# HONOURS ANALYSIS IV FINAL REVIEW PROBLEMS

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In this text we shall solve problems suggested by Professor Toth as practice/review problems for the final exam. These problems are taken from Stein and Shakarchi's *Measure Theory, Integration and Hilbert Spaces* book. Detailed solutions to most of the problems assigned are given below:

Contents



## 1. Chapter 1 Problems

**Problem 16.** *Prove the Borel-Cantelli Lemma. Let $\{E_k\}_{k\in\mathbb{N}}$ be a countable family of measurable sets and suppose that*

$$\sum_{k\in\mathbb{N}} \mu(E_k) < \infty$$

*Then, if $E$ denotes the set of points belonging to infinitely many of the $E_k$ one must have $\mu(E) = 0$.*

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*Solution.* The key step is to note that

$$E = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k$$

This type of decomposition is widely encountered in probability theory but we explain it here nonetheless. Clearly, if $x$ lies in infinitely many of the $E_k$ there is no largest $k$ for which $x \in E_k$ whence $E \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k$. For the reverse inclusion, if $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k$ then $x \in \bigcup_{k\geq n} E_k$ for all $n \in \mathbb{N}$ and therefore lies in $E$.

Now, each term of the sum above is non-negative and therefore for each $\varepsilon > 0$ we may find $N \in \mathbb{N}$ so large that

$$\sum_{k\geq N} \mu(E_k) \leq \varepsilon \tag{1}$$

This follows from the fact that the tail of $\sum_{k\in\mathbb{N}} \mu(E_k)$ must tend to 0 since the sum is convergent. Thus, from (1)

$$\mu(E) \leq \mu\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k\right) \leq \sum_{k\geq N} \mu(E_k) \leq \varepsilon$$

Since $\varepsilon > 0$ was arbitrary we have $\mu(E) = 0$. □

**Problem 17.** *Let $\{f_n\}$ be a sequence of measurable functions $f_n : [0,1] \to \mathbb{C}$ with $|f_n(x)| < \infty$ for almost all $x \in [0,1]$. Show that there exists a sequence of real numbers $\{c_n\}_{n\in\mathbb{N}}$ such that*

$$\frac{f_n(x)}{c_n} \xrightarrow{n\to\infty} 0$$

*for almost all $x \in [0,1]$.*

*Solution.* We first begin by selecting these $c_n$ "carefully". Define for $m \in \mathbb{N}$ the set:

$$E_m := \left\{x \in [0,1] : \frac{|f_n(x)|}{m} > \frac{1}{n}\right\} \tag{2}$$

Note that $E_{m+1} \subseteq E_m$ for all $m$. We claim that $\mu\left(\bigcap_{m\in\mathbb{N}} E_m\right) = 0$. Indeed, if this $x$ lies in this intersection then $f(x)$ is surely not-finite and therefore the collection of all such $x$ has measure 0. This implies (since these sets are decreasing) that

$$\lim_{m\to\infty} \mu(E_m) = 0$$

hence, for each $n \in \mathbb{N}$ we may associate $c_n \in \mathbb{N}$ (equal to some $m$ in the limit) such that:

$$\mu\left(\left\{x \in [0,1] : \frac{|f_n(x)|}{c_n} > \frac{1}{n}\right\}\right) \leq 2^{-n}$$

Now, call $F_n := \left\{x \in [0,1] : \frac{|f_n(x)|}{c_n} > \frac{1}{n}\right\}$. We claim that if

$$\lim_{n\to\infty} \frac{f_n(x)}{c_n} \neq 0 \tag{3}$$

then $x$ belongs to infinitely many $F_n$. Certainly, if $x$ belonged to only finitely many $F_n$ there is some $N$ such that $x \notin F_n$ for all $n \geq N$. This would imply that

$$\left|\frac{f_n(x)}{c_n}\right| \leq \frac{1}{n}, \quad \forall n \geq N$$

which is absurd. Then, letting $F$ denote the set of all points such that (3) holds we find that $F \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} F_k$ whence the result follows by invoking the Borel-Cantelli lemma (proven in the previous problem). $\square$

