ABOUT COMPLEX VALUED DISTRIBUTIONS ON $\mathbb{R}$

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Contents

In these notes we introduce the notion of a distribution which serves as a generalized function, so to speak. Throughout this document we shall let $\mathcal{D}(\mathbb{R})$ denote $C_c^\infty(\mathbb{R})$, i.e. the space of smooth functions of compact support

$$\phi : \mathbb{R} \longrightarrow \mathbb{K}.$$

Here, we shall write $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. We shall show how the Fourier transform is useful in the study of partial differential equations.

1. Distributions

It is a basic fact from the Lebesgue theory of integration that $\mathcal{D}(\mathbb{R})$ is a dense subspace of $L^p(\mathbb{R})$ for all $p \geq 1$. This suggests a good "starting point" for the generalization of functions since even L^p-functions can be approximated nicely by elements of $\mathcal{D}(\mathbb{R})$. Before we proceed it is desirable to endow $\mathcal{D}(\mathbb{R})$ with a notion of convergence. If $\{\phi_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathcal{D}(\mathbb{R})$ we shall say that ϕ_k **converges** to $\phi \in \mathcal{D}(\mathbb{R})$ provided

$$\phi_k^{(n)} \xrightarrow{k\to\infty} \phi^{(n)}$$

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uniformly for all $n \in \mathbb{N}$, i.e. if

$$\left\|\phi_k^{(n)} - \phi^{(n)}\right\|_\infty = \sup_{x\in\mathbb{R}} \left|\phi_k^{(n)} - \phi^{(n)}\right| \xrightarrow{k\to\infty} 0, \quad \forall n \in \mathbb{N}.$$

In this case we shall write $\phi_k \to \phi$ or $\lim \phi_k = \phi$.

Definition 1. A distribution F is a continuous linear functional $F : \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{K}$.

By continuous we mean that

$$\lim_{k\to\infty} F(\phi_k) = F(\phi)$$

whenever $\{\phi_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathcal{D}(\mathbb{R})$ converging to $\phi \in \mathcal{D}(\mathbb{R})$. To ease notation, we shall write $\langle F, \phi\rangle$ to mean $F(\phi)$. Note that the quantity $\langle F, \phi\rangle$ is always a scalar (provided F is a distribution). Hence, if F is a distribution and $\{\phi_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathcal{D}(\mathbb{R})$ converging to $\phi \in \mathcal{D}(\mathbb{R})$, one has

$$\langle F, \phi_k\rangle \xrightarrow{k\to\infty} \langle F, \phi\rangle.$$

It is not too difficult to see that the set of all distributions forms a vector space over $\mathbb{K}$, and the proof of this is left as a simple exercise to the reader.

Example. Dirac delta is the distribution defined by:

$$\left\langle \delta_{x_0}, \phi\right\rangle := \phi(x_0).$$

Often, we take $x_0 = 0$.

1.1. **Functions as Distributions.** It is an important observation that very general classes of functions can be expressed as distributions. Let $f \in L^1_{\text{loc}}(\mathbb{R})$, we define the *distribution associated to f*, denoted F_f, by

$$\left\langle F_f, \phi\right\rangle := \int_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x, \quad \forall \phi \in \mathcal{D}(\mathbb{R}).$$

Note that since f is locally integrable and the $\phi(x)$ have compact support this is always a finite complex number. It should be noted that two "distinct" functions can give rise to the "same" distribution. Indeed, if $f, g \in L^1_{\text{loc}}(\mathbb{R})$ are equal almost everywhere then $\left\langle F_f, \phi\right\rangle = \left\langle F_g, \phi\right\rangle$. In fact, this is an if and only if statement (to see this one needs only use the fact that $\mathcal{D}(\mathbb{R})$ is a dense subspace of $L^1(\mathbb{R})$ and the Lebesgue differentiation theorem).

1.2. **Weak and Distributional Derivatives.** Let now $f \in C^1(\mathbb{R})$ (i.e. f is at least once differentiable and f' is continuous on $\mathbb{R}$). If $\phi \in \mathcal{D}(\mathbb{R})$ then we can integrate by parts to obtain

$$\int_{\mathbb{R}} f'(x)\phi(x)\,\mathrm{d}x = -\int_{\mathbb{R}} f(x)\phi'(x)\,\mathrm{d}x$$

since ϕ vanishes outside a closed interval of the form $[-R, R]$. Now, if $f \in L^1_{\text{loc}}(\mathbb{R})$ and there exists $g \in L^1_{\text{loc}}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} f(x)\phi'(x)\,\mathrm{d}x = -\int_{\mathbb{R}} g(x)\phi(x)\,\mathrm{d}x$$

for all $\phi \in \mathcal{D}(\mathbb{R})$, we shall say that g is the **weak derivative** of f. Note that if f is $C^1(\mathbb{R})$ in the traditional sense then f' is a weak derivative of f. We say **a** weak derivative as a function may have multiple. Indeed, if $f \in C^1(\mathbb{R})$ then f' is a weak derivative of f–as is any function g equal to f outside a set of measure zero.

