MATH 475 – SOLUTIONS TO IMPORTANT EXERCISES

EDWARD CHERNYSH

This document will be updated in the near future and more problems will be added.

CONTENTS

This document comprises of solutions to assignments and problems from the midterm examination for Math 475 (Honours Partial Differential Equations) at McGill university in the Fall of 2016. This text grew out of preparation for the final examination of this same course in the Fall of 2017.

PART 1: TRANSPORT AND WAVE EQUATIONS

Problem 1. *Suppose that $u(x,t)$ is a smooth solution to the given PDE for $x \in \mathbb{R}$ and $t > 0$. Assume also that $u(4,1) = 1$. For each of the following, find a point $x_0 \in \mathbb{R}$ for which we know the value of $u(x_0, 0)$ and give this value.*

(i) $u_t + u_x \equiv 0$;
(ii) $u_t + uu_x \equiv 0$;
(iii) $u_t + t^2 u_x \equiv 0$;
(iv) $u_t + u_x + u \equiv 0$.

Solution. For each of these we shall employ the method of characteristics. In the case of (i), our characteristics are solutions to the following system of ODE:

$$\dot{t}(s) = 1, \quad \dot{x}(s) = 1$$

for some dummy variable s. If we let $z(s) = u(x(s), t(s))$ then $\dot{z}(s) = 0$ whence $z(s) = C$, for some constant C. That is, u is constant along these characteristics. Now, we solve the ODE system (choosing the constants of integration wisely) to obtain

$$t(s) = s + 1, \quad x(s) = s + 4.$$

Hence, $(x(0), t(0)) = (4, 1)$ so that $z(0) = 1$. Taking $s = -1$ we find that the point $(3, 0)$ lies on this curve as well. This means that $u(3, 0) = u(4, 1) = 1$.

A similar process applies to (ii). Once again, we define

$$\dot{t}(s) = 1, \quad \dot{x}(s) = z(s)$$

Date: December 12, 2017.

where $\dot{z}(s) = 0$ along these curves. This means that $z(s) = C$ and

$$t(s) = s + 1, \quad x(s) = Cs + 4.$$

Once again, we have chosen our constants of integration such that $x(0) = 4$ and $t(0) = 1$. Knowing the value of $z(0)$, we recover $C = 1$ so that

$$t(s) = s + 1, \quad x(s) = s + 4.$$

Hence, $u(3, 0) = 1$, as in the first case.

For (iii), we follow the same procedure. We choose the same characteristic for $t(s)$:

$$t(s) = s + c_1$$

and set $\dot{x}(s) := (s + c_1)^2$. This yields

$$x(s) = \frac{(s + c_1)^3}{3} + c_2.$$

Now, we take $c_1 = 1$ so that $t(0) = 1$ whence

$$x(s) = \frac{(s + 1)^3}{3} + c_2.$$

We wish to take c_2 such that $x(0) = 4$; this involves solving

$$4 = \frac{1}{3} + c_2$$

which has $c_2 = 11/3$ as a solution. Our characteristics are therefore

$$t(s) = s + 1, \quad x(s) = \frac{(s + 1)^3 + 11}{3}.$$

Since u will be constant along these characteristics we know that $z(-1) = z(0) = 1$. Taking $s = -1$ gives us the point

$$t(-1) = 0, \quad x(-1) = \frac{11}{3}.$$

This means $u(11/3, 0) = 1$.

In our final case, we have the characteristics

$$t(s) = s + 1, \quad x(s) = s + 4$$

but instead $\dot{z}(s) = -z(s)$. This has solution $z(s) = Ce^{-s}$. At $s = 0$, we know that $z(0) = u(x(0), t(0)) = 1$ which gives $C = 1$, i.e. $z(s) = e^{-s}$. The point $(3, 0)$ corresponds to $(x(-1), t(-1))$ which lies on the characteristic. This implies that $u(3, 0) = e^{-1}$. This concludes our first problem. □

Problem 2. *Let $u(x, t)$ be a smooth solution to the 1D wave equation:*

$$u_{tt} = u_{xx}, \quad (x, t) \in \mathbb{R} \times (0, \infty)$$

subject to the constraints $u(x, 0) = 0$ and $u_t(x, 0) = \psi(x)$ where

$$\psi(x) = \begin{cases} 1, & \textit{if } |x - 3| \le 1 \textit{ or } |x + 3| \le 1, \\ 0, & \textit{otherwise.} \end{cases}$$

(i) At $t \in \{1, 4\}$, for which x is the displacement non-zero?