## 2. Chapter 2 Problems

**Problem 4.** *Let $b > 0$ and $f \in L^1([0,b])$, define $g : (0,b] \to \mathbb{C}$ by setting*

$$g(x) := \int_x^b \frac{f(t)}{t}\,\mathrm{d}t$$

*Prove that $g$ is integrable on $[0,b]$ and that $\int_0^b g(x)\,\mathrm{d}x = \int_0^b f(t)\,\mathrm{d}t$.*

*Solution.* The idea here is to use Fubini to "swap the order of integration". Typically, I've seen this done with algebraic manipulations of indicator functions but I believe a geometric argument is much better. We "compute":

$$\int_0^b g(x)\,\mathrm{d}x = \int_0^b \int_x^b \frac{f(t)}{t}\,\mathrm{d}t\mathrm{d}x$$

The region of $[0,b] \times [0,b]$ we integrate over here is described in Figure (1) below

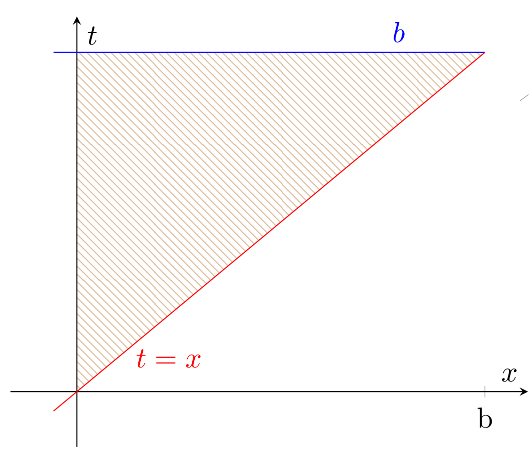


Figure 1. Integration Domain for Problem 4.

From the above we see that we may rewrite this integral as

$$\int_0^b g(x)\,\mathrm{d}x = \int_0^b \int_x^b \frac{f(t)}{t}\,\mathrm{d}t\mathrm{d}x = \int_0^b \int_0^t \frac{f(t)}{t}\,\mathrm{d}x\mathrm{d}t = \int_0^b f(t)\,\mathrm{d}t$$

since $f \in L^1([0,b])$ the proof is complete. (Short eh?)

**Problem 6-(b).** *Prove that if $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and integrable then*

$$\lim_{|x|\to\infty} f(x) = 0 \tag{$\dagger$}$$

*Solution.* We argue by contradiction. Suppose this is not the case, then for some $\varepsilon_0 > 0$ we may find a sequence $(x_n)$ in $\mathbb{R}$ with $|x_n| \to \infty$ such that $|f(x_n)| \geq 2\varepsilon_0$ for all $n$. Now, since $f$ is uniformly continuous for this same $\varepsilon_0 > 0$ there is a $\delta > 0$ such that for all $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon_0$. Note that since there are infinitely many $x_n$ as above we may construct the sequence in such a way that the intervals $(x_n - \delta, x_n + \delta)$ are disjoint for distinct $n$.

Now, for any $x \in (x_n - \delta, x_n + \delta)$ we note $|x - x_n| < \delta$ whence $|f(x) - f(x_n)| < \varepsilon_0$. On the other-hand:

$$|f(x_n)| - |f(x)| < \varepsilon_0$$

which implies that $|f(x)| \geq \varepsilon_0$. This implies that

$$\begin{aligned} \|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)| \, \mathrm{d}x &\geq \int_{\bigsqcup_{n=1}^{\infty}(x_n-\delta,x_n+\delta)} |f(x)| \, \mathrm{d}x \\ &\geq \varepsilon_0 \int_{\bigsqcup_{n=1}^{\infty}(x_n-\delta,x_n+\delta)} \mathrm{d}x \\ &= \varepsilon_0 \sum_{n\in\mathbb{N}} 2\delta = \infty \end{aligned}$$

contradicting the integrability of $f$.

