SOLUTIONS TO EXERCISES IN PARTIAL DIFFERENTIAL EQUATIONS

EDWARD CHERNYSH

In this document we solve multiple exercises in preparation for the midterm exam in Math 475 at McGill university. The problems were taken from either the course notes or the assignments. We focus on the following topics:

- The method of characteristics,
- 1D wave equation,
- 3D wave equation,
- distributions (generalized functions),
- Fourier transforms.

1. Method of Characteristics

Exercise 1.1. We know from the lectures that the general solution to the PDE

$$au_x + bu_y + u = 0, \quad a \neq 0$$

is $u(x,y) = f(bx-ay)e^{-\frac{ax+by}{a^2+b^2}}$ for a smooth function f of a single variable. Show that this is equivalent to saying that there exists a function g such that

$$u(x,y) = g(bx-ay)e^{-\frac{x}{a}}.$$

Solution. This follows at once from the observation that

$$-\frac{ax+by}{a^2+b^2} + \frac{x}{a} - \frac{x}{a} = \frac{-a^2x - bya + xa^2 + xb^2}{a(a^2+b^2)} - \frac{x}{a} = -\frac{bx-ay}{a(a^2+b^2)} - \frac{x}{a}.$$

$\square$

Exercise 1.2. Suppose that every solution to $au_x + bu_y = 0$ satisfies $u(1,2) = u(3,6)$. What is $\frac{a}{b}$?

Solution. The PDE may be rewritten as $\nabla u \cdot (a,b) = 0$. This means that all solutions are constant along curves with direction (a,b) at all points. These are precisely curves of the form:

$$y(x) = \frac{b}{a}x + c, \quad c \in \mathbb{R}.$$

Date: October 31, 2017 – Spooky!

Since $u(1,2) = u(3,6)$ for all solutions, it follows that $(1,2)$ and $(3,6)$ live on the same characteristic curve. Hence,

$$4 = 6 - 2 = \frac{b}{a}(3-1)$$

which implies that $\frac{b}{a} = 2$. □

Exercise 1.3. Using the methods of characteristics, solve

$$u_{x_1} + x_1 u_{x_2} + u_{x_3} = u, \quad u(x_1, x_2, 0) = x_1 + x_2.$$

Solution. The characteristic curves of the equation are given by:

$$\dot{x}_1(s) = 1, \quad \dot{x}_2(s) = x_1(s), \quad \dot{x}_3(s) = 1$$

with $z(s) = u(x_1(s), x_2(s), x_3(s))$. Along these curves, by the chain rule, $\dot{z}(s) = z(s)$. This means that

$$z(s) = Ce^s$$

along these curves. The characteristics can then be solves by taking

$$x_1(s) = s + c_1, \quad x_3(s) = s, \quad x_2(s) = \frac{s^2}{2} + c_1 s + c_2.$$

Here we have taken $x_3(s) = s$ so that $x_3(0) = 0$. Therefore,

$$z(0) = Ce^0 = C = x_1(0) + x_2(0) = c_1 + c_2.$$

Hence, $u(x_1, x_2, x_3) = (c_1 + c_2)e^{x_3}$ along these curves. Clearly,

$$c_1 = x_1 - x_3$$

so that

$$c_2 = x_2 - \frac{x_3^2}{2} + (x_3 - x_1)x_3 = x_2 + \frac{x_3^2}{2} - x_1 x_3.$$

Putting all this together yields

$$u(x_1, x_2, x_3) = \left(x_1 - x_3 + x_2 - x_1 x_3 + \frac{x_3^2}{2}\right)e^{x_3}.$$

□

Exercise 1.4. Suppose $u(x,t)$ solves $u_t + u u_x = 0$ in $\mathbb{R} \times (0, \infty)$ with $u(1,1) = 5$. Along which curve in the $x - t$ plane do we know the values of u?

Solution. Along the characteristics we know that the solution u will be constant. Thus, we are asked to determine which characteristic curve the point $(1,1)$ lies. Introducing a dummy variable s

$$\dot{t}(s) = s, \quad z(s) = C, \quad \dot{x}(s) = C$$

where $z(s) = u(x(s), t(s))$. This gives $t(s) = s + 1$ and $x(s) = Cs + 1$ (where we have chosen our arbitrary constants so that $(x(0), t(0)) = (1, 1)$). It follows that $C = 5$. Hence, we know the values of u on

$$x(t) = 5(t - 1) + 1 = 5t - 4.$$

$\square$

Exercise 1.5. Use the method of characteristics to solve the equation

$$u_t + u u_x = 1, \quad u(x, 0) = x \quad \text{for } x, t \geq 0.$$

Solution. As usual, we take $t(s) = s$ as a dummy parameter so that $t(0) = 0$. If $\dot{x}(s) = z(s)$ then $\dot{z}(s) = 1$ along these curves. Hence,

$$z(s) = s + C$$

so that $x(s) = \frac{s^2}{2} + Cs + D$. Observe that for $s = 0$:

$$z(0) = u(x(0), t(0)) = x(0) = D.$$

Therefore $C = D$ which allows us to write

$$x(s) \leftrightarrow x(t) = \frac{t^2}{2} + C(t + 1).$$

This implies that

$$C = \frac{2x - t}{2(t + 1)}.$$

Thus, along these curves

$$u(x, t) = t + \frac{2x - t}{2(t + 1)}.$$

$\square$

Exercise 1.6. Employ the method of characteristics to find a general solution to the PDE $u_x + u_y + u_z = 0$. Find a particular solution when $u(x, y, 0) = x^2 + y^2$.