(ii) *At $t = 10$, where is the displacement* ***maximal****.*
(iii) *Determine $u(0,t)$ explicitly as a function of t.*

Solution. Before we begin we will make a crucial observation. Recalling d'Alembert's formula, we see that $u(x,t)$ will be of the form

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\sigma)\,\mathrm{d}\sigma. \tag{1}$$

In our case, (1) becomes

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \psi(\sigma)\,\mathrm{d}\sigma. \tag{2}$$

Let us now break down this function $\psi(\sigma)$. ψ is non-zero if and only if $|\sigma - 3| \leq 1$ or $|\sigma + 3| \leq 1$. That is, if and only if

$$2 \leq \sigma \leq 4 \quad \text{or} \quad -4 \leq \sigma \leq -2.$$

(i). From (2) it is clear that $u(x,1)$ is non-zero if and only if $[x-1, x+1]$ intersects $(-4,-2) \cup (2,4)$. Now, $[x-1,x+1]$ intersects $(-4,-2)$ if and only if

$$-4 < x+1 \quad \text{and} \quad x-1 < -2.$$

This gives that $x > -5$ and $x < -1$, i.e. $x \in (-5,-1)$. Similarly, $[x-1,x+1]$ intersects $(2,4)$ if and only if

$$x - 1 < 4 \quad \text{and} \quad x+1 > 2$$

which is to say that $x \in (1,5)$. This means that $u(x,1)$ is non-zero on $(-5,-1) \cup (1,5)$. We now repeat this procedure for $t = 4$. Our previous arguments tell us that $u(x,4)$ is non-zero if and only if $[x-4,x+4]$ intersects $(-4,-2) \cup (2,4)$. Now, $[x-4,x+4] \cap (-4,-2)$ is non-empty if and only if

$$x+4 > -4 \quad \text{and} \quad x-4 < -2$$

which is equivalent to $x \in (-8,2)$. Likewise, $[x-4,x+4] \cap (2,4)$ is non-empty if and only if

$$x-4 < 4 \quad \text{and} \quad x+4 > 2$$

which is to say that $x \in (-2,8)$. This gives $u(x,4) > 0$ if and only if $x \in (-8,8)$.

(ii). From (2) and the observations made in (i), it is obvious that $u(x,10)$ is maximal when $[x-10,x+10]$ covers the most of $(-4,-2) \cup (2,4)$. Now, it is natural to ask for which x one has $(-4,-2) \cup (2,4) \subseteq [x-10,x+10]$. This occurs if and only if

$$x - 10 \leq -4 \quad \text{and} \quad x+10 \geq 4.$$

These are precisely the x that belong to $[-6,6]$.

(iii). Again, (2) gives us

$$u(0,t) = \frac{1}{2}\int_{-t}^{t} \psi(\sigma)\,\mathrm{d}\sigma.$$

Our previous calculations also imply that $u(0,t)$ is non-zero if and only if $[-t,t]$ intersects $(-4,-2)\cup(2,4)$. Equivalently, $u(0,t)$ vanishes if and only if $[-t,t]$ does not intersect $(-4,-2)\cup(2,4)$. That is, if $[-t,t]\subseteq[-2,2]$. Hence,

$$\boxed{u(0,t)=0, \quad t\in[-2,2].}$$

Now, the other extreme is when $(-4,-2)\cup(2,4)\subseteq[-t,t]$, which occurs if and only if $t\geq 4$. If this is true, then

$$u(0,t)=\frac{1}{2}\int_{(-4,-2)\cup(2,4)} 1\,\mathrm{d}\sigma=\frac{2+2}{2}=2.$$

