the asymptotic identities in (2)-(4). This turns out to be quite easy since Lemma 3.1 directly implies both (3) and (4). As for (2), we have by this same result that

$$\lim_{n\to\infty} \left(\|u_n\|^p - \left\|u_n^1\right\|^p\right) = \|v_0\|^p .$$

It follows that

$$\begin{aligned} \left\|u_n^1\right\|^p + \|v_0\|^p - \|u_n\|^p &= \|v_0\|^p - \left(\|u_n\|^p - \left\|u_n^1\right\|^p\right) \\ &= \|v_0\|^p - \|v_0\|^p + o(1) = o(1). \end{aligned}$$

We also point out that (u_n^1) is a sequence in $W_0^{1,p}(\Omega)$ which may be considered (using standard extension results) as a sequence in $W^{1,p}(\mathbb{R}^N)$. In fact, the auxiliary sequence $(u_n^1)_{n\in\mathbb{N}}$ will also be bounded in $W^{1,p}(\mathbb{R}^N)$. □

Step 2. *Theorem 1.1 holds true in the case where* $u_n^1 \to 0$ *strongly in* $L^{p^*}(\Omega)$*.*

Proof of Step 2. In light of Step 1, we know that $\phi'_\infty(u_n^1)$ converges to 0 in $W^{-1,p'}(\Omega)$ as $n \to \infty$. Furthermore, for every integer $n \geq 1$ there holds

$$\begin{aligned}\left|\langle \phi'_\infty(u_n^1), u_n^1\rangle\right| &\leq \left\|\phi'_\infty(u_n^1)\right\|_{W^{-1,p'}(\Omega)} \sup_{n\geqslant 1} \left\|u_n^1\right\| \\ &\leq \left\|\phi'_\infty(u_n^1)\right\|_{W^{-1,p'}(\Omega)} \sup_{n\geqslant 1} \left\|u_n^1\right\|_{W^{1,p}(\Omega)} \to 0,\end{aligned}$$

as $n \to \infty$. In this last step, we have used the boundedness of (u_n^1) in $W_0^{1,p}(\Omega)$. Returning to our expression for $\langle \phi'_\infty(u_n^1), h\rangle$, we see that

$$\langle \phi'_\infty(u_n^1), u_n^1\rangle = \int_{\mathbb{R}^N} \left(\left|\nabla u_n^1\right|^p - \mu \left|u_n^1\right|^{p^*}\right) \to 0, \quad \text{as } n \to \infty.$$

However, $u_n^1 \to 0$ in $L^{p^*}(\Omega)$ would imply that, as $n \to \infty$,

$$\int_{\mathbb{R}^N} \mu \left|u_n^1\right|^{p^*} \to 0.$$

From the last two equations, we infer that

$$\int_{\mathbb{R}^N} \left|\nabla u_n^1\right|^p \to 0, \quad \text{as } n \to \infty.$$

In other words, $\nabla u_n \to \nabla v_0$ in $L^p(\Omega)$. It follows that

$$\|u_n - v_0\| \to 0 \quad \text{and} \quad \|u_n\| \to \|v_0\|$$

as $n \to \infty$. Since $u_n \to v_0$ in $L^p(\Omega)$ by Step 1, we have that $u_n^1 \to 0$ in $L^p(\Omega)$. Combining these facts shows that $u_n^1 \to 0$ in $W_0^{1,p}(\Omega)$. That is,

$$\lim_{n\to\infty} u_n = v_0 \quad \text{strongly in } W_0^{1,p}(\Omega)$$

whence $\phi(v_0) = \lim_{n\to\infty} \phi(u_n)$. Furthermore, Step 1 states that v_0 solves problem (1.3) weakly whence Theorem 1.1 follows with $k = 0$. □

Next, we will consider a family of Levý concentration functions. For every index $n \geq 1$, we define a real valued function Q_n on $[0, \infty)$ via the formula

$$Q_n(r) := \sup_{y\in\bar{\Omega}} \int_{B(y,r)} \left|u_n^1\right|^{p^*}.$$

Note that each Q_n is well defined since $u_n^1 \in L^{p^*}(\mathbb{R}^N)$ by grace of the Sobolev embedding theorem. By the same token, since the $\{u_n^1\}_{n\in\mathbb{N}}$ are uniformly bounded in $W_0^{1,p}(\mathbb{R}^N)$, the functional family

$$\mathcal{F} := \{Q_n(\cdot)\}_{n\in\mathbb{N}}$$

defines (through the use of Proposition 2.4) a collection of continuous functions on $[0,\infty)$. Now, in light of Step 2, we may assume that $u_n^1 \not\to 0$ strongly in $L^{p^*}(\Omega)$. After passing to a subsequence, this means that there exists $\delta > 0$ such that

$$\inf_{n\in\mathbb{N}} \int_\Omega \left|u_n^1\right|^{p^*} > \delta.$$

Without loss of generality, we may assume that

$$0 < \delta < \left(\frac{S_p}{2\mu}\right)^{N/p} \tag{4.1}$$

where $S_p > 0$ is the best constant for which the Gagliardo-Nirenberg-Sobolev inequality (GNS) holds in all of $\mathcal{D}^{1,p}(\mathbb{R}^N)$:

$$S_p \left\|w\right\|_{L^{p^*}(\mathbb{R}^N)}^p \leq \left\|\nabla w\right\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}^p, \quad \forall w \in \mathcal{D}^{1,p}(\mathbb{R}^N),$$