Solution. The characteristic curves are

$$x(s) = s + c_1, \quad y(s) = s + c_2, \quad z(s) = s + c_3.$$

Without loss of generality we take $c_3 = 0$. The PDE then states that any $C^1(\mathbb{R}^3)$ solution is constant along these curves. Hence, given a triple (x, y, z) in $\mathbb{R}^3$ we need only determine which characteristic curve it lies upon.[1] Each of the curves is determined completely, and uniquely, by c_1 and c_2. Hence,

$$u(x, y, z) = f(c_1, c_2) = f(x - z, y - z)$$

---

[1] It is easy to see that, even with our choice of c_3, every point in $\mathbb{R}^3$ lies on a characteristic.

for some function f in two variables. Now assume that the solution u satisfies the auxiliary condition $u(x, y, 0) = x^2 + y^2$. Then

$$u(x, y, 0) = f(x, y) = x^2 + y^2.$$

Therefore $u(x, y, z) = (x - z)^2 + (y - z)^2$. □

Exercise 1.7. In each of the following, let $u(x, t)$ be a smooth solution to the stated PDE such that $u(4, 1) = 1$. In each case, give a point $x \in \mathbb{R}$ such that we know the value of $u(x, 0)$.

(i) $u_t + u_x = 0$,
(ii) $u_t + uu_x = 0$,
(iii) $u_t + t^2 u_x = 0$,
(iv) $u_t + u_x + u = 0$.

Solution.

(i) The characteristics are of the form $t(s) = s + c_1$ and $x(s) = s + c_2$. An appropriate choice of $c_{1,2}$ yields

$$t(s) = s + 1, \quad x(s) = s + 4.$$

Thus, at $s = -1$ we find that $u(x(-1), t(-1)) = u(3, 0) = 1$ since any solution u must be constant along these characteristics.

(ii) Again, take $t(s) = s + 1$ and $\dot{x}(s) = z(s)$. We know that $z(s) = C$ for some constant C (as implied by the PDE) so that $x(s) = Cs + 4$. The initial condition $u(4, 1) = 1$ implies that $c = 1$. Therefore we are left with the characteristics

$$t(s) = s + 1, \quad x(s) = s + 4.$$

Thus we know that $u(3, 0) = 1$.

(iii) Choose $t(s) = s$ and $\dot{x}(s) = s^2$. We then find that

$$x(s) = \frac{s^3}{3} + C$$

where we take $C = 4$ so that $x(0) = 4$. Hence,

$$x(s) = \frac{s^3}{3} + 4.$$

At $s = -1$:

$$x(-1) = 4 - \frac{1}{3} = \frac{11}{3}.$$

It follows (since u will be constant along such a characteristic and $(4, 1)$ is a point on this curve) that $u(11/3, 0) = 1$.

(iv) Here we choose $t(s) = s + 1$ and $x(s) = s + 4$ so that $(x(0), t(0)) = (4, 1)$ as per our initial data. However, we instead have

$$\dot{z}(s) = -z(s) \implies z(s) = Ce^{-s}.$$

At $s = 0$ we know that $z(0) = u(x(0), t(0)) = Ce^0 = C = 1$. Therefore,

$$z(0) = e^{-s}$$

which shows that $u(3, 0) = e$ (take $s = -1$ in our parametrized curves).

□

2. Wave Equation

Exercise 2.1. Let $\mathbf{E}(x, y, z, t)$ and $\mathbf{B}(x, y, z, t)$ be smooth electric and magnetic fields, respectively. They are governed by Maxwell's equations:

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \times \mathbf{B} = \mu\varepsilon \partial_t E, \quad \nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0.$$

in the above μ and ε are constants (that you likely encountered in physics). Show that if

$$\mathbf{E}(x, y, z, t) = (0, 0, E(x, t)) \quad \text{and} \quad \mathbf{B}(x, y, z, t) = (0, B(x, t), 0)$$

then both $E(x, t)$ and $B(x, t)$ satisfy the wave equation:

$$\partial_{tt} u - c^2 \partial_{xx} u \equiv 0, \quad c = (\mu\varepsilon)^{-1/2}.$$