This gives

$$\boxed{u(0,t)=2, \quad t\geq 4.}$$

Suppose now that $2<t<4$. Then,

$$\begin{aligned} u(0,t)=\frac{1}{2}\int_{-t}^{t}\psi(\sigma)\,\mathrm{d}\sigma &= \frac{1}{2}\int_{-t}^{0}\psi(\sigma)\,\mathrm{d}\sigma+\frac{1}{2}\int_{0}^{t}\psi(\sigma)\,\mathrm{d}\sigma \\ &= \frac{1}{2}\int_{-t}^{-2}1\,\mathrm{d}\sigma+\frac{1}{2}\int_{2}^{t}1\,\mathrm{d}\sigma \\ &= \frac{-2+t}{2}+\frac{t-2}{2} \\ &= t-2. \end{aligned}$$

All this combined implies that

$$u(0,t)=\begin{cases} 0, & \text{if } t\leq 2, \\ t-2, & \text{if } t\in(2,4), \\ 4, & \text{if } t\geq 4. \end{cases}$$

□

Problem 3. *Consider a smooth solution $u(x,t)$ to the 3D wave equation with $c^2=1$, i.e. suppose u solves*

$$u_{tt}=\Delta u, \quad \textit{for } (x,t)\in\mathbb{R}\times[0,\infty)$$

subject to the initial constraints $u(\mathbf{x},0)=\phi(\mathbf{x})\equiv 0$ and $u_t(\mathbf{x},0)=\psi(\mathbf{x})$ where

$$\psi(\mathbf{x})=\begin{cases} 1, & \textit{if } |\mathbf{x}|\leq 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Determine $u(\mathbf{0},t)$ as a function of t for $t>0$.

Solution. This problem makes heavy use of Kirchoff's formula. For a general solution u to the wave equation subject to initial constraints, we recall that u is given by

$$u(\mathbf{x},t)=\frac{1}{4\pi c^2t^2}\iint_{\partial B(\mathbf{x},ct)}\left[\phi(\mathbf{y})+(\mathbf{y}-\mathbf{x})\cdot\nabla\phi(\mathbf{x})+t\psi(\mathbf{y})\right]\,\mathrm{d}\mathbf{S}.$$

In our case, this becomes

$$u(\mathbf{0},t) = \frac{1}{4\pi t}\iint_{\partial B(\mathbf{0},t)} \mathbb{1}_{\overline{B}(\mathbf{0},1)}(\mathbf{y})\,\mathrm{d}\mathbf{S}.$$

Here, $\overline{B}(\mathbf{0},1)$ denotes the closed unit ball, centered at the origin, in $\mathbb{R}^3$. For $t > 0$, this reduces to

$$u(\mathbf{0},t) = \frac{1}{4\pi t} m\left(\partial B(\mathbf{0},t) \cap \overline{B}(\mathbf{0},t)\right)$$

where $m(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^2$. Therefore,

$$u(\mathbf{0},t) = \begin{cases} t, & \text{if } t \leq 1, \\ 0, & \text{else.} \end{cases}$$

□

PART 2: RANDOM WALKS AND DISTRIBUTIONS

Problem 4. *Using the Fourier transform, solve the ODE*

$$y''(x) - y(x) = f(x) \tag{3}$$

where $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ is given. ***Hint****: use the fact that*

$$\mathcal{F}\left[\frac{e^{-|x|}}{2}\right](k) = \frac{1}{k^2+1}.$$

Solution. Using the linearity of the Fourier transform and taking the Fourier transform of both sides of (3), we recover

$$\mathcal{F}\left[y''\right] - \mathcal{F}(y) = \mathcal{F}(y).$$

Now, since $\mathcal{F}[y''] = i^2k^2\mathcal{F}[y]$ it follows from the above that

$$-\widehat{y}(k)\left(1+k^2\right) = \widehat{f}(k)$$

whence

$$\widehat{y}(k) = -\frac{\widehat{f}}{1+k^2} = -\widehat{f}(k)\cdot\widehat{g}(k)$$

where we define

$$g(x) := \frac{e^{-|x|}}{2}.$$

Using now that the Fourier transform of $f * g$ is merely $\widehat{f}\cdot\widehat{g}$, by taking the Fourier inverse transform of the above equation we obtain the following integral expression for $y(x)$:

$$y(x) = -(f*g)(x) = -\frac{1}{2}\int_{-\infty}^{\infty} f(y)e^{-|x-y|}\,\mathrm{d}y.$$