**Problem 11.** *If $f \in L^1(\mathbb{R}^d)$ and for all measurable sets $E \subseteq \mathbb{R}^d$ one has $\int_E f \geq 0$ then $f \geq 0$ almost-everywhere. Moreover, if $\int_E f = 0$ for all measurable sets $E$ we have $f = 0$ almost-everywhere.*

*Solution.* We first assume $\int_E f \geq 0$ for all measurable sets $E$. Let $n \in \mathbb{N}$ be given and define

$$A_n := \left\{ x \in \mathbb{R}^d : f(x) < -\frac{1}{n} \right\}$$

then, $A := \{f < 0\} = \bigcup_{n\in\mathbb{N}} A_n$. Now, since $A_n$ is measurable and

$$-\frac{1}{n}\mu(A_n) \geq \int_{A_n} f(x) \, \mathrm{d}x = 0$$

we conclude $\mu(A_n) =$ so that $\mu(A) \leq \sum_{n\in\mathbb{N}} \mu(A_n) = 0$. So, $f \geq 0$ almost-everywhere. Assume further that $\int_E f = 0$ for all measurable sets $E$. We still have $f \geq 0$ almost everywhere. Let now $B_n := \left\{ x \in \mathbb{R}^d : f(x) > \frac{1}{n} \right\}$ so that

$$\{f > 0\} = \bigcup_{n\in\mathbb{N}} B_n$$

by the argument above it suffices to prove that $\mu(B_n) = 0$ for all $n$. To see this write,

$$0 = \int_{B_n} f(x)\,\mathrm{d}x \geq \frac{1}{n}\mu(B_n)$$

**Problem 12.** *Show that there is $f \in L^1(\mathbb{R})$ and a sequence $\{f_n\}$ in $L^1(\mathbb{R})$ with $\|f_n - f\| \to 0$ but $f_n(x) \not\to f(x)$ for all $x$.*

*Solution.* For all $m$ define

$$-2^{2m} \leq k \leq 2^{2m}, \quad k \in \mathbb{Z} \tag{4}$$

Thus, for $m \in \mathbb{N}$ a *finite range* for $k$ is induced. We first construct a finite "subsequence" for $m \in \mathbb{N}$. Let $m \in \mathbb{N}$ be given, define

$$(f_{m,k}) := \mathbb{1}_{\left(\frac{k}{2^m}, \frac{k+1}{2^m}\right]}$$

for $k$ satisfying (4) for the associated $m \in \mathbb{N}$. This defines a sequence $(f_{m,k})_{(m,k)}$, as $\mathbb{Z} \times \mathbb{Z}$ is itself countable. Index these by $n \in \mathbb{N}$ and call the resulting sequence $(f_n)$. The crucial step is to note that for any pair of indices $(m, k)$ (note $k$ always has finite bounds):

$$\int f_{m,k} = \int_{\mathbb{R}} \mathbb{1}_{\left(\frac{k}{2^m}, \frac{k+1}{2^m}\right]} = \mu\left(\left(\frac{k}{2^m}, \frac{k+1}{2^m}\right]\right) = \frac{1}{2^m}$$

It is then obvious that

$$\lim_{n\to\infty} \int f_n = \lim_{m\to\infty} \int f_{m,k} = 0 \tag{5}$$

This is because if $n \to \infty$, as $k$ always has a finite range, one has $m \to \infty$. It then follows, if we set $f \equiv 0$ on $\mathbb{R}$, that

$$\lim_{n\to\infty} \int |f_n - f| = 0 \tag{6}$$

On the other hand, if we let $x \in \mathbb{R}$ be given, we see that $\lim_{n\to\infty} f_n(x)$ is undefined. This is easy to see, since there are infinitely many $f_{m,k}$ where $f_{m,k}(x) = 0$ and likewise there are infinitely many where $f_{m,k}(x) = 1$ (as the intervals were taken to be disjoint for fixed $m$). If we had pointwise convergence, we would have two subsequences with different limits, which is impossible for a convergent sequence.

This is valid for any $x \in \mathbb{R}$, as for large enough $n$ the intervals $\pm\left(\frac{k}{2^m}, \frac{k+1}{2^m}\right]$ will eventually contain $x$ for some $k$ given by (4).