Solution. We first relate $B(x, t)$ to $E(x, t)$ using Maxwell's equations. Observe that

$$(0, -B_t(x, t), 0) = -\partial_t \mathbf{B}(x, y, z, t) = \nabla \times \mathbf{E}(x, y, z, t) = (0, -E_x(x, t), 0).$$

Thus, $\boxed{B_t(x, t) = E_x(x, t)}$. In like,

$$\mu\varepsilon(0, 0, E_t(x, t)) = \nabla \times \mathbf{B}(x, y, z, t) = (0, 0, B_x(x, t)).$$

Thus, $\boxed{\mu\varepsilon E_t(x, t)} = B_x(x, t)$. Finally,

$$E_{xx}(x, t) = \partial_x B_t(x, t) = B_{tx}(x, t) = \partial_t B_x(x, t) = \mu\varepsilon \partial E_t(x, t) = \mu\varepsilon E_{tt}(x, t).$$

Similarly, $B(x, t)$ satisfies the wave equation. □

Exercise 2.2. Fix a time $t > 0$ and assume $\phi_{1,2}, \psi_{1,2}$ are bounded functions defined on $\mathbb{R}$. Let u_i, for $i = 1, 2$, denote the solution to

$$\partial_{tt} u_i - c^2 \partial_{xx} u \equiv 0, \quad u_i(x, 0) = \phi_i(x), \quad \partial_t u_i(x, 0) = \psi(x).$$

Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left\| \phi_1 - \phi_2 \right\| < \delta \text{ and } \left\| \phi_1 - \phi_2 \right\| < \delta \implies \| u_1 - u_2 \|_\infty < \varepsilon.$$

Solution. Applying d'Albembert's formula, we find that

$$u_1(x,t) = \frac{1}{2}\left(\phi_1(x+ct)+\phi_1(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_1(\sigma)\,\mathrm{d}\sigma$$
$$u_2(x,t) = \frac{1}{2}\left(\phi_2(x+ct)+\phi_2(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_2(\sigma)\,\mathrm{d}\sigma.$$

This implies, in particular, that for $t > 0$ fixed:

$$|u_1(x,t)-u_2(x,t)| \le \frac{\left|\phi_1(x+ct)-\phi_2(x+ct)\right| + \left|\phi_1(x-ct)-\phi_2(x-ct)\right|}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\left|\psi_1(\sigma)-\psi_2(\sigma)\right|\mathrm{d}\sigma.$$

Especially,

$$|u_1(x,t)-u_2(x,t)| \le \left\|\phi_1-\phi_2\right\|_\infty + \frac{1}{2c}\int_{x-ct}^{x+ct}\left\|\psi_1-\psi_2\right\|_\infty \mathrm{d}\sigma.$$

Fix $\varepsilon > 0$ and take $\delta > 0$ such that $\delta(1+t) < \varepsilon$. Then, for each x:

$$|u_1(x,t)-u_2(x,t)| \le \delta + \frac{\delta}{2c}\cdot 2ct = \delta(1+t) < \varepsilon.$$

Take now the supremum over $x \in \mathbb{R}$; this yields

$$\|u_1-u_2\|_\infty \le \delta(1+t) < \varepsilon.$$

□

Exercise 2.3. Suppose that the propagation due to a pressure disturbance in 3D is modeled by the 3D wave equation: $u_{tt} = \Delta u$. At time $t = 0$ an explosion occurs at position $\mathbf{x} = 0$ inducing the initial conditions:

$$\phi(\mathbf{x}) \equiv 0, \quad \psi(\mathbf{x}) = \mathbb{1}_{|\mathbf{x}|\le 1}(\mathbf{x}).$$

(i) What is the value of $u(\mathbf{x},10)$ where $\mathbf{x} = (10,0,0)$?
(ii) At $t = 10$, what is the value of u at the point $\mathbf{x} = (20,8,17)$?
(iii) At what times will $u((20,20,20,),t)$ be non-zero?

Solution. Before we proceed, we apply Kirchoff's formula to obtain an explicit representation of u:

$$u(\mathbf{x},t) = \frac{1}{4\pi t}\iint_{\partial B(\mathbf{x},t)} \psi(\mathbf{y})\,\mathrm{d}\mathbf{S}.$$

(i). Directly taking $\mathbf{x} = (10,0,0)$ and $t = 10$:

$$(\mathbf{x},t) \mapsto \frac{1}{40\pi}\int_{\partial B((10,0,0),10)\cap\overline{B(\mathbf{0},1)}} \mathrm{d}\mathbf{S}$$

(ii). We shall show that $|\mathbf{y}-\mathbf{x}| = 10$ implies that $|\mathbf{y}| > 1$. This will imply that u evaluates to zero at $(\mathbf{x}, 10)$. Indeed,

$$|\mathbf{x}| - |\mathbf{y}| \leq 10 \implies |\mathbf{y}| > 1.$$

(iii). It is easier to determine at which points the displacement vanishes. This is equivalent to saying that

$$|\mathbf{y}-\mathbf{x}| = t \implies |\mathbf{y}| \geq 1.$$

This can occur only if

$$0 \leq t <\leq \sqrt{1200} - 1, \quad t \geq \sqrt{1200} + 1.$$

☐

Exercise 2.4. Consider a vibrating infinite string with initial disturbance at $t = 0$ in the intervals $[1,2]$ and $[4,5]$. At $t = 10$, at which positions will one feel these disturbances?