Now, every Q_n is a continuous function on $[0,\infty)$ satisfying both

$$Q_n(0) = 0 \quad \text{and} \quad \lim_{r\to\infty} Q_n(r) > \delta.$$

By the intermediate value theorem, there exists $r_n \in [0,\infty)$ such that $Q_n(r_n) = \delta$. Letting R_n be the set of all such r_n, we put $\lambda_n^1 := \inf R_n \geq 0$. Fixing n, let $(r_l)_{l\in\mathbb{N}}$ be a sequence in R_n converging to λ_n^1 as $l \to \infty$. By continuity, it is clear that

$$Q_n(\lambda_n^1) = \lim_{l\to\infty} Q_n(r_l) = \delta.$$

Furthermore, since $Q_n(0) = 0$, we necessarily have $\lambda_n^1 > 0$. Proceeding inductively, we obtain a sequence (λ_n^1) in $(0,\infty)$ such that

$$Q_n(\lambda_n^1) = \sup_{y\in\bar{\Omega}} \int_{B(y,\lambda_n^1)} \left|u_n^1\right|^{p^*} = \delta,$$

for each $n \in \mathbb{N}$. Once again fixing $n \geq 1$, we may find a sequence $(\xi_{m,n}^1)_{m=1}^\infty$ in $\bar{\Omega}$ such that

$$\delta = \lim_{m\to\infty} \int_{B(\xi_{m,n}^1,\lambda_n^1)} \left|u_n^1\right|^{p^*}.$$

Passing to a subsequence if necessary, we may assume that $\xi^1_{m,n} \to y^1_n \in \bar{\Omega}$ as we let $m \to \infty$. By the dominated convergence theorem,

$$\delta = \lim_{m\to\infty} \int_{B(\xi^1_{m,n},\lambda^1_n)} \left|u^1_n\right|^{p^*} = \int_{B(y^1_n,\lambda^1_n)} \left|u^1_n\right|^{p^*}, \quad \forall n \in \mathbb{N}.$$

Remark 4.1. Since Ω is bounded, the sequence of radii (λ^1_n) must also be bounded. To be more explicit, we must have $0 < \lambda^1_n \leq 2\,\mathrm{diam}(\Omega)$ for each $n \geq 1$. Furthermore, the sequence of centers (y^1_n) lives in $\bar{\Omega}$ and must also be bounded. Thus, after passing to another subsequence, we may find y^1 in $\bar{\Omega}$ and $\lambda^1 \geq 0$ such that

$$\lim_{n\to\infty} y^1_n = y^1 \quad \text{and} \quad \lim_{n\to\infty} \lambda^1_n = \lambda^1.$$

Step 3. *For each $n \in \mathbb{N}$ consider the domain*

$$\Omega_n := \frac{1}{\lambda^1_n}\left(\Omega - y^1_n\right).$$

It is obvious that $x \in \Omega_n$ if and only if

$$\lambda^1_n x + y^1_n \in \Omega.$$

Next, we define a sequence $(v^1_n)_{n\in\mathbb{N}}$ in $W^{1,p}_0(\Omega_n) \subseteq W^{1,p}(\mathbb{R}^N)$ by the rule

$$v^1_n(x) := \left(\lambda^1_n\right)^{\frac{N-p}{p}} u^1_n(\lambda^1_n x + y^1_n).$$

The family $\{v^1_n\}_{n\in\mathbb{N}}$ belongs to the space $W^{1,p}(\mathbb{R}^N)$ and is uniformly bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Moreover, one has

$$\delta = \sup_{y\in\overline{\Omega_n}} \int_{B(y,1)} \left|v^1_n\right|^{p^*} = \int_{B(0,1)} \left|v^1_n\right|^{p^*}. \tag{4.2}$$

Proof of Step 3. Since every v^1_n is a rescaling of a function in $W^{1,p}(\mathbb{R}^N)$, it is clear that $\{v^1_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N) \subseteq \mathcal{D}^{1,p}(\mathbb{R}^N)$. To see that $\{v^1_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, it suffices to use the homogeneity of $\mathcal{D}^{1,p}(\mathbb{R}^N)$:

$$\begin{aligned}
\left\|v^1_n\right\|^p = \left\|\nabla v^1_n\right\|^p_{L^p(\mathbb{R}^N;\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \left|\lambda^1_n \cdot \left(\lambda^1_n\right)^{\frac{N-p}{p}} \nabla u^1_n(\lambda^1_n x + y^1_n)\right|^p \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} (\lambda^1_n)^N \left|\nabla u^1_n(\lambda^1_n x + y^1_n)\right|^p \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left|\nabla u^1_n\right|^p \\
&\leq \left\|u^1_n\right\|^p_{W^{1,p}(\mathbb{R}^N)}.
\end{aligned}$$

Since the $\{u_n^1\}_{n\in\mathbb{N}}$ are bounded in $W^{1,p}(\mathbb{R}^N)$ by Step 1, the family $\left\{v_n^1\right\}_{n\in\mathbb{N}}$ is therefore bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Next, we verify (4.2). Given $n \in \mathbb{N}$, we have by construction that

$$\begin{aligned}\int_{B(0,1)} \left|v_n^1\right|^{p^*} &= \int_{B(0,1)} \left|(\lambda_n^1)^{\frac{N-p}{p}} u_n^1(\lambda_n^1 x + y_n^1)\right|^{\frac{Np}{N-p}} \,\mathrm{d}x \\ &= \int_{B(0,1)} (\lambda_n^1)^N \left|u_n^1(\lambda_n^1 x + y_n^1)\right|^{p^*} \,\mathrm{d}x \\ &= \int_{B(y_n^1,\lambda_n^1)} \left|u_n^1\right|^{p^*} = \delta.\end{aligned}$$