This concludes the solution. □

Problem 5. *Consider a random walk on a line with time and distance increments $\triangle x$ and $\triangle t$ and let $p(x,t)$ denote the probability of being at the position x at the given time t. Suppose that at a time t we move to the right by $\triangle x$ with probability $\frac{1}{2}$ and move to the left by $\triangle x$ with probability $\frac{1}{2}$. If as our increments are such that*

$$\frac{(\triangle x)^2}{\triangle t} = 4, \tag{$\dagger$}$$

determine a diffusion equation that governs $p(x,t)$.

Solution. We must first determine a difference equation for $p(x,t)$. Fix a position x and a time t. The probability that we move to x at a time $t + \triangle t$ is given by

$$p(x, t+\triangle t) = \frac{1}{2}p(x-\triangle x, t) + \frac{1}{2}p(x+\triangle x, t). \tag{4}$$

Equation (4) is all that we need to solve this problem. The above implies that, for $\triangle t > 0$, one has

$$\frac{p(x,t+\triangle t) - p(x,t)}{\triangle t} = \frac{p(x-\triangle x,t) + p(x+\triangle x,t) - 2p(x,t)}{2\triangle t}.$$

Using assumption ($\dagger$), the above may alternatively be written as

$$\frac{p(x,t+\triangle t) - p(x,t)}{\triangle t} = 2\frac{p(x-\triangle x,t) + p(x+\triangle x,t) - 2p(x,t)}{(\triangle x)^2}.$$

Recall that for a smooth function $f : \mathbb{R} \to \mathbb{R}$ one has the identity:

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x). \tag{$\star$}$$

Notice also that, by ($\dagger$), $\triangle x \to 0$ if and only if $\triangle t \to 0$. We therefore have (if p is smooth)

$$\begin{aligned} p_t(x,t) = \lim_{\triangle t\to 0} \frac{p(x,t+\triangle t) - p(x,t)}{\triangle t} &= \lim_{\triangle x\to 0} \frac{p(x,t+\triangle t) - p(x,t)}{\triangle t} \\ &= 2\lim_{\triangle x\to 0} \frac{p(x-\triangle x,t) + p(x+\triangle x,t) - 2p(x,t)}{(\triangle x)^2} \\ &= 2p_{xx}(x,t). \end{aligned}$$

We therefore recover the diffusion equation

$$\boxed{\partial_t p(x,t) = 2\partial_x^2 p(x,t).}$$

□

Problem 6. *Define*

$$g(x) := \begin{cases} x^3, & \text{if } x \geq 0, \\ -x^3, & \text{if } x < 0. \end{cases}$$

Calculate $g^{(4)}$ in the sense of distributions. ***Hint****: recall that distributional derivatives agree with the regular derivative for $C^1(\mathbb{R})$ functions.*

Solution. It is easy to check that $g \in C^2(\mathbb{R})$. This implies that,

$$g'' = \begin{cases} 6x, & \text{if } x \geq 0, \\ -6x, & \text{if } x < 0. \end{cases}$$

Now, let $\phi \in \mathcal{D}(\mathbb{R})$ and calculate

$$\begin{aligned} \left\langle g^{(3)}, \phi \right\rangle = -\left\langle g'', \phi' \right\rangle &= \int_{-\infty}^{0} 6x\phi'(x)\,\mathrm{d}x - \int_{0}^{\infty} 6x\phi'(x)\,\mathrm{d}x \\ &= 6x\phi(x)|_{-\infty}^{0} - \int_{-\infty}^{0} 6\phi(x)\,\mathrm{d}x - 6x\phi(x)|_{0}^{\infty} + \int_{0}^{\infty} 6\phi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} 6\phi(x)\,\mathrm{d}x \end{aligned}$$

where we have used that ϕ has compact support. This implies that, in the sense of distributions,

$$g^{(3)} = \begin{cases} 6, & \text{if } x \geq 0, \\ -6, & \text{if } x < 0. \end{cases}$$