Solution. Applying d'Alembert's formula with $t = 10$ shows that we wish to know for which x:

$$[x-10, x+10] \cap ([1,2] \cap [4,5]) \neq \emptyset.$$

First, note that $[x-10, x+10]$ intersects $[1,2]$ if and only if

$$x + 10 \geq 1 \quad \text{and} \quad x - 10 \leq 2.$$

That is, if and only if $x \in [-9, 12]$. Similarly, $[x-10, x+10]$ intersects $[4,5]$ if and only if $x \in [-6, 15]$. Hence, we feel the disturbance for $x \in [-9, 15]$. ☐

Exercise 2.5. Consider the 1D wave equation $u_t t = u_x x$ for $(x,t) \in \mathbb{R} \times (0, \infty)$ together with the initial constraints

$$\phi(x) = 0, \quad u_t(x, 0) = \psi(x)$$

where

$$\psi(x) = \begin{cases} 1, & \text{if } |x-3| \leq 1 \text{ or } |x+3| \leq 1, \\ 0, & \text{else.} \end{cases}$$

(i) At the time $t = 1$ where on the string is the displacement non-zero?
(ii) At time $t = 10$ which points have maximal displacement?
(iii) Compute $u(0,t)$.

Solution. Before we proceed, we note that by d'Alembert's formula any solution is of the form

$$u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} \psi(\sigma) \, \mathrm{d}\sigma.$$

(i). At time $t = 1$ we obtain $u(x, 1) = \frac{1}{2} \int_{x-t}^{x+t} \psi(\sigma) \, \mathrm{d}\sigma$ which is non-zero if and only if

$$(x-1, x+1) \cap \{x : |x-3| \leq 1 \text{ or } |x+3| \leq 1\} \neq \emptyset.$$

Note that $|x-3| \leq 1$ if and only if$-1 \leq x-3 \leq 1$ which is equivalent to saying $2 \leq x \leq 4$. Similarly, $|x+3| \leq 1$ if and only if $-4 \leq x \leq -2$. Therefore, we seek points x such that

$$(x-1, x+1) \cap [[-4,-2] \sqcup [2,4]] \neq \emptyset.$$

Clearly, $(x-1,x+1) \cap (-4,-2)$ is non-empty if and only if $x-1 < -2$ and $x+1 > -4$. That is, if and only if $x \in (-5,-1)$. In like, $(x-1,x+1)$ intersects $(2,4)$ if and only if $x-1 < 4$ and $x+1 > 2$. Thus, if and only if $x \in (1,5)$.

(ii) The solution $u(x,10)$ will have maximal displacement whenever $(x-10,x+10)$ covers $[[-4,-2] \sqcup [2,4]]$. This will occur if and only if $x-10 \leq -4$ and $x+10 \geq 4$. Thus, if and only if $x \in [-6,6]$.

(iii). The "difficulty" lies in computing $\int_{-t}^{t} \psi(\sigma)\,\mathrm{d}\sigma$. If $t \leq 2$ then clearly $(-t,t)$ does not intersect $(-4,-2) \sqcup (2,4)$ which implies that $u(0,t) = 0$. On the other-hand, if $t \geq 4$ then $(-t,t)$ engulfs $(-4,-2) \sqcup (2,4)$ which would imply that

$$\frac{1}{2}\int_{-t}^{t} \psi(\sigma)\,\mathrm{d}\sigma = \frac{2+2}{2} = 2.$$

Suppose now that $t \in (2,4)$. Then,

$$u(0,t) = \frac{t-2+(-2+t)}{2} = t-2.$$

That is,

$$u(0,t) = \begin{cases} 0, & \text{if } t \leq 2, \\ t-2, & \text{if } 2 < t < 4, \\ 2, & \text{if } t \geq 4. \end{cases}$$

□

Exercise 2.6. Assume a phenomenon propagates in 3-dimensional space according to the equation $u_{tt} - \Delta u \equiv 0$. Suppose a disturbance at $x = 0$ creates an initial change in velocity according to

$$\psi(\mathbf{x}) = \begin{cases} 1, & \text{if } |\mathbf{x}| \leq 1, \\ 0, & \text{else.} \end{cases}.$$