Since $y \in \overline{\Omega_n}$ if and only if

$$y = \frac{z - y_n^1}{\lambda_n^1}$$

for a unique $z \in \bar{\Omega}$, a similar argument yields

$$\begin{aligned}\sup_{y\in\overline{\Omega_n}} \int_{B(y,1)} \left|v_n^1\right|^{p^*} &= \sup_{y\in\overline{\Omega_n}} \int_{B(y,1)} \left|(\lambda_n^1)^{\frac{N-p}{p}} u_n^1(\lambda_n^1 x + y_n^1)\right|^{\frac{Np}{N-p}} \,\mathrm{d}x \\ &= \sup_{y\in\overline{\Omega_n}} \int_{B(y,1)} (\lambda_n^1)^N \left|u_n^1(\lambda_n^1 x + y_n^1)\right|^{p^*} \,\mathrm{d}x \\ &= \sup_{z\in\bar{\Omega}} \int_{B(z,\lambda_n^1)} \left|u_n^1\right|^{p^*} = \delta.\end{aligned}$$

This directly implies (4.2) and the proof is complete. □

For each $n \in \mathbb{N}$, let us now consider the linear functional $\phi'_\infty(u_n^1)$ on $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Since $W_0^{1,p}(\Omega)$ is a closed subspace of $\mathcal{D}^{1,p}(\mathbb{R}^N)$, every $\phi'_\infty(u_n^1)$ restricts to an element of the dual $W^{-1,p'}(\Omega)$. By a duality result analogous to the Riesz Representation Theorem for

L^p, we can find $f_n^1, \dots, f_n^N \in L^{p'}(\Omega)$ such that[6]

$$\langle \phi'_\infty(u_n^1), h\rangle = \sum_{i=1}^{N} \int_\Omega f_n^i \partial_i h$$

for all $h \in W_0^{1,p}(\Omega)$. Next, consider the functions given by

$$g_n^i = \left(\lambda_n^1\right)^{N-\frac{N}{p}} f_n^i(\lambda_n^1 x + y_n^1), \quad \forall i = 1, \dots, N.$$

Step 4. *Adhering to the notation and terminology developed above, one has*

$$\langle \phi'_\infty(v_n^1), h\rangle = \sum_{i=1}^{N} \int_{\Omega_n} g_n^i \partial_i h, \quad \forall h \in W_0^{1,p}(\Omega_n).$$

Proof of Step 4. First, we fix a function $h \in W_0^{1,p}(\Omega_n)$ and define

$$\tilde{h}(z) = h\left(\frac{z - y_n^1}{\lambda_n^1}\right).$$

Notice that $\tilde{h} \in W_0^{1,p}(\Omega)$. Furthermore, we have the following equalities;

$$\lambda_n^1 \partial_i \tilde{h}(z) = \partial_i h\left(\frac{z - y_n^1}{\lambda_n^1}\right) \quad \text{and} \quad \lambda_n^1 \nabla \tilde{h}(z) = \nabla h\left(\frac{z - y_n^1}{\lambda_n^1}\right).$$

Similarly, we see that

$$\nabla v_n^1(x) = (\lambda_n^1)^{(N-p)/p+1} \nabla u_n^1(\lambda_n^1 x + y_n^1) = (\lambda_n^1)^{N/p} \nabla u_n^1(\lambda_n^1 x + y_n^1).$$

6. Given general domain Ω and a continuous linear functional φ on $W_0^{1,p}(\Omega)$, we can find $f_0, f_1, \dots, f_N$ belonging to $L^{p'}(\Omega)$ such that

$$\varphi(w) = \int_\Omega f_0 w + \sum_{i=1}^{N} \int_\Omega f_i \partial_i w$$

for all $w \in W_0^{1,p}(\Omega)$. However, when Ω is bounded, one can dispense with the f_0-term in the above. That is, we have $\varphi(w) = \sum_{i=1}^N \int_\Omega f_i^p \partial_i w$ for suitable $f_1, \dots, f_N$ in $L^{p'}(\Omega)$. This is not too surprising since $w \mapsto \|\nabla w\|_{L^p(\Omega)}$ is an equivalent norm on $W_0^{1,p}(\Omega)$. We refer the reader to Adams-Fournier [1] and Brézis [2] for a precise statement and proof.

Fix now an integer $n \geq 1$ and let $h \in W_0^{1,p}(\Omega_n)$ be given. It is easily seen that

$$\begin{aligned}
\sum_{i=1}^{N} \int_{\Omega_n} g_n^i \partial_i h &= \sum_{i=1}^{N} \int_{\Omega_n} (\lambda_n^1)^{N-\frac{N}{p}} f_n^i(\lambda_n^1 x + y_n^1) \partial_i h(x) \, \mathrm{d}x \\
&= (\lambda_n^1)^{N-\frac{N}{p}-N} \sum_{i=1}^{N} \int_{\Omega} f_n^i(z) \partial_i h \left(\frac{z - y_n^1}{\lambda_n^1} \right) \mathrm{d}z \\
&= (\lambda_n^1)^{-\frac{N}{p}} \sum_{i=1}^{N} \int_{\Omega} f_n^i(z) \partial_i h \left(\frac{z - y_n^1}{\lambda_n^1} \right) \mathrm{d}z \\
&= (\lambda_n^1)^{1-N/p} \sum_{i=1}^{N} \int_{\Omega} f_n^i(z) \partial_i \tilde{h}(z) \, \mathrm{d}z \\
&= (\lambda_n^1)^{1-N/p} \langle \phi_\infty'(u_n^1), \tilde{h} \rangle.
\end{aligned}$$