Once again taking the derivative (in the sense of distributions), we obtain for each $\phi \in \mathcal{D}(\mathbb{R})$:

$$\left\langle g^{(4)}, \phi \right\rangle = -\int_{0}^{\infty} 6\phi'(x)\,\mathrm{d}x + \int_{-\infty}^{0} 6\phi'(x)\,\mathrm{d}x = 6\phi(0) + 6\phi(0)$$

whence $g^{(4)} = 12\delta_0$, where δ_0 is the Dirac delta 'function'. □

Problem 7. *Let g be defined as in the previous problem and consider the smooth (you do not have to prove this) function $f(x) := \int_{\mathbb{R}} \frac{g(x-y)}{1+y^6}\,\mathrm{d}y$. Compute $f^{(4)}$.*

Solution. Denote by $h(x) := 1/(1+x^6)$. Then $f(x) = (h * g)(x) = (g * h)(x)$. In more explicit terms, this is to say that

$$f(x) = \int_{\mathbb{R}} \frac{g(y)}{1+(x-y)^6}\,\mathrm{d}y.$$

Using that f is smooth, we may differentiate under the integral sign to obtain

$$\begin{aligned} f^{(4)}(x) = \int_{\mathbb{R}} \frac{\partial^4}{\partial x^4}\left[\frac{g(y)}{1+(x-y)^6}\right]\mathrm{d}y &= \int_{\mathbb{R}} g(y)\frac{\partial^4}{\partial x^4}\left[\frac{1}{1+(x-y)^6}\right]\mathrm{d}y \\ &= \left\langle g^{(4)}, h(x-y) \right\rangle \end{aligned}$$

which, by the previous problem, is precisely

$$\frac{12}{1+x^6}.$$

□

Problem 8. *Consider a random walk where at integer multiples of $\triangle t$ we move to the left by $\triangle x$ with probability $1/4$, move to the right by $\triangle x$ with probability $1/4$, and stay at our position with probability $1/2$. Let $p(x,t)$ be the probability of having the position x at a time t and express p as a solution to a diffusion equation. You may assume that*

$$\frac{(\triangle x)^2}{\triangle t} \to \sigma^2 \qquad \text{as } \triangle x, \triangle t \to 0.$$

Solution. Fix a pair (x,t) and observe that the setup gives us the following:

$$p(x, t+\triangle t) = \frac{1}{2}p(x,t) + \frac{1}{4}p(x-\triangle x, t) + \frac{1}{4}p(x+\triangle x, t).$$

Hence,

$$\begin{aligned}\frac{p(x,t+\triangle t) - p(x,t)}{\triangle t} &= \frac{p(x-\triangle x,t) + p(x+\triangle x,t)}{4\triangle t} - \frac{p(x,t)}{2\triangle t} \\ &= \frac{p(x+\triangle x,t) + p(x-\triangle x,t) - 2p(x,t)}{4\triangle t}.\end{aligned}$$

This means that, as $\triangle x$ and $\triangle t$ decrease to 0,

$$\begin{aligned}p_t(x,t) &= \frac{\sigma^2}{4}\lim_{\triangle x\to 0}\frac{p(x+\triangle x,t) + p(x-\triangle x,t) - 2p(x,t)}{(\triangle x)^2} \\ &= \frac{\sigma^2}{4}p_{xx}(x,t).\end{aligned}$$

This yields the following diffusion equation:

$$\boxed{\partial_t p(x,t) \equiv \frac{\sigma^2}{4}\partial_{xx}p(x,t).}$$

□

PART 3: LAPLACE'S EQUATION AND HARMONIC FUNCTIONS

Recall that a domain Ω in $\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ that is non-empty, bounded, and connected. The Laplacian operator is given by $\Delta := \sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}$ and a function $f \in C^2(\Omega)$ is called harmonic in Ω provided $\Delta f \equiv 0$ in Ω.