Evaluate $u(\mathbf{0},t)$ for $t \geq 0$

Solution. Kirchoff's formula yields that for $(x,t) \in \mathbb{R} \times (0,\infty)$:

$$u(\mathbf{0},t) = \frac{1}{4\pi t}\iint_{\partial B(\mathbf{0},t)} \psi(\mathbf{y})\,\mathrm{d}\mathbf{S} = \frac{1}{4\pi t}\iint_{\partial B(\mathbf{0},t)\cap\overline{B(\mathbf{0},1)}} \mathrm{d}\mathbf{S}$$

Thus, since $\partial(\mathbf{0},t)$ does not intersect $\overline{B(\mathbf{0},1)}$ for $t>1$, it follows that $u(\mathbf{0},t)$ vanishes for $t>1$. Otherwise, if $0<t<1$:

$$u(\mathbf{0},t)=\frac{4\pi t^2}{4\pi t}=t.$$

$$\square$$

3. Distributions

Henceforth, we shall denote by $\mathcal{D}(\mathbb{R})$ the space of all smooth (infinitely differentiable) functions of compact support defined on $\mathbb{R}$. The functions in $\mathcal{D}(\mathbb{R})$ are allowed to take complex values. We endow $\mathcal{D}(\mathbb{R})$ with the metric topology by defining

$$\|\cdot\|_\infty:\mathcal{D}(\mathbb{R})\longrightarrow[0,\infty),\quad \phi\mapsto\sup_{x\in\mathbb{R}}\left|\phi(x)\right|.$$

It is easy to show that $\|\cdot\|_\infty$ is indeed a valid norm on $\mathcal{D}(\mathbb{R})$, and it therefore induces a metric on the space. A distribution is then defined to be a continuous linear functional

$$F:\mathcal{D}(\mathbb{R})\longrightarrow\mathbb{C}.$$

If $f\in L^1_{\text{loc}}(\mathbb{R})$ it **induces** a distribution:

$$\langle F_f,\phi\rangle.$$

Exercise 3.1. *Give an example of a continuous function f who derivatives f',f'' in the sense of distributions are generated by functions such that $f'''=\delta_0$.*[2]

Solution. We define

$$f(x):=\begin{cases}x^2/2, & \text{if } x\geq 0,\\ 0, & \text{else.}\end{cases}$$

This function is clearly continuous on $\mathbb{R}$ and has derivative (in the classical sense):

$$f'(x)=\begin{cases}x, & \text{if } x\geq 0,\\ 0, & \text{else.}\end{cases}$$

---

[2] We denote by δ_0 the Dirac delta "function". As a distribution, this is the functional given by $\langle\delta_0,\phi\rangle=\phi(0)$ for all $\phi\in\mathcal{D}(\mathbb{R})$.

Now, for each $\phi \in \mathcal{D}(\mathbb{R})$:

$$\begin{aligned}\langle F_f'', \phi\rangle = \langle F_f, \phi''\rangle = \int_{\mathbb{R}} f(x)\phi'(x)\,\mathrm{d}x &= \int_0^\infty \frac{x^2}{2}\phi''(x)\,\mathrm{d}x \\ &= -\int_0^\infty x\phi'(x)\,\mathrm{d}x \\ &= \int_0^\infty \phi(x)\,\mathrm{d}x.\end{aligned}$$

In the sense of distributions, $f'' = H_0(x)$; where $H_0(x) = \mathbb{1}_{[0,\infty)}(x)$ is the Heaviside function. Since $H_0'(x) = \delta_0$ in the sense of distributions, we are done. □

Exercise 3.2. Prove that the *sequence of hat functions*:

$$f_n(x) := \begin{cases} \frac{n}{2}, & \text{if } |x| \le \frac{1}{n}, \\ 0, & \text{else,} \end{cases}$$

converges to δ_0 in the sense of distributions.

Solution. Let $\varepsilon > 0$ be given and fix $\phi \in \mathcal{D}(\mathbb{R})$. We must show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\phi(x)\,\mathrm{d}x = \phi(0).$$

For each $n \in \mathbb{N}$

$$\begin{aligned}\left|\int_{\mathbb{R}} f_n(x)\phi(x)\,\mathrm{d}x - \phi(0)\right| &= \left|\int_{-1/n}^{1/n} \frac{n}{2}\phi(x)\,\mathrm{d}x - \frac{n}{2}\int_{-1/n}^{1/n} \phi(0)\,\mathrm{d}x\right| \\ &\le \frac{n}{2}\int_{-1/n}^{1/n} \left|\phi(x) - \phi(0)\right|\mathrm{d}x \\ &\le \sup_{|x|\le 1/n} \left|\phi(x) - \phi(0)\right|.\end{aligned}$$

Since ϕ is continuous, there exists $\delta > 0$ such that $\left|\phi(x) - \phi(0)\right| < \varepsilon$ whenever $|x| < \delta$. Now, for all n sufficiently large it is clear that $|x| < 1/n < \delta$ whence

$$\left|\int_{\mathbb{R}} f_n(x)\phi(x)\,\mathrm{d}x - \phi(0)\right| \le \varepsilon.$$