On the other hand, $\langle \phi_\infty'(v_n^1), h \rangle$ is given by

$$\int_{\mathbb{R}^N} \left[\left| \nabla v_n^1 \right|^{p-2} \nabla v_n^1 \cdot \nabla h - \mu \left| v_n^1 \right|^{p^*-2} v_n^1 h \right]$$

which, when expanded, becomes

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left| (\lambda_n^1)^{N/p} \nabla u_n^1(\lambda_n^1 x + y_n^1) \right|^{p-2} (\lambda_n^1)^{N/p} \nabla u_n^1(\lambda_n^1 x + y_n^1) \cdot \nabla h(x) \, \mathrm{d}x - \int_{\mathbb{R}^N} \mu \left| v_n^1 \right|^{p^*-2} v_n^1 h \\
&= (\lambda_n^1)^{N(p-1)/p} \int_{\mathbb{R}^N} \left| \nabla u_n^1(\lambda_n^1 x + y_n^1) \right|^{p-2} \nabla u_n^1(\lambda_n^1 x + y_n^1) \cdot \nabla h(x) \, \mathrm{d}x \\
&\qquad - \int_{\mathbb{R}^N} \mu \left| (\lambda_n^1)^{(N-p)/p} u_n^1(\lambda_n^1 x + y_n^1) \right|^{p^*-2} (\lambda_n^1)^{(N-p)/p} u_n^1(\lambda_n^1 x + y_n^1) h(x) \, \mathrm{d}x \\
&= (\lambda_n^1)^{N(p-1)/p-N} \int_{\mathbb{R}^N} \left| \nabla u_n^1(z) \right|^{p-2} \nabla u_n^1(z) \cdot \nabla h \left(\frac{z - y_n^1}{\lambda_n^1} \right) \mathrm{d}z \\
&\qquad - (\lambda_n^1)^{(N-p)(p^*-1)/p} \int_{\mathbb{R}^N} \mu \left| u_n^1(\lambda_n^1 x + y_n^1) \right|^{p^*-2} u_n^1(\lambda_n^1 x + y_n^1) h(x) \, \mathrm{d}x \\
&= (\lambda_n^1)^{-N/p} \int_{\mathbb{R}^N} \left| \nabla u_n^1(z) \right|^{p-2} \nabla u_n^1(z) \cdot \nabla h \left(\frac{z - y_n^1}{\lambda_n^1} \right) \mathrm{d}z \\
&\qquad - (\lambda_n^1)^{(N-p)(p^*-1)/p-N} \int_{\mathbb{R}^N} \mu \left| u_n^1(z) \right|^{p^*-2} u_n^1(z) h \left(\frac{z - y_n^1}{\lambda_n^1} \right) \mathrm{d}z \\
&= (\lambda_n^1)^{1-N/p} \int_{\mathbb{R}^N} \left| \nabla u_n^1(z) \right|^{p-2} \nabla u_n^1(z) \cdot \nabla \tilde{h}(z) \, \mathrm{d}z - (\lambda_n^1)^{1-N/p} \int_{\mathbb{R}^N} \mu \left| u_n^1(z) \right|^{p^*-2} u_n^1(z) \tilde{h}(z) \, \mathrm{d}z.
\end{aligned}$$

After some cleaning up, this last expression is equal to

$$(\lambda_n^1)^{1-N/p} \int_{\mathbb{R}^N} \left[\left| \nabla u_n^1(z) \right|^{p-2} \nabla u_n^1 \cdot \nabla \tilde{h} - \mu \left| u_n^1 \right|^{p^*-2} u_n^1 \tilde{h} \right],$$

which is precisely

$$(\lambda_n^1)^{1-N/p} \langle \phi_\infty'(u_n^1), \tilde{h} \rangle.$$

Combining these identities, we find that for all $h \in W_0^{1,p}(\Omega_n)$:

$$\langle \phi_\infty'(v_n^1), h \rangle = (\lambda_n^1)^{1-N/p} \langle \phi_\infty'(u_n^1), \tilde{h} \rangle = \sum_{i=1}^N \int_{\Omega_n} g_n^i \partial_i h.$$

□

In light of the observations made in Remark 4.1, we may assume that $\Omega_n \to \Omega_\infty$, as $n \to \infty$, where $\Omega_\infty \subseteq \mathbb{R}^N$ is

- a rescaled translation of Ω if $\lambda^1 > 0$;
- all of $\mathbb{R}^N$ if $\lambda^1 = 0$ and $y^1 \in \Omega$; or
- a half space if $\lambda^1 = 0$ and $y^1 \in \partial\Omega$.

Here, the convergence should be understood in the sense of indicator functions, which is equivalent to the compact-open topology. However, a geometric picture of this convergence is enough for our purposes.

Step 5. *Passing to a subsequence if necessary, we may assume that $v_n^1 \rightharpoonup v_1$ for some $v_1 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and that*

$$\lim_{n\to\infty} v_n^1(x) = v_1(x)$$

for almost every $x \in \mathbb{R}^N$.

Proof of Step 5. We already know from Step 3 that (v_n^1) is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Thus, the remaining assertions follow at once from Theorem 2.9. □

Next, we show that our first bubble $v^1 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ is non-trivial.

Step 6. *The weak $\mathcal{D}^{1,p}(\mathbb{R}^N)$-limit v_1 of v_n^1 is non-zero.*

Proof of Step 6. We argue by contradiction. If $v_1 = 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, then v_1 vanishes almost everywhere in $\mathbb{R}^N$. Recall that the $\{v_n^1\}_{n\in\mathbb{N}}$ are uniformly bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Since the family $\{v_n^1\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, the Gagliardo-Nirenberg-Sobolev inequality (GNS) ensures that the $\{v_n^1\}_{n\in\mathbb{N}}$ also form a bounded family of functions in $L^{p^*}(\mathbb{R}^N)$.