However, this is not general enough. There are plenty of locally integrable functions without weak derivatives. This is where distributions come in handy. As we shall see, all distributions have (infinitely many) 'derivatives'.

Definition 2. Let $\langle F, \phi \rangle$ be a distribution, we define the distributional derivative of $\langle F, \phi \rangle$ to be the distribution given by

$$\langle F', \phi \rangle := -\langle F, \phi' \rangle$$

for all $\phi \in \mathcal{D}(\mathbb{R})$.

Obviously, given $n \in \mathbb{N}$, we can go so far as to define the n^{th} derivative of F by

$$\left\langle F^{(n)}, \phi \right\rangle := (-1)^n \left\langle F, \phi^{(n)} \right\rangle.$$

This is also useful in defining derivatives of locally integrable functions. Indeed, given $f \in L^1_{\text{loc}}(\mathbb{R})$ we say a distribution F is the distributional derivative of f provided F is the derivative of F_f.

Example. Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be the Heaviside step function defined by:

$$h(x) := \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Note that h' exists and is equal to zero except at $x = 0$, where it is undefined in the classical sense. Intuitively, $h(x)$ makes a "jump" at $x = 0$ and many (physicists, typically) argue (heuristically) that

$$h'(x) := \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0. \end{cases}$$

We give more precise meaning to this statement in this example. We know from the above discussion that h has a distributional derivative. Let us compute it. Let H be the distribution induced by h; by definition for all $\phi \in \mathcal{D}(\mathbb{R})$ one has

$$\begin{aligned}\langle H', \phi \rangle = -\langle H, \phi \rangle = -\int_{\mathbb{R}} h(x)\phi'(x)\,\mathrm{d}x &= -\int_0^\infty \phi'(x)\,\mathrm{d}x \\ &= -\lim_{x\to\infty} \phi(x) + \phi(0) \\ &= \phi(0) \\ &= \langle \delta_0, \phi \rangle.\end{aligned}$$

Hence, $H' = \delta_0$ in the sense of distributions.

This allows us to discuss "new" and weaker notions of convergence. Given a sequence $\{F_n\}_{n\in\mathbb{N}}$ of distributions we say that F_n converges to a distribution F as $n \to \infty$ provided

$$\lim_{n\to\infty} \langle F_n, \phi \rangle = \langle F, \phi \rangle$$

for all $\phi \in \mathcal{D}(\mathbb{R})$. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $L^1_{\mathrm{loc}}(\mathbb{R})$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ we shall say that f_n converges to f in distribution provided

$$\int_{\mathbb{R}} f_n(x)\phi(x)\,\mathrm{d}x \xrightarrow{n\to\infty} \int_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x$$

for each $\phi \in \mathcal{D}(\mathbb{R})$. Let $F_{f,n}$ denote the distribution generated by f_n, the above is then equivalent to saying that $F_{f,n} \to F_f$ in the sense of distributions as $n \to \infty$.

Proposition 1.1. *Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of distributions converging to a distribution F. Then for all $k \in \mathbb{N}$*

$$\left\langle F_n^{(k)}, \phi \right\rangle \xrightarrow{n\to\infty} \left\langle F^{(k)}, \phi \right\rangle.$$

Proof. By induction in suffices to prove the claim for $k = 1$. Now we need only fix $\phi \in \mathcal{D}(\mathbb{R})$ and note that

$$\langle F_n', \phi \rangle = -\langle F_n, \phi' \rangle$$

in which letting $n \to \infty$ yields

$$\lim_{n\to\infty} \langle F_n', \phi \rangle = -\lim_{n\to\infty} \langle F_n, \phi' \rangle = -\langle F, \phi' \rangle = \langle F', \phi \rangle.$$

QED.

2. The Fourier Transform

In this section we survey basic facts regarding the Fourier transform on $\mathbb{R}$ and attempt to extend it to distributions (this will be particularly useful when attempting to "solve" PDE). We should first like to stress that this will be, in the most part, a mere overview of the subject and not a rigorous development. One can consult these notes for proofs of the identities that will follow. Recall that we denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$.

Definition 3. Let $f : \mathbb{R} \longrightarrow \mathbb{K}$ be intgrable, i.e. $\int_{\mathbb{R}} |f| < \infty$. We define the Fourier transform of f, denoted $\widehat{f}$, by

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-i\xi x}\, \mathrm{d}x.$$

This function $\widehat{f}$ is sometimes denoted by $\mathcal{F}[f]$.