Letting $n \to \infty$ we obtain

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}} f_n(x)\phi(x)\,\mathrm{d}x - \phi(0)\right| \le \varepsilon$$

for each $\varepsilon > 0$. It follows that

$$\int_{\mathbb{R}} f_n(x)\phi(x)\,\mathrm{d}x \xrightarrow{n\to\infty} \phi(0).$$

□

Exercise 3.3. Define

$$f(x) := \begin{cases} e^{-x}, & \text{if } x > 0, \\ -e^{x}, & \text{if } x \leq 0. \end{cases}$$

Determine f' in the sense of distributions and show that as distributions:

$$f'' = 2\delta_0' + f.$$

Solution. Fix a test function ϕ and evaluate

$$\begin{aligned} \langle F_f', \phi \rangle = -\langle F_f, \phi' \rangle &= \int_{-\infty}^{0} e^{x} \phi'(x)\,\mathrm{d}x - \int_{0}^{\infty} e^{-x} \phi'(x)\,\mathrm{d}x \\ &= 2\phi(0) - \int_{-\infty}^{0} e^{x} \phi(x)\,\mathrm{d}x - \int_{0}^{\infty} e^{-x} \phi(x)\,\mathrm{d}x. \\ &= 2\phi(0) + \int_{\mathbb{R}} g(x)\phi(x)\,\mathrm{d}x \end{aligned}$$

where $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ is given by

$$g(x) := \begin{cases} -e^{-x}, & \text{if } x > 0, \\ -e^{x}, & \text{if } x \leq 0. \end{cases}$$

Taking, again, the derivative in the sense of distributions we find that

$$\langle F_f'', \phi \rangle = -\langle F_f', \phi' \rangle = -\langle \delta_0, \phi' \rangle - \langle F_g, \phi' \rangle$$

where

$$\begin{aligned} -\langle F_g, \phi' \rangle = \int_{0}^{\infty} e^{-x} \phi'(x)\,\mathrm{d}x + \int_{-\infty}^{0} e^{x} \phi'(x)\,\mathrm{d}x = -\phi(x) + \int_{0}^{\infty} e^{-x} \phi(x)\,\mathrm{d}x + \phi(0) \\ - \int_{-\infty}^{0} e^{x} \phi(x)\,\mathrm{d}x. \end{aligned}$$

This completes the problem. □

Exercise 3.4. Find a locally integrable function $f(x)$ such that

$$f'(x) = x^2 + 4x + \delta_2$$

in the sense of distributions.

Solution. The approach here is classical. Consider the function

$$f(x) := \frac{x^3}{3} + 2x^2 + H_2(x), \quad H_2(x) := \mathbb{1}_{[2,\infty)}(x).$$

Fix a function $\phi \in \mathcal{D}(\mathbb{R})$ and compute

$$\langle F_f', \phi\rangle = -\langle F_f, \phi'\rangle = -\int_{-\infty}^{2} \left(\frac{x^3}{3} + 2x^2\right)\phi'(x)\,\mathrm{d}x - \int_{2}^{\infty} \left(\frac{x^3}{3} + 2x^2 + 1\right)\phi'(x)\,\mathrm{d}x$$

where we label these integrals by I_1 and I_2 respectively. Clearly,

$$I_1 = -\int_{-\infty}^{2} \left(\frac{x^3}{3} + 2x^2\right)\phi'(x)\,\mathrm{d}x = -\left[8 + \frac{8}{3}\right]\phi(2) + \int_{-\infty}^{2} (x^2 + 4x)\phi(x)\,\mathrm{d}x$$

whilst

$$I_2 = \left[8 + \frac{8}{3} + 1\right]\phi(2) + \int_{2}^{\infty} (x^2 + 4x)\phi(x)\,\mathrm{d}x.$$

Thus,

$$\langle F_f', \phi\rangle = \langle \delta_2, \phi\rangle + \int_{\mathbb{R}} (x^2 + 4x)\phi(x)\,\mathrm{d}x.$$

□

Exercise 3.5. Suppose $\{F_n\}_{n\in\mathbb{N}}$ is a sequence of distributions converging (in distribution) to a distribution F. Show that for each $k \in \mathbb{N}$ one has that $F_n^{(k)} \to F^{(k)}$ in distribution.

Proof. It suffices to prove the claim for $k = 1$. Note that for each $\phi \in \mathcal{D}(\mathbb{R})$:

$$\lim_{n\to\infty} \langle F_n', \phi\rangle = -\lim_{n\to\infty} \langle F_n, \phi'\rangle = -\langle F, \phi'\rangle = \langle F', \phi\rangle.$$

□

Exercise 3.6. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a continuously differentiable function. Show that f' is a derivative of f in the sense of distributions.