Now, given an open ball $B \subset \mathbb{R}^N$, we see that $\{v_n^1\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $W^{1,p}(B)$. By the Rellich-Kondrachov theorem, we may assume that $v_n^1 \to w$ strongly in $L^p(B)$. Then, passing to a subsequence yet again, we must have $v_n^1 \to w$ pointwise almost everywhere in this ball B. Hence, $w = v_1$ almost everywhere on B. In short, we have found a subsequence (v_n^1) that converges to 0 strongly in $L^p(B)$. By taking an exhaustion of $\mathbb{R}^N$ by open balls of radius $m \geq 1$ and using a diagonal argument, we find a subsequence (also denoted by (v_n^1)) that converges *strongly* to 0 in $L^p(B_m)$, for each $m \in \mathbb{N}$. Clearly, this means that $v_n^1 \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ as $n \to \infty$.

By using Step 1 together with a duality result from Brézis [2] (Proposition 9.20 therein), it follows that

$$\left\|\phi'_\infty(u_n^1)\right\|_{W^{-1,p'}(\Omega)} = \sum_{i=1}^N \int_\Omega \left|f_n^i\right|^{p'} \tag{4.3}$$

$$= \sum_{i=1}^N \int_{\Omega_n} \left|g_n^i\right|^{p'} \to 0, \quad \text{as } n \to \infty. \tag{4.4}$$

Next, recall that Ω_∞ denotes the "limiting domain" of the Ω_n. By definition, if $y \in \Omega_\infty$, then y belongs to Ω_n for all sufficiently large n. Fix now a point $y \in \Omega_\infty$ and consider the ball $B(y, 1)$, which may not be contained in Ω_∞. For a given test function $h \in C_c^\infty(\mathbb{R}^N)$ with $\text{supp}(h) \subset B(y, 1)$, Hölder's inequality with exponent $\frac{N}{p}$, gives

$$\begin{aligned}\int_{\mathbb{R}^N} |h|^p \left|v_n^1\right|^{p^*} &= \int_{\mathbb{R}^N} \left|v_n^1\right|^{p^*-p} \cdot \left|hv_n^1\right|^p \\ &\leq \left(\int_{\text{supp}(h)} \left|v_n^1\right|^{p^*}\right)^{p/N} \left(\int_{\mathbb{R}^N} \left|hv_n^1\right|^{p^*}\right)^{\frac{N-p}{N}}.\end{aligned}$$

From this, an application of Sobolev's inequality shows that

$$\int_{\mathbb{R}^N} |h|^p \left|v_n^1\right|^{p^*} \leq \left(\int_{\text{supp}(h)} \left|v_n^1\right|^{p^*}\right)^{p/N} \left(\int_{\mathbb{R}^N} \left|hv_n^1\right|^{p^*}\right)^{\frac{N-p}{N}} \tag{4.5}$$

$$\leq S_p^{-1} \left(\int_{\text{supp}(h)} \left|v_n^1\right|^{p^*}\right)^{p/N} \left(\int_{\mathbb{R}^N} \left|\nabla(hv_n^1)\right|^p\right). \tag{4.6}$$

Because v_n^1 is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and converges *strongly* to 0 in $L^p_{\text{loc}}(\mathbb{R}^N)$, one has

$$\int_{\Omega_n} \left|\nabla(hv_n^1)\right|^p = \int_{\mathbb{R}^N} |h|^p \left|\nabla v_n^1\right|^p + o(1) \tag{4.7}$$
$$= \int_{\mathbb{R}^N} \left|\nabla v_n^1\right|^{p-2} \nabla v_n^1 \cdot \nabla\left(|h|^p\, v_n^1\right) + o(1) \tag{4.8}$$
$$= \langle \phi'_\infty(v_n^1), |h|^p\, v_n^1\rangle + \mu \int_{\mathbb{R}^N} \left|v_n^1\right|^{p^*-2}\left(v_n^1\right)^2 |h|^p + o(1) \tag{4.9}$$
$$= \sum_{i=1}^N \int_{\Omega_n} g_n^i \partial_i(|h|^p\, v_n^1) + \mu \int_{\mathbb{R}^N} |h|^p \left|v_n^1\right|^{p^*} + o(1) \tag{4.10}$$
$$\leq \mu \int_{\mathbb{R}^N} |h|^p \left|v_n^1\right|^{p^*} + o(1). \tag{4.11}$$

Let us take a brief moment to justify this series of estimates. Equality (4.7)-(4.8) follows from Hölder's inequality. Thereafter, equation (4.11) follows from (4.3)-(4.4) together with an application of Hölder's inequality after observing that $\partial_i(|h|^p\, v_n^1)$ is uniformly bounded in $L^p(\mathbb{R}^N)$.[7] Applying (4.2) and (4.5)-(4.6), we actually obtain

$$\int_{\Omega_n} \left|\nabla(hv_n^1)\right|^p \leq \mu S_p^{-1} \delta^{\frac{p}{N}} \int_{\mathbb{R}^N} \left|\nabla(hv_n^1)\right|^p + o(1)$$
$$\leq \frac{1}{2} \int_{\mathbb{R}^N} \left|\nabla(hv_n^1)\right|^p + o(1)$$

where the last inequality follows from property (4.1). Whence we conclude that

$$\lim_{n\to\infty} \int_{\Omega_n} \left|\nabla(hv_n^1)\right|^p = 0.$$

This means that $\nabla v_n^1 \to 0$ in $L^p(B)$, for any ball B of radius 1 with center in Ω_∞. After covering Ω_∞ with countably many balls, each of radius 1 with center in Ω_∞, we infer that ∇v_n^1 converges to 0 in $L^p_{\text{loc}}(\Omega_\infty)$, as $n \to \infty$. Similarly, $\left\|\nabla v_n^1\right\|_{L^p(B)} \to 0$ on any ball B compactly contained in the complement of Ω_∞, as $n \to \infty$. To summarize, we have shown that $\nabla v_n^1 \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ whence $v_n^1 \to 0$ in $W^{1,p}_{\text{loc}}(\mathbb{R}^N)$. Finally, the Sobolev inequality implies that