**Problem 24.** *Recall that for $f, g \in L^1(\mathbb{R}^d)$ we define the convolution:*

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x - \xi)g(\xi)\,\mathrm{d}\xi$$

*Show the following:*

(a) *If $g$ is bounded then $(f * g)$ is uniformly continuous.*
(b) *If $g$ is integrable then $(f * g)(x) \to 0$ as $|x| \to \infty$.*

*Solution.*

(a) Let $x, y \in \mathbb{R}^d$ and label $h := x - y$. We shall then write out

$$\begin{aligned}|(f * g)(x) - (f * g)(y)| &= \left| \int_{\mathbb{R}^d} (f(x-\xi) - f(y-\xi))\, g(\xi)\, \mathrm{d}\xi \right| \\ &\leq M \int_{\mathbb{R}} |f(y+h-\xi) - f(y-\xi)|\, \mathrm{d}\xi \\ &= M \|f_h - f\|\end{aligned}$$

where $M > 0$ is a bound on $g$ and $f_h := f(x+h)$. It then suffices to show that $\|f_h - f\|$ may be made arbitrarily small by choosing $h$ small enough. To see this, recall that $C_0(\mathbb{R}^d)$ is a dense subspace of $L^1(\mathbb{R}^d)$. We may therefore choose a continuous function of compact support, say, $p(x)$ such that

$$\|f - p\| = \|f_h - p_h\| \leq \varepsilon$$

for given $\varepsilon > 0$.[1] Now by the triangle inequality we have

$$\|f_h - f\| \leq \|f_h - g_h\| + \|g_h - g\| + \|g - f\| \leq 2\varepsilon + \|g_h - g\|$$

Since $g_h, g$ are continuous functions of compact support they are bounded, and an application of the bounded convergence theorem shows that $\|g_h - g\| \to 0$ as $h \to \infty$. Therefore, there exists $\delta > 0$ such that $\|f_h - f\| \leq 3\varepsilon$ whenever $|h| < \delta$.

(b) Having established the above, we observe that it is a consequence of Problem-6-(b) that it suffices to show that $(f * g)$ is integrable whenever $f, g$ are. This is an easy application of Fubini-Tonelli:

$$\begin{aligned}\|f * g\| \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-\xi)|\, |g(\xi)|\, \mathrm{d}\xi \mathrm{d}x &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-\xi)|\, |g(\xi)|\, \mathrm{d}x \mathrm{d}\xi \\ &= \int_{\mathbb{R}} |g(\xi)| \int_{\mathbb{R}} |f(x-\xi)|\, \mathrm{d}x \mathrm{d}\xi \\ &= \int_{\mathbb{R}} |g(\xi)| \int_{\mathbb{R}} |f(x)|\, \mathrm{d}x \mathrm{d}\xi \\ &= \|f\| \cdot \|g\| < \infty\end{aligned}$$

## 3. Chapter 4 Problems

**Problem 5.** *Prove that*

*(a) Neither $L^2(\mathbb{R}^d) \subseteq L^1(\mathbb{R}^d)$ nor $L^2(\mathbb{R}^d) \supseteq L^1(\mathbb{R}^d)$ holds true.*
*(b) However, if $f \in L^2(\mathbb{R}^d)$ is supported on a set of finite measure then $f \in L^1(\mathbb{R}^d)$.*
*(c) Similarly, if $f \in L^1(\mathbb{R}^d)$ is bounded then $f \in L^2(\mathbb{R}^d)$.*

*Solution.*

(a) Consider $f = 1/x$ on $(1, \infty)$. Clearly, $\int_1^\infty \frac{1}{x^2}\, \mathrm{d}x < \infty$ but $\int_1^x \frac{1}{x}\, \mathrm{d}x = \infty$. Similarly, let $g(x) = 1/\sqrt{x}$ on $(0, 1)$. Then, $g$ is integrable but not square-integrable, as is seen by usual calculus methods together with monotone convergence.

[1] This last equality follows from translation invariance of the Lebesgue integral.

(b) Let $E \supseteq \operatorname{supp}(f)$ be a measurable set. It suffices to estimate $\|f\|_{L^1(\mathbb{R}^d)}$ as follows:

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)|\, \mathrm{d}x = \int_{\mathbb{R}^d} |f(x)|\, \mathbb{1}_E(x)\, \mathrm{d}x &= \int_{\mathbb{R}^d} |f(x)|\, \overline{\mathbb{1}_E(x)}\, \mathrm{d}x \\ &= \langle |f|, \mathbb{1}_E \rangle \\ &\leq \|f\|_{L^2} \left( \int_{\mathbb{R}^d} \mathbb{1}_E(x)\, \mathrm{d}x \right)^{1/2}\end{aligned}$$

whence $\|f\|_{L^1} \leq \|f\|_{L^1} \sqrt{\mu(E)}$.