Proof. Fix a function $\phi \in \mathcal{D}(\mathbb{R})$ and observe that

$$\langle F_f', \phi\rangle = -\int_{\mathbb{R}} f(x)\phi'(x)\,\mathrm{d}x = \int_{\mathbb{R}} f'(x)\phi(x)\,\mathrm{d}x.$$

□

Exercise 3.7. Define

$$g(x) = \begin{cases} x^3, & \text{if } x \geq 0, \\ -x^3, & \text{if } x < 0. \end{cases}$$

(i) Compute $g^{(4)}$ in the sense of distributions.

(ii) Define

$$f(x) = \int_{\mathbb{R}} \frac{g(x-y)}{1+y^6}\,\mathrm{d}y.$$

Accept that $f(x)$ is smooth and compute $f^{(4)}(x)$.

Solution. (i). Let F_g denote the distribution induced by g (this makes sense since $g \in L^1_{\mathrm{loc}}(\mathbb{R})$). For a function $\phi \in \mathcal{D}(\mathbb{R})$ we find that

$$\begin{aligned}\langle F_g^{(4)}, \phi\rangle = \langle F_g, \phi^{(4)}\rangle &= \int_{\mathbb{R}} g(x)\phi^{(4)}(x)\,\mathrm{d}x \\ &= \int_0^\infty x^3\phi^{(4)}(x)\,\mathrm{d}x - \int_{-\infty}^0 x^3\phi^{(4)}(x)\,\mathrm{d}x \\ &= -3\int_0^\infty x^2\phi^{(3)}(x)\,\mathrm{d}x + 3\int_{-\infty}^0 x^2\phi^{(3)}(x)\,\mathrm{d}x \\ &= 6\int_0^\infty x\phi^{(2)}(x)\,\mathrm{d}x - 6\int_{-\infty}^0 x\phi^{(2)}(x)\,\mathrm{d}x \\ &= -6\int_0^\infty \phi'(x)\,\mathrm{d}x + 6\int_{-\infty}^0 \phi'(x)\,\mathrm{d}x\end{aligned}$$

which reduces to $12\phi(0)$. Hence, $g^{(4)} = 12\delta_0$ in the sense of distributions.

(ii). Using the fact that convolution is commutative:

$$f(x) = \int_{\mathbb{R}} \frac{g(y)}{1+(x-y)^6}\,\mathrm{d}y$$

which implies that

$$f^{(4)}(x) = \frac{\mathrm{d}^4}{\mathrm{d}x^4}\int_{\mathbb{R}} \frac{g(y)}{1+(x-y)^6}\,\mathrm{d}y = \int_{\mathbb{R}} \frac{\partial^4}{\partial x^4}\frac{g(y)}{1+(x-y)^6}\,\mathrm{d}y.$$

By symmetry,

$$\int_{\mathbb{R}} \frac{\partial^4}{\partial x^4}\frac{g(y)}{1+(x-y)^6}\,\mathrm{d}y = \int_{\mathbb{R}} g(y)\frac{\partial^4}{\partial y^4}\frac{1}{1+(x-y)^6}\,\mathrm{d}y$$

which is <u>by very definition</u>

$$\langle F_g^{(4)}, \phi\rangle$$

where $\phi(y) = \frac{1}{1+(x-y)^6}$ with x fixed. We know from the first part that this is precisely $12\delta_0$ which implies that

$$f^{(4)}(x) = 12\phi(0) = \frac{12}{1+x^6}.$$

□

4. THE FOURIER TRANSFORM

Exercise 4.1. Suppose that $y(x)$ is a smooth function that satisfies

$$y^{(4)} + 4y^{(2)} + y = f(x), \quad f \in C_c(\mathbb{R}).$$

Using the Fourier transform, write an expression for $\hat{y}$ in-terms of $\hat{f}$.

Solution. Using the linearity of $\mathcal{F}[\cdot]$, we take the Fourier transform of both sides of the ODE to obtain

$$\hat{f}(\xi) = \widehat{y^{(4)}}(\xi) + 4\widehat{y^{(2)}}(\xi) + \hat{y}(\xi) = i^4\xi^4\hat{y}(\xi) + 4i^2\xi^2\hat{y}(\xi) + \hat{y}(\xi).$$

Therefore, for $\xi \in \mathbb{R}$:

$$\hat{f}(\xi) = \hat{y}(\xi)\left(\xi^4 - 4\xi^2 + 1\right)$$

so that

$$\hat{y}(\xi) = \frac{\hat{f}(\xi)}{\xi^4 - 4\xi^2 + 1}.$$

☐

Exercise 4.2. Define $f(x) := e^{-|x|}/2$. Compute the Fourier transform of f.