$$\lim_{n\to\infty} v_n^1 = 0 \quad \text{in } L^{p^*}_{\text{loc}}(\mathbb{R}^N),$$

which directly contradicts (4.2). □

7. A simple calculation shows that $\partial_i(|h|^p\, v_n^1) = v_n^1 \partial_i(|h|^p) + |h|^p\, \partial_i v_n^1$. To see that $v_n^1 \partial_i(|h|^p)$ is bounded in $L^p(\mathbb{R}^N)$, we recall that v_n^1 is bounded in $L^{p^*}(\mathbb{R}^N)$ and $h \in C_c^\infty(\mathbb{R}^N)$. Similarly, $|h|^p\, \partial_i v_n^1$ is bounded in $L^p(\mathbb{R}^N)$ since $\partial_i v_n^1$ is bounded in $L^p(\mathbb{R}^N)$ and $h \in C_c^\infty(\mathbb{R}^N)$.

Step 7. *The radii converge to* 0 *as* $n \to \infty$, *i.e.* $\lambda^1 = \lim_{n\to\infty} \lambda_n^1 = 0$.

Proof of Step 7. Arguing by contradiction, let us assume that $\lambda^1 > 0$. Since $v_n^1 \rightharpoonup v_1$ weakly in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, Proposition 2.6 states that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla v_n^1 \cdot g = \int_{\mathbb{R}^N} \nabla v_1 \cdot g, \quad \forall g \in L^{p'}(\mathbb{R}^N; \mathbb{R}^N). \tag{4.12}$$

Fix now $\varphi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$; a straightforward calculation shows that

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla v_n^1 \cdot \varphi &= \int_{\mathbb{R}^N} (\lambda_n^1)^{\frac{N-p}{p}} \cdot \lambda_n^1 \nabla u_n^1(\lambda_n^1 x + y_n^1) \cdot \varphi(x)\, \mathrm{d}x \\ &= \int_{\mathbb{R}^N} (\lambda_n^1)^{N/P} \nabla u_n^1(\lambda_n^1 x + y_n^1) \cdot \varphi(x)\, \mathrm{d}x \\ &= (\lambda_n^1)^{N/p-N} \int_{\mathbb{R}^N} \nabla u_n^1(z) \cdot \varphi\left(\frac{z - y_n^1}{\lambda_n^1}\right) \mathrm{d}z.\end{aligned}$$

Now, Step 1 ensures that $\nabla u_n^1 \to 0$ pointwise almost everywhere on Ω. Actually, since $\{u_n^1\}_{n\geqslant 1} \subset W_0^{1,p}(\Omega)$, we know that $\nabla u_n^1 \to 0$ pointwise almost everywhere on $\mathbb{R}^N$. Moreover, it is not hard to see that the sequence (∇u_n^1) is uniformly bounded in $L^p(\mathbb{R}^N; \mathbb{R}^N)$. Since $\lambda_n^1 \to \lambda^1 > 0$ and $y_n^1 \to y^1 \in \bar{\Omega}$, there exists a compact set $\Lambda \subset \mathbb{R}^N$ such that

$$\operatorname{supp}\left(\varphi\left(\frac{\cdot - y_n^1}{\lambda_n^1}\right)\right) \subseteq \Lambda, \quad \forall n \geq 1.$$

Using now that φ is bounded, an application of Theorem 2.1 gives

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla v_n^1 \cdot \varphi = \lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla u_n^1(z) \cdot \varphi\left(\frac{z - y_n^1}{\lambda_n^1}\right) \mathrm{d}z = 0.$$

Indeed, this is because for $M := \sup_{\mathbb{R}^N} |\varphi|$ one has

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \nabla u_n^1(z) \cdot \varphi\left(\frac{z - y_n^1}{\lambda_n^1}\right) \mathrm{d}z\right| &\leq \int_{\mathbb{R}^N} \left|\nabla u_n^1(z) \cdot \varphi\left(\frac{z - y_n^1}{\lambda_n^1}\right)\right| \mathrm{d}z \\ &\leq \int_{\mathbb{R}^N} \left|\nabla u_n^1\right| M \mathbf{1}_\Lambda.\end{aligned}$$

Now, we relax the assumption that $\varphi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$. Fix $g \in L^{p'}(\mathbb{R}^N; \mathbb{R}^N)$ and let $\varepsilon > 0$ be given. By density, there exists $\varphi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$ such that

$$\|g - \varphi\|_{L^{p'}(\mathbb{R}^N;\mathbb{R}^N)} < \varepsilon.$$

Clearly, one has

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \nabla v_n^1 \cdot g\right| &\leq \left|\int_{\mathbb{R}^N} \nabla v_n^1 \cdot \varphi\right| + \left|\int_{\mathbb{R}^N} \nabla v_n^1 \cdot (g-\varphi)\right| \\ &\leq \left|\int_{\mathbb{R}^N} \nabla v_n^1 \cdot \varphi\right| + \left\|v_n^1\right\| \|g-\varphi\|_{L^{p'}(\mathbb{R}^N;\mathbb{R}^N)} .\end{aligned}$$