(c) Assume there exists $M > 0$ such that $|f(x)| \leq M$ on $\mathbb{R}^d$. We write

$$\|f\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)|^2\, \mathrm{d}x \leq M \int_{\mathbb{R}^d} |f(x)|\, \mathrm{d}x = M\, \|f\|_{L^1(\mathbb{R}^d)}$$

**Problem 26.** *Suppose $\omega : \mathbb{R}^d \to (0, \infty]$ is a measurable function that is finite almost-everywhere. and that $K$ is a measurable function defined on $\mathbb{R}^{2d}$ that satisfies*

$$\int_{\mathbb{R}^d} |K(x,y)|\, \omega(y)\, \mathrm{d}y \leq A\omega(x), \qquad \textit{for almost all } x \in \mathbb{R}^d \tag{7}$$

$$\int_{\mathbb{R}^d} |K(x,y)|\, \omega(x)\, \mathrm{d}x \leq A\omega(y), \qquad \textit{for almost all } y \in \mathbb{R}^d \tag{8}$$

*where $A > 0$ is a fixed constant. Prove that the integral operator*

$$Tf(x) := \int_{\mathbb{R}^d} K(x,y) f(y)\, \mathrm{d}y$$

*is bounded, and hence continuous.*

*Solution.* We first estimate for almost all $x \in \mathbb{R}^d$:

$$\begin{aligned}\int_{\mathbb{R}^d} |K(x,y)|\, |f(y)|\, \mathrm{d}y &= \int_{\mathbb{R}^d} |K(x,y)|^{1/2} \omega(y)^{1/2} \omega(y)^{-1/2} |K(x,y)|^{1/2} |f(y)|\, \mathrm{d}y \\ &= \int_{\mathbb{R}^d} |K(x,y)|^{1/2} \omega(y)^{1/2} \overline{\omega(y)^{-1/2} |K(x,y)|^{1/2} |f(y)|}\, \mathrm{d}y \\ &\leq \sqrt{A\omega(x)} \left( \int_{\mathbb{R}^d} \omega(y)^{-1} |K(x,y)|\, |f(y)|^2\, \mathrm{d}y \right)^{1/2} \qquad \text{(by (7))}\end{aligned}$$

Now, we may calculate:

$$\begin{aligned}\|Tf\|^2_{L^2(\mathbb{R}^d)} &\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |K(x,y)|\, |f(y)|\, \mathrm{d}y \right)^2 \mathrm{d}x \leq \int_{\mathbb{R}^d} A\omega(x) \int_{\mathbb{R}^d} \omega(y)^{-1} |K(x,y)|\, |f(y)|^2\, \mathrm{d}y\, \mathrm{d}x \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} A\omega(x)\omega(y)^{-1} |K(x,y)|\, |f(y)|^2\, \mathrm{d}x\mathrm{d}y \\ &\leq \int_{\mathbb{R}} A^2 \omega(y)\omega(y)^{-1} |f(y)|^2\, \mathrm{d}y\end{aligned}$$

by (8); which is no-greater than $A^2 \|f\|^2_{L^2}$. Note that we also get $\|T\| \leq A$ as a consequence.

Before we tackle problem 28 we shall prove a lemma, which may be found in Stein and Shakarchi. We give it here, nonetheless, for the sake of completeness.

**Lemma.** *Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces over a field $\mathbb{F}$ and $\{T_n\}_{n=1}^{\infty}, T : \mathcal{B}_1 \to \mathcal{B}_2$ bounded linear operators. If the $T_n$ are all compact, and $\|T_n - T\| \to 0$ as $n \to \infty$, then $T$ is compact.*