Problem 9. *Assuming that a smooth function u satisfies the mean value property for shells, deduce the mean property for balls in $\mathbb{R}^3$.*

Solution. Let $B(\mathbf{x},\varepsilon)$ be an open ball and let u be smooth in a domain containing $B(\mathbf{x},\varepsilon)$. By passing to polar coordinates:

$$\iiint_{B(x,\varepsilon)} u(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_0^{\varepsilon}\left(\iint_{\partial B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}\mathbf{S}\right)\,\mathrm{d}\rho.$$

The mean value property for shells states that

$$4\pi\rho^2 u(\mathbf{x}) = \iint_{\partial B(\mathbf{x},\rho)} u(\mathbf{y})\,\mathrm{d}\mathbf{S}$$

whence

$$\iiint_{B(x,\varepsilon)} u(\mathbf{y})\, \mathrm{d}\mathbf{y} = 4\pi u(\mathbf{x}) \int_0^{\varepsilon} \rho^2 \, \mathrm{d}\rho = \frac{4\pi\varepsilon^3}{3} u(\mathbf{x}),$$

as was asserted. □

Problem 10. *Let $u(r,\theta) \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in a domain $\Omega \supset \mathbb{D}_2$, where $\mathbb{D}_2$ is the closed disk of radius 2 centered at the origin in $\mathbb{R}^2$. Suppose that $u(r,\theta) \equiv 3\sin 2\theta + 1$ on $\partial\mathbb{D}_2$.*

(i) Compute the maximum value of u on $\mathbb{D}_2$.
(ii) Compute $u(\mathbf{0})$.

Solution. We attack each part individually (they are easy).

(i) The weak maximum principle states that $\max_{\mathbf{x}\in\mathbb{D}_2} u(\mathbf{x}) = \max_{\mathbf{x}\in\partial\mathbb{D}_2} u(\mathbf{x})$; it thus suffices to maximize u on the circle where we are given explicit data for u. Now, by observation, it is clear that $u(2,\theta) = 3\sin 2\theta + 1$ achieves its maximum value of 4 at the point $(2, \pi/4)$. Therefore,

$$\max_{\mathbf{x}\in\mathbb{D}_2} u(\mathbf{x}) = 4.$$

(ii) Using the mean value property over shells,

$$u(\mathbf{0}) = \fint_{\partial\mathbb{D}_2} u(\mathbf{y})\, \mathrm{d}\mathbf{S} = \frac{1}{4\pi} \int_0^{2\pi} (3\sin 2\theta + 1)\, \mathrm{d}\theta = \frac{1}{2}.$$

□

Problem 11. *In $\mathbb{R}^3$, show that $|\mathbf{x}|^2 \Delta\delta_0 = 6\delta_0$ in the sense of distributions.*

Solution. Fix $\phi \in \mathcal{D}(\mathbb{R}^3)$ and note that, by definition:

$$\left\langle |\mathbf{x}|^2 \Delta\delta_0, \phi \right\rangle = \left\langle \Delta\delta_0, |\mathbf{x}|^2 \phi \right\rangle = \left\langle \delta_0, \Delta(|\mathbf{x}|^2 \phi) \right\rangle.$$

Represent $\mathbf{x} = (x_1, x_2, x_3)$ for $x_j \in \mathbb{R}$. Fix $j \in \{1,2,3\}$ and note that

$$\frac{\partial}{\partial x_j}(|\mathbf{x}|^2 \phi(\mathbf{x})) = 2x_j\phi(\mathbf{x}) + |\mathbf{x}|^2 \phi_{x_j}(\mathbf{x})$$

whence

$$\frac{\partial^2}{\partial x_j^2}(|\mathbf{x}|^2 \phi(\mathbf{x})) = 2\phi(\mathbf{x}) + 2x_j\phi_{x_j}(\mathbf{x}) + |\mathbf{x}|^2 \phi_{x_j x_j}(\mathbf{x}).$$

Since distributions are linear functionals, this means that

$$\left\langle |\mathbf{x}|^2 \Delta\delta_0, \phi \right\rangle = \sum_{j=1}^{3} \left\langle \delta_0, \frac{\partial^2}{\partial x_j^2}(|\mathbf{x}|^2 \phi(\mathbf{x})) \right\rangle = \sum_{j=1}^{3} 2\phi(\mathbf{0}).$$

This means that $|\mathbf{x}|^2 \Delta\delta_0 = 6\delta_0$ in the sense of distributions. □

Problem 12. *Let $f \in L^1_{loc}(\mathbb{R}^2)$. Show that $f_{xy} \equiv f_{yx}$ in the sense of distributions. Then, consider the function*

$$g(x,y) := \begin{cases} 1, & \text{if } y \le x^3, \\ 0, & \text{if } y > x^3. \end{cases}$$

Compute g_{xy} in the sense of distributions.