Since $\{v_n^1\}_{n\geqslant 1}$ is a bounded family in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \nabla v_n^1 \cdot \varphi \to 0$ as $n \to \infty$, we infer that

$$\left|\int_{\mathbb{R}^N} \nabla v_n^1 \cdot g\right| < \varepsilon + \sup_{n\geqslant 1} \left\|v_n^1\right\| \varepsilon$$

for all n large. Or, rather, that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla v_n^1 \cdot g = 0, \quad \forall g \in L^{p'}(\mathbb{R}^N;\mathbb{R}^N).$$

But by Proposition 2.6 this means that $v_n^1 \rightharpoonup 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, which directly contradicts Step 6. It follows that $\lambda_n^1 \to 0$ as $n \to \infty$. □

Step 8. *The function v^1 is almost everywhere non-negative in $\mathbb{R}^N$, i.e. $v^1 \geq 0$.*

Proof of Step 8. Recall from Step 5 that $v_n^1 \to v^1$ a.e. on $\mathbb{R}^N$. Hence, it would be enough to check that $\lim_{n\to\infty} v_n^1(x) \geq 0$ for almost every $x \in \mathbb{R}^N$. In fact, because

$$v_n^1(x) = \left(\lambda_n^1\right)^{\frac{N-p}{p}} u_n(\lambda_n^1 x + y_n^1) - \left(\lambda_n^1\right)^{\frac{N-p}{p}} v_0(\lambda_n^1 x + y_n^1)$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[\left(\lambda_n^1\right)^{\frac{N-p}{p}} u_n(\lambda_n^1 x + y_n^1)\right]_-^{p^*} \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^N} (u_n)_-{}^{p^*} = 0,$$

it would be enough to show that the sequence

$$w_n(x) := \left(\lambda_n^1\right)^{\frac{N-p}{p}} v_0(\lambda_n^1 x + y_n^1)$$

has a subsequence converging a.e. to zero on $\mathbb{R}^N$. Fix now $R > 0$ and consider the open ball $B_R := B(0,R)$. By homogeneity, we have that

$$\int_{B_R} |w_n|^{p^*} = \int_{B(y_n^1, R\lambda_n^1)} |v_0|^{p^*} \to 0,$$

because $m(B(y_n^1, R\lambda_n^1)) \to 0$ as $n \to \infty$. Hence, (w_n) has a subsequence converging pointwise a.e. to 0 on $B_R(0)$, for any $R > 0$. By applying this argument iteratively with the sequence $R_m := m$ with $m \in \mathbb{N}$ and using a diagonal argument, we obtain a subsequence (w_n) convegring pointwise a.e. on $\mathbb{R}^N$ to 0. □

Since $\lambda_n^1 \to 0$ as $n \to \infty$, we can rule out the case where Ω_∞ is a rescaled translation of Ω. This leaves us with only two possibilities:

- $y^1 \in \Omega$ and $\Omega_\infty = \mathbb{R}^N$; or
- $y_1 \in \partial\Omega$ and Ω_∞ is a half-space.

If Ω_∞ is a half-space, then an argument similar to that used in Lemma 3.2 shows that v^1 is a non-negative (by virtue of Step 8) solution to the limiting problem (1.9) in a half-space. However, this together with Step 6 would contradict Theorem C. Therefore, $\Omega_\infty = \mathbb{R}^N$, $y^1 \in \Omega$, and v_0 solves (1.4) by Lemma 3.2. In particular, after passing to a subsequence, we may assume that $y_n \in \Omega$ for each $n \geq 1$.

Step 9. *If $w \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ is a non-trivial critical point of ϕ_∞, one has*

$$\phi_\infty(w) \geq \frac{\mu}{N}\left(\frac{S_p}{\mu}\right)^{N/p} > 0. \tag{4.13}$$

Proof of Step 9. Let $w \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a non-trivial critical point of ϕ_∞. By definition, this forces

$$0 = \langle \phi'_\infty(w), w\rangle = \int_{\mathbb{R}^N} \left(|\nabla w|^p - \mu\,|w|^{p^*}\right)$$

whence, by the Gagliardo-Nirenberg-Sobolev inequality (GNS)

$$S_p\,\|w\|^p_{L^{p^*}(\mathbb{R}^N)} \leq \|\nabla w\|^p_{L^p(\mathbb{R}^N;\mathbb{R}^N)} = \|w\| = \mu\,\|w\|^{p^*}_{L^{p^*}(\mathbb{R}^N)}\,.$$

Especially, this gives

$$\frac{S_p}{\mu} \leq \|w\|^{p^*-p}_{L^{p^*}(\mathbb{R}^N)}\,.$$

Put otherwise,

$$\|w\|_{L^{p^*}(\mathbb{R}^N)} \geq \left(\frac{S_p}{\mu}\right)^{\frac{1}{p^*-p}}\,.$$

From this, we infer that

$$\phi_\infty(w) = \frac{\|w\|^p}{p} - \frac{\mu \|w\|^{p^*}_{L^{p^*}}}{p^*} = \mu \|w\|^{p^*}_{L^{p^*}} \left(\frac{1}{p} - \frac{1}{p^*}\right)$$
$$\geq \mu \left[\left(\frac{S_p}{\mu}\right)^{\frac{1}{p^*-p}}\right]^{p^*} \left(\frac{1}{N}\right)$$
$$= \frac{\mu}{N}\left(\frac{S_p}{\mu}\right)^{N/p} > 0.$$

□

Next, we complete the proof of Theorem 1.1. By invoking Lemma 3.2, the sequence

$$u_n^2(x) := u_n^1(x) - \left(\lambda_n^1\right)^{\frac{p-N}{N}} v_1\left(\frac{x - y_n^1}{\lambda_n^1}\right)$$

satisfies

1. $\left\|u_n^2\right\|^p = \|u_n\|^p - \|v_0\|^p - \|v_1\|^p + o(1)$;
2. $\phi_\infty(u_n^2) \to \lim_{n\to\infty} \phi(u_n) - \phi(v_0) - \phi_\infty(v_1)$; and
3. $\phi'_\infty(u_n^2) \to 0$ in $W^{-1,p'}(\Omega)$.