Solution. For fixed $\xi \in \mathbb{R}$ we have

$$\begin{aligned}\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-|x|}}{2} e^{-ix\xi}\,\mathrm{d}x &= \frac{1}{2}\int_{-\infty}^{0} e^{x(1-i\xi)}\,\mathrm{d}x + \frac{1}{2}\int_{0}^{\infty} e^{-x(1+i\xi)}\,\mathrm{d}xs \\ &= \left.\frac{e^{x(1-i\xi)}}{2(1-i\xi)}\right|_{-\infty}^{0} + \left.\frac{-e^{-x(1+i\xi)}}{2(1+i\xi)}\right|_{0}^{\infty} \\ &= \frac{1}{2}\left(\frac{1}{1-i\xi} + \frac{1}{1+i\xi}\right) \\ &= \frac{1}{(1-i\xi)(1+i\xi)} = \frac{1}{1+\xi^2}.\end{aligned}$$

☐

Exercise 4.3. Using the Fourier transform, solve the ODE

$$y''(x) - y(x) = f(x), \quad f \in C(\mathbb{R}) \cap L^1(\mathbb{R}).$$

Solution. By linearity of the Fourier transform, for fixed $\xi \in \mathbb{R}$

$$\hat{f}(\xi) = \widehat{f''}(\xi) - \hat{y}(\xi) = i^2\xi^2\hat{y}(\xi) - \hat{y}(\xi) = -(1+\xi^2)\hat{y}(\xi).$$

Hence,

$$\hat{y}(\xi) = -\frac{\hat{f}(\xi)}{1+\xi^2}.$$

By virtue of the previous problem, we know that $g(x) = e^{-|x|}2$ is the Fourier inverse to $\frac{1}{1+\xi^2}$. We may therefore write,

$$y(x) = -\left(\hat{f}(\xi) \cdot \hat{g}(\xi)\right)^{\vee} = -\left(\widehat{f * g}\right)^{\vee} = -(f * g)(x) = -\frac{1}{2}\int_{\mathbb{R}} f(y) e^{-|x-y|}\,\mathrm{d}y.$$

□

Prove the Riemann-Lebesgue lemma:

Lemma 1. *Let* $f \in L^1(\mathbb{R})$, *then*

$$\lim_{n\to\infty} \hat{f}(n) = 0.$$

Proof. We proceed in 2 steps.

STEP 1. The claim holds when f is the characteristic function of an interval. Suppose that $f(x) = \mathbb{1}_{[a,b]}(x)$ where a, b are real numbers. By direct calculation,

$$\hat{f}(n) = \int_{\mathbb{R}} f(x) e^{-inx}\,\mathrm{d}x = \int_a^b e^{-inx}\,\mathrm{d}x = \left.\frac{ie^{-inx}}{n}\right|_a^b = \frac{i(e^{-inb} - e^{-ina})}{n}.$$

Thus,

$$\left|\hat{f}(\xi)\right| \le \frac{2}{n} \xrightarrow{n\to\infty} 0.$$

Therefore, this must also hold for finite linear combination of characteristic functions of intervals (i.e. step functions).

STEP 2. This holds for general $f \in L^1(\mathbb{R})$. Certainly, let $\varepsilon > 0$ be given. By their density in $L^1(\mathbb{R})$, there exists a step function $\varphi = \sum_{j=1}^N a_j \mathbb{1}_{[a_j,b_j]}$ such that

$$\int_{\mathbb{R}} |f(x) - \varphi(x)|\,\mathrm{d}x < \frac{\varepsilon}{2}.$$

Therefore,

$$\int_{\mathbb{R}} f(x) e^{-inx}\,\mathrm{d}x = \int_{\mathbb{R}} (f(x) - \varphi(x)) e^{-inx}\,\mathrm{d}x + \int_{\mathbb{R}} \varphi(x) e^{-inx}\,\mathrm{d}x.$$

Which then implies

$$\begin{aligned}\left|\int_{\mathbb{R}} f(x) e^{-inx}\,\mathrm{d}x\right| &\le \left|\int_{\mathbb{R}} (f(x) - \varphi(x)) e^{-inx}\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}} \varphi(x) e^{-inx}\,\mathrm{d}x\right| \\ &\le \int_{\mathbb{R}} |f(x) - \varphi(x)|\,\mathrm{d}x + \left|\int_{\mathbb{R}} \varphi(x) e^{-inx}\,\mathrm{d}x\right|.\end{aligned}$$

Now, there exists $K \in \mathbb{N}$ such that $n \geq N$ implies, by step 1,

$$\left| \int_{\mathbb{R}} \varphi(x) e^{-inx} \, \mathrm{d}x \right| < \frac{\varepsilon}{2}.$$

Putting all this together:

$$\left| \int_{\mathbb{R}} f(x) e^{-inx} \, \mathrm{d}x \right| < \varepsilon, \quad \forall n \geq K.$$

This is, by definition, equivalent to saying that

$$\lim_{n \to \infty} \hat{f}(n) = 0.$$

□