Remark 1. By linearity of the integral it follows immediately that for two functions $f, g \in L^1(\mathbb{R})$ and $\alpha, \beta \in \mathbb{K}$ there holds

$$\mathcal{F}[\alpha f + \beta g](\xi) = \alpha \mathcal{F}[f](\xi) + \beta \mathcal{F}[g](\xi).$$

That is, $\mathcal{F}[\cdot]$ is a linear functional on $L^1(\mathbb{R})$.

Example. We compute the Fourier transform of

$$f(x) = \begin{cases} 1, & |x| \le a, \\ 0, & |x| > a. \end{cases}$$

This is a straightforward computation:

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, \mathrm{d}x = \int_{-a}^{a} e^{-ix\xi}\, \mathrm{d}x = -\frac{e^{-ia\xi} - e^{ia\xi}}{i\xi} = \frac{i\left(e^{-ia\xi} - e^{ia\xi}\right)}{\xi}$$

Recall Euler's identity: $e^{i\theta} = \cos\theta + i \sin\theta$ for all $\theta \in \mathbb{R}$. Observe that the above then becomes

$$\widehat{f}(\xi) = \frac{i(-i\sin a\xi - i \sin x\xi)}{\xi} = \frac{2\sin a\xi}{\xi}.$$

Based off of this example we can make several remarks.

(i) Clearly $f \in L^1(\mathbb{R})$ (it is the characteristic function of a compact set). Nonetheless, $\widehat{f} \notin L^1(\mathbb{R})$.
(ii) $\widehat{f} \in L^2(\mathbb{R})$.
(iii) f does not have compact support and its image has the cardinality of the continuum.

These first two points illustrate the main drawback of defining the Fourier transform on $L^1(\mathbb{R})$: it does not map functions back into this same space. The usual "fix" for this problem is to define $\mathcal{F}[\cdot]$ on $L^2(\mathbb{R})$: the space of square integrable functions. This has the advantage that, unlike all other L^p-spaces, $L^2(\mathbb{R}^n)$ is a Hilbert Space for all $n \in \mathbb{N}$.

2.1. **Differentiation and Inversion.** Let f be a differentiable function. For the sake of simplicity, we shall suppose additionally that f is smooth, i.e. $f \in C^\infty(\mathbb{R})$. This implies in particular that $f'(x)$ exists and is continuous. Suppose that $f, f' \in L^1(\mathbb{R})$ so that they both have well defined Fourier transforms. How can we relate $\mathcal{F}[f]$ and $\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right]$?

We require a more refined setting. We shall denote by $\mathcal{S}$ the **Schwarz space** consisting of all smooth functions vanishing at infinity quicker than any polynomial, i.e.

$$\mathcal{S} := \left\{ f \in C^\infty(\mathbb{R}) : \forall a, n \in \mathbb{N}_0, \exists C_{a,n} > 0 \text{ s.t } \left\| x^a f^{(n)} \right\|_\infty \leq C_{a,n} \right\}.$$

It follows, in particular, that for each $f \in \mathcal{S}$

$$\lim_{x \to \pm\infty} x^a f^{(n)}(x) = 0$$

for all non-negative integers a and n. This is the property we shall frequently exploit.[1]

Now let $f \in \mathcal{S}$ be given; it follows that $f' \in \mathcal{S}$ as well. Then, $f' \in L^1(\mathbb{R})$ has a valid Fourier transform given by

$$\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = \int_{\mathbb{R}} f'(x) e^{-ix\xi} \, \mathrm{d}x = \int_{-\infty}^{\infty} f'(x) e^{-ix\xi} \, \mathrm{d}x$$

whence an integration by parts yields:

$$\int_{-\infty}^{\infty} f'(x) e^{-ix\xi} \, \mathrm{d}x = f(x) e^{-ix\xi} \Big|_{x=-\infty}^{x=\infty} + i\xi \int_{-\infty}^{\infty} f(x) e^{-ix\xi} \, \mathrm{d}x = i\xi \int_{\mathbb{R}} f(x) e^{-ix\xi} \, \mathrm{d}x.$$

Thus,

$$\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = i\xi \mathcal{F}[f].$$

It is then clear that for $f \in \mathcal{S}$ there holds

$$\boxed{\mathcal{F}\left[f^{(n)}\right] = (i\xi)^n \mathcal{F}[f].} \tag{2.1}$$

[1] The Schwarz space $\mathcal{S}$ is ideal in the sense that it is dense in $L^p(\mathbb{R})$ for all $p \geq 1$ (it contains $C_c^\infty(\mathbb{R})$) and thus we can extend $\mathcal{F}[\cdot]$ to a bounded linear operator on $L^2(\mathbb{R})$.