In light of this and Step 9, the iterative procedure described above can only construct finitely many sequences (v_n^i), (λ_n^i), and (y_n^i). Afterwards, we would find ourselves in Step 2 thereby terminating the proof. We note that this part of the argument is standard and can be found explicitly in Struwe [8]-[9], Willem [10]-[11], and Mercuri-Willem [5].

4.1 Further Observations

The assumption that $(u_n)_- \to 0$ in $L^{p^*}(\Omega)$ is only used to establish the non-negativity of the v_i extracted in Step 5. Then, by using Theorem C, it was possible to show that the v_i satisfied (1.4) as opposed to, possibly, the limiting problem (1.9) for a half-space. By relaxing this assumption on (u_n), a stronger and more precise variant of Theorem 1.1 continues to hold true. Formally, we have the following theorem:

Theorem 4.1. *Let (u_n) be a Palais-Smale sequence for ϕ. After passing to a subsequence, there exists a weak solution $v_0 \in W_0^{1,p}(\Omega)$ to the problem*

$$\begin{cases} -\Delta_p u + a\,|u|^{p-2}\,u \equiv \mu\,|u|^{p^*-2}\,u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and a finite (possibly empty) family $v_1, \ldots, v_k$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ of weak solutions to either

$$-\Delta_p u + a\,|u|^{p-2}\,u \equiv \mu\,|u|^{p^*-2}\,u \quad \text{in } \mathbb{R}^N$$

or

$$\begin{cases} -\Delta_p u \equiv \mu\,|u|^{p^*-2}\,u \text{ in a halfspace } \mathbb{H}, \\ u \in \mathcal{D}_0^{1,p}(\mathbb{H}), \end{cases}$$

together with associated sequences $(y_n^i)_{n\in\mathbb{N}} \subset \overline{\Omega}$ and $(\lambda_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$\left\| u_n - v_0 - \sum\nolimits_{i=1}^{k} \left(\lambda_n^i\right)^{\frac{p-N}{p}} v_i \left(\frac{\cdot - y_n^i}{\lambda_n^i}\right) \right\| \to 0 \quad \text{as } n \to \infty,$$

$$\|u_n\|^p \to \sum_{i=0}^{k} \|v_i\|^p \quad \text{as } n \to \infty,$$

$$\phi(v_0) + \sum_{i=1}^{k} \phi_\infty(v_i) = \lim_{n\to\infty} \phi(u_n).$$

Since Lemmas 3.1-3.2 were established without any assumption on the limiting behaviour of $(u_n)_-$, the proof of this theorem is simply an easy adaptation of the argument used in the proof of Theorem 1.1.

A Appendix

In this section, we state or establish results necessary for this paper that are now either standard or considered elementary. Unlike the results presented in §2, these are not directly related to the main idea behind the proof of Theorem 1.1.

Proposition A.1. *Let $(X, \mathfrak{M}, \mu)$ be a finite measure space and let (f_n) be a sequence of measurable functions converging almost everywhere to a measurable function f on X. Then, f_n converges to f in measure on X.*

Proof. Let $\delta > 0$ be given; we must show that $\mu(\{x \in X : |f_n(x) - f(x)| \geq \delta\}) \to 0$ as $n \to \infty$. To this end, we fix $\varepsilon > 0$. Through Egoroff's theorem, we can find a measurable set $E \subseteq X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ *uniformly* on $X \setminus E$. Let $N \in \mathbb{N}$ be such that

$$|f_n(x) - f(x)| < \delta$$

for all $x \in X \setminus E$ and every $n \geq N$. Then, for all $n \geq N$ one necessarily has

$$\{x \in X : |f_n(x) - f(x)| \geq \delta\} \subseteq E$$

whence $\mu(\{x \in X : |f_n(x) - f(x)| \geq \delta\}) \leq \mu(E) < \varepsilon$. □

Next, we state a nearly trivial inequality for real numbers:

Lemma A.2. *Let* $1 < p < \infty$*; then,*

$$(a - b)\left(|a|^{p-2} a - |b|^{p-2} b\right) \geq 0$$

for all $a, b \in \mathbb{R}$.

Proof. Clearly, we may assume without loss of generality that $a > b$. Then, it would be enough to show that $|a|^{p-2} a \geq |b|^{p-2} b$. Now, this inequality is obvious if $a > b \geq 0$, $a \geq 0 > b$, or $a > 0 \geq b$. It remains only to verify that case where $b < a < 0$. However, this case is also trivial because

$$\left|a\,|a|^{p-2}\right| = |a|^{p-1} < |b|^{p-1} = \left|b\,|b|^{p-2}\right|$$

gives $a\,|a|^{p-2} \geq b\,|b|^{p-2}$. □

Finally, we state (without proof) the Brézis-Lieb lemma (see Theorem 1 in [4]).

Theorem A.3. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $0 < p < \infty$. *Assume that* (f_n) *is a bounded sequence in* $L^p(X, \mu)$ *and that* $f_n \to f$ *a.e. on* X*, where* f *is measurable. Then,* $f \in L^p(X, \mu)$ *and*

$$\lim_{n\to\infty}\left(\int_X |f_n|^p \, \mathrm{d}\mu - \int_X |f_n - f|^p \, \mathrm{d}\mu\right) = \int_X |f|^p \, \mathrm{d}\mu.$$

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