*Proof.* Let $(f_n)_n$ be a sequence in $\mathcal{B}_1$ that is bounded. There is a subsequence $(f_{1n})$ such that $\{T_1(f_{1n})\}$ converges. From this same subsequence we extract a subsequence such that $\{T_2(f_{2n})\}$ converges and so forth. This grants us a sequence of subsequences:

$$\{f_{kn}\}_{k\in\mathbb{N}}$$

such that $\{T_k(f_{kn})\}$ converges, for $k \in \mathbb{N}$. Consider the subsequence $(g_k)_{k\in\mathbb{N}}$ where we take the $k^{\text{th}}$ entry of the sequence $(f_{kn})$. Then, for all $k, \ell$ large we estimate:

$$\|T(g_k) - T(g_\ell)\| \leq \|T(g_k) - T_n(g_k)\| + \|T_n(g_k) - T_n(g_\ell)\| + \|T_n(g_\ell) - T(g_\ell)\|$$

Since $\|T - T_n\| \to 0$, by taking $n$ large enough we may these first and last terms $\leq \varepsilon$. Now, since $T_n(g_k)$ converges by construction, as $k \to \infty$ the sequence $\{T(g_k)\}_k$ is Cauchy, and by taking $k, \ell$ large we make this term $\leq \varepsilon$. $\square$

**Problem 28.** Let $B$ be the unit ball in $\mathbb{R}^d$, and let $\mathcal{H} = L^2(B)$. Suppose $K(x, y)$ is a measurable function on $B \times B$ satisfying

$$|K(x,y)| \leq A\,|x-y|^{-d+\alpha}$$

for some $\alpha > 0$. Define an operator:

$$Tf(x) := \int_B K(x,y)f(y)\,\mathrm{d}y$$

Prove the following:

(a) $T$ is a bounded operator.
(b) $T$ is compact.

*Solution.*

(a) We first calculate:

$$\int_B |K(x,y)|\,\mathrm{d}y \leq \int_B \frac{1}{|x-y|^{d-\alpha}}\,\mathrm{d}y = \int_{B(x,1)} \frac{1}{|y|^{d-\alpha}}\,\mathrm{d}y < \infty$$

where we know that this is finite because $\alpha > 0$. The key is to observe that there is a bound independent of $x$. Certainly, since $|x| \leq 1$ we find that

$$\int_B |K(x,y)|\,\mathrm{d}y \leq \int_{|y|\leq 10} \frac{1}{|y|^{d-\alpha}}\,\mathrm{d}y = C < \infty$$

In the above we could have probably chosen a bound on $|y|$ less than 10, but this works and who really cares anyways. Hence, it follows from Problem 26 with $\omega := 1$ that $T[\cdot]$ is a bounded operator.

(b) Let us define a series of kernels by

$$K_n(x,y) := \begin{cases} K(x,y) & |x-y| \geq \frac{1}{n} \\ 0 & \text{else} \end{cases}$$

This induces an operator: $T_n f(x) := \int_B K_n(x,y) f(y)\,\mathrm{d}y$. We claim that the $T_n$'s are compact. To see this, we need only show they are Hilbert-Schmidt and this follows from the following observation:

$$\begin{aligned}\int_{B^2} |K_n(x,y)|^2 \,\mathrm{d}y\mathrm{d}x &= \int_{\{|x-y|\geq 1/n\}\cap B^2} |K(x,y)|^2 \,\mathrm{d}y\mathrm{d}x \\ &\leq \int_{\{|x-y|\geq 1/n\}\cap B^2} \frac{1}{|x-y|^{2(\alpha-d)}} \,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

which is clearly finite, for any $\alpha > 0$. Thus, $K \in L^2(B^2)$ and is compact. In light of the previous lemma, we need only show $\|T_n - T\| \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$ let us calculate:

$$\begin{aligned}\int_B |K_n(x,y) - K(x,y)| \,\mathrm{d}y &= \int_{|x-y|\leq 1/n} A\,|x-y|^{-d+\alpha} \,\mathrm{d}y \\ &= A \int_{B(0,n)} \frac{1}{|x|^{-d+\alpha}} \,\mathrm{d}x \\ &= AC_n\end{aligned}$$

with $C_n := \int_{B(0,n)} \frac{1}{|x|^{-d+\alpha}}\,\mathrm{d}x$. By Problem 26, we get as a corollary that $\|T_n - T\| \leq C_n$ for all $n$. This integral may be evaluated directly, and it is not hard to see that it tends to 0 as $n \to \infty$. Thus, $T$ is compact.