Solution. We fix a test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ and write

$$\begin{aligned}\langle f_{xy}, \phi\rangle = \langle \partial_y \partial_x f, \phi\rangle = (-1)^2 \langle f, \partial_y \partial_x \phi\rangle &= (-1)^2 \langle f, \partial_x \partial_y \phi\rangle \\ &= \langle \partial_x \partial_y f, \phi\rangle \\ &= \langle f_{yx}, \phi\rangle .\end{aligned}$$

For the second part, we proceed directly

$$\begin{aligned}\langle g_{xy}, \phi\rangle = \langle g, \phi_{xy}\rangle = \iint_{\mathbb{R}^2} g(x,y)\phi_{xy}\,\mathrm{d}A &= \int_{-\infty}^{\infty}\int_{-\infty}^{x^3} \phi_{xy}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \phi_x(x, x^3)\,\mathrm{d}x.\end{aligned}$$

Thus, g_{xy} is the linear functional $\mathcal{D}(\mathbb{R}^2) \to \mathbb{C}$ which takes ϕ to $\int_{\mathbb{R}} \phi_x(x, x^3)\,\mathrm{d}x$. □

Problem 13. *Here we work with "harmonic" functions in a single variable, i.e. $C^2(\mathbb{R})$ solutions to $Lu = 0$ where $L = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$.*

(i) *Define $\Phi(x) = \dfrac{|x|}{2}$. Prove that $L\Phi = \delta_0$ in the sense of distributions. Conclude that Φ is the fundamental solution to L on $\mathbb{R}$.*

(ii) *Find a Green's function for $\Phi(x, x_0) = \Phi(x - x_0)$ on the interval $(-1, 1)$.*

Solution. We first point out that Φ, considered as a distribution, makes sense since Φ is locally integrable (in fact, it is uniformly continuous).

(i). Let $\phi : \mathbb{R} \to \mathbb{C}$ be a smooth function of compact support and notice that

$$\begin{aligned}\langle L\Phi, \phi\rangle = \langle \Phi, L\phi\rangle = \int_{\mathbb{R}} \Phi(x)\phi''(x)\,\mathrm{d}x &= \frac{1}{2}\int_0^{\infty} x\phi''(x)\,\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{0} x\phi''(x)\,\mathrm{d}x \\ &= -\frac{1}{2}\int_0^{\infty} \phi'(x)\,\mathrm{d}x + \frac{1}{2}\int_{-\infty}^{0} \phi'(x)\,\mathrm{d}x \\ &= \phi(0).\end{aligned}$$

Thus, $L\Phi = \delta_0$ in the sense of distributions.

(ii). We would now like to compute the Green's function for Φ on the domain $(-1, 1)$. Let us fix a point $x_0 \in (-1, 1)$; the typical Green's function algorithm will have us determine a smooth function $H(x, x_0)$, that is harmonic in Ω, and equal to $-\Phi(x - x_0)$ on the boundary of $(-1, 1)$, i.e. at the points $\{\pm 1\}$. Naturally, we first calculate

$$\Phi(1, x_0) = \frac{|1 - x_0|}{2} = \frac{1 - x_0}{2} \quad \text{and} \quad \Phi(-1, x_0) = \frac{|-1 - x_0|}{2} = \frac{1 + x_0}{2}.$$

This gives us a good idea of how to proceed. Consider the function

$$H(x, x_0) := \frac{x x_0 - 1}{2}.$$

For fixed x_0, this is a linear function (and therefore smooth and harmonic in $\mathbb{R}$). By construction, $H(x, x_0)$ agrees with $-\Phi(x, x_0)$ for $x = \pm 1$. Hence, the Green's function is given by.

$$\mathcal{G}(x, x_0) := H(x, x_0) + \Phi(x, x_0) = \frac{xx_0 - 1}{2} + \frac{|x - x_0|}{2}$$

□