A natural question that follows is how to "go back"? That is, given a function $f \in \mathcal{S}$, when can we find $g \in \mathcal{S}$ so that $\widehat{g} = f$? The answer to this question appears in the form of an elegant equation.

Theorem 2.1 (Fourier Inversion Formula). *Suppose f and $\widehat{f}$ are integrable and continuous. Then,*

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) e^{ix\xi} \, \mathrm{d}\xi. \tag{2.2}$$

We will not prove this theorem here, the proof is non-elementary and requires more analysis than we assume here. We once again refer the interested reader to this text for the proof for functions in $\mathcal{S}$.

2.2. **Convolutions.** The Fourier transform is closely related to the notion of a convolution. Despite being an elementary concept, we reiterate the definition below.

Definition 4. Let $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ we define their convolution, denoted $f * g$, by

$$(f * g)(x) := \int_{\mathbb{R}} f(y) g(x-y) \, \mathrm{d}y = \int_{\mathbb{R}} f(x-y) g(y) \, \mathrm{d}y.$$

This second equality follows immediately from a simple change of variables and translation invariance. It is possible to view the process of convolution as a method of "smoothing out" functions. What we are interested in here is the following fundamental relation:

$$\widehat{f * g}(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi). \tag{2.3}$$

To prove this formula, we employ a straightforward computation:

$$\begin{aligned}
\widehat{f * g}(\xi) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) g(y) e^{-ix\xi} \, \mathrm{d}y \, \mathrm{d}x &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) g(y) e^{-ix\xi} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) g(y) e^{-i(x-y+y)\xi} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-y) e^{-i(x-y)\xi} \, \mathrm{d}x \right) g(y) e^{-iy\xi} \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(\zeta) e^{-i(\zeta)\xi} \, \mathrm{d}\zeta \right) g(y) e^{-iy\xi} \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \widehat{f}(\xi) g(y) e^{-iy\xi} \, \mathrm{d}y \\
&= \widehat{f}(\xi) \cdot \widehat{g}(\xi).
\end{aligned}$$

Theorem 2.2 (Plancherel). *Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then*

$$\|f\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi}\|\mathcal{F}[f]\|_{L^2(\mathbb{R})}^2. \tag{2.4}$$

3. Fourier Transforms of Distributions

It is clear from the title of the section that it is our intention to generalize $\mathcal{F}[\cdot]$ to distributions. This, however, also requires us to move to "a wider class" of distributions. Given our discussion of differentiation for distributions one would hope to define

$$\left\langle \widehat{F}, \phi \right\rangle = \left\langle F, \widehat{\phi} \right\rangle$$

for a distribution F acting upon $\phi \in C_c^\infty(\mathbb{R})$. There is a glaring issue here: $\widehat{\phi}$ need not have compact support. To correct this, we instead consider functions in the more general space $\mathcal{S}$, which we have defined in the previous section. It is also not too difficult to verify that $\mathcal{S}$ is a vector space over $\mathbb{K}$.

Definition 5. A tempered distribution is a continuous linear functional on $\mathcal{S}$.

We again denote a tempered distribution by

$$\langle F, \phi \rangle, \quad \text{for } \phi \in \mathcal{S}$$

where $\langle F, \phi \rangle \mapsto F(\phi) \in \mathbb{C}$.

Remark 2. Note that any tempered distribution is a distribution (in the traditional sense) since $\mathcal{S} \supseteq \mathcal{D}(\mathbb{R})$.

Now, given a tempered distribution $\langle F, \phi \rangle$ we define the **Fourier transform** of $\langle F, \phi \rangle$ to be the tempered distribution given by

$$\left\langle \widehat{F}, \phi \right\rangle := \left\langle F, \widehat{\phi} \right\rangle.$$

This indeed makes sense since $\mathcal{F}[\cdot]$ is an isomorphism $\mathcal{S} \longrightarrow \mathcal{S}$.[2]

Example. Fix a point mass $a \in \mathbb{R}$ and let δ_a be the Dirac (tempered) distribution given by

$$\langle \delta_a, \phi \rangle = \phi(a).$$

Note now that

$$\left\langle \widehat{\delta_a}, \phi \right\rangle = \left\langle \delta_a, \widehat{\phi} \right\rangle = \widehat{\phi}(a) = \int_{\mathbb{R}} \phi(x) e^{-iax}\, \mathrm{d}x.$$

Thus, $\left\langle \widehat{\delta_a}, \phi \right\rangle = \langle F, \phi \rangle$, where F is the distribution induced by e^{-iax}. We shall then say that $\widehat{\delta_a}$ is equal to e^{-iax} in the <u>sense of distributions</u>.

[2]For a proof of this consult the notes that I have linked